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On fractional differential equations and fixed point theory

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Abstract

In this work first we establish some fixed point theorems for $\perp-$Mizoguchi-Takahashi contractions mappings in the setting of orthogonal metric spaces. Next, we investigate the existence of solution for certain fractional differential equation via some integral boundary value conditions and obtained fixed point results.

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1. Introduction

In the line of research of multi-valued mappings, one of the most important results was given by Nadler [8]. He extended the Banach contraction principle to multi-valued mappings and then proved the following famous theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping, where $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X. We say T is a multi-valued contraction, if there exists $r \in [0, 1)$ such that,*

$$H(Tx, Ty) \leq rd(x, y)$$

holds for all $x, y \in X$, where H is the Pompeiu-Hausdorff metric on $CB(X)$ defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Then there exists $x^* \in X$ such that $x^* \in Tx^*$ (i.e., T has a fixed point).

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Reich [10] established and proved some results for multi-valued nonlinear contractions as a generalization of Theorem 1.1. He then asked whether his results can be extended to multi-valued mappings whose range consists of nonempty bounded closed sets (see [11]). This problem is called Reich conjecture. There are some imprecise answers to this conjecture. One of the famous answers to Reich conjecture was given by Mizoguchi and Takahashi with the substitution $[0, \infty)$ instead of $(0, \infty)$ in hypothesis of results of Reich.

Eshaghi et.al. [4] introduced the notion of orthogonal set and gave a real generalization of Banach contraction principle in orthogonal metric spaces (For more details on orthogonal set, also see [2]).

Definition 1.2. *Let $X \neq \emptyset$ and $\perp \in X \times X$ be an binary relation. Assume that there exists $x_0 \in X$ such that $x_0 \perp x$ for all $x \in X$. Hence we say that X is an orthogonal set (briefly O-set). We denote orthogonal set by $(X, \perp)$. Also, suppose that $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if $(\forall n; x_n \perp x_{n+1})$.*

Definition 1.3. *Let X be a metric space and $M \subseteq X$.*

- *M is an orthogonal metric space if $(M, \perp)$ is an O-set.*
- *$T : M \to M$ is $\perp-$continuous in $x \in M$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, $\lim_{n\to\infty} d(x_n, x) = 0$, implies, $\lim_{n\to\infty} d(Tx_n, Tx) = 0$. Furthermore, T is $\perp$-continuous when T is $\perp$-continuous in each $x \in M$.*
- *$T : M \to CB(M)$ is $\perp^\star-$continuous in $x \in M$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, $\lim_{n\to\infty} d(x_n, x) = 0$, implies, $\lim_{n\to\infty} H(Tx_n, Tx) = 0$. Also, T is $\perp^\star$-continuous when T is $\perp^\star$-continuous in each $x \in M$.*
- *We say $T : M \to M$ is $\perp$-preserving if $Tx \perp Ty$ whence $x \perp y$.*
- *We say $T : M \to CB(M)$ is $\perp^\star$-preserving, when $x \perp y$ implies $u \perp v$ for all $u \in Tx$ and $v \in Ty$.*
- *Finally, X is orthogonal complete if every Cauchy O-sequence is convergent.*

Example 1.4. *If $0 < \rho \leq 1$, let $\Lambda_\rho([0, 1])$ be the space of Hölder continuous functions of the exponent ρ in $[0; 1]$. That is, $f \in \Lambda_\rho([0, 1])$ if and only if $\|f\|_{\Lambda_\rho} < \infty$, where*

$$\|f\|_{\Lambda_\rho} = |f(0)| + \sup_{x,y\in[0,1],\ x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\rho}.$$

For $0 < \rho \leq 1$, assume that

$$\lambda_\rho([0, 1]) = \{f \in \Lambda_\rho([0, 1])|\ \lim_{x\to y} \frac{|f(x) - f(y)|}{|x - y|^\rho} = 0,\ \forall y \in [0, 1]\}.$$

Now for all $\rho, \tau \in (0, 1]$, define $\lambda_\rho([0, 1]) \perp \lambda_\tau([0, 1])$ if and only if $\lambda_{\frac{\rho\tau}{2}}([0, 1])$ be an infinite-dimensional closed subspace of $\Lambda_{\frac{\rho\tau}{2}}([0, 1])$. Therefore, $(\{\lambda_\rho([0, 1])\}_{\rho\in(0,1]}, \perp)$ is an O$-$ set.

Fractional differential equations(FDEs for short) has generated much interest in recent years. Many researchers have investigated on FDEs by utilizing different methods and techniques(see [1, 3, 14, 13, 15] and references therein). It is well known that The Riemann$-$Liouville fractional integral of order

α of a function f is defined by $I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds(\alpha > 0)$ and the Caputo derivative of order α for a function f is defined by

$$cD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds$$

where $n = [\alpha] + 1$(for more detailes on Riemann−Liouville fractional integral and Caputo derivative see, [6, 9, 12]).

In 2007, Xinwei and Landong [13] investigated the existence and uniqueness of solutions for the fractional differential equation

$$cD^{\alpha}u(t) = f(t, u(t), cD^{\beta}u(t)),\ 0 \leq t \leq 1$$

with boundary values $u(0) = u'(1) = 0$ or $u'(0) = u(1) = 0$ or $u(0) = u(1) = 0$, whence $1 < \alpha \leq 2$, $0 < \beta \leq 1$ and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

In 2009, Su and Zhang [14] reviewed the existence of solutions for the following fractional differential equation

$$cD^{\alpha}u(t) = f(t, u(t), cD^{\beta}u(t)),\ 0 \leq t \leq 1$$

with boundary values $a_1u(0) - a_2u'(0) = A$ and $b_1u(1) - b_2u'(1) = B$, where α, a_1, a_2, b_1, b_2 satisfy some conditions.

Motivated by the above results, we consider the following fractional differential equation

$$cD^{\alpha}u(t) = f(t, u(t), cD^{\beta_1}u(t), cD^{\beta_2}u(t), ...cD^{\beta_n}u(t)),\ 0 \leq t \leq 1,\ 0 < \beta_i < 1,\ i = 1, 2, ...n$$

with boundary values

$$\begin{cases} a_1u(0) + a_2(cD^{\gamma}u(0)) = \int_0^1 g_0(s, u(s))ds, \\ b_1u(1) + b_2(cD^{\gamma}u(1)) = \int_0^1 g_1(s, u(s))ds \\ b_1u(\eta) + b_2(cD^{\gamma}u(\eta)) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

where, $2 < \alpha \leq 3$, $0 < \gamma < 1$, $0 < \eta$, $a_1 \neq 0$, $a_2, b_1, b_2 \in \mathbb{R}$ and $g_0, g_1, g_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $f : [0, 1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous.

2. Fixed point results

In this section we state and prove two fixed point theorems for Mizoguchi-Takahash type contractions mappings in the setting of orthogonal metric spaces.

We denote by Δ_{δ} the set of all functions $\delta : (0, +\infty) \to (0, 1)$ satisfying the following condition:

$$\limsup_{t \to s^+} \delta(t) < 1,\ \forall s \in [0, \infty).$$

Theorem 2.1. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^{\star}-$preserving multifunction. There exists $\delta \in \Delta_{\delta}$ such that*

$$\begin{cases} x \neq y, \\ x \perp y, \end{cases} \implies H(Tx, Ty) \leq \delta(d(x, y))d(x, y). \tag{2.1}$$

Also, T is $\perp^{\star}-$continuous. Then T has a fixed point.

Proof . From Definition 1.2 there exists $x_0 \in X$ such that $x_0 \perp y$ for all $y \in X$. Let $x_1 \in Tx_0$, so $x_0 \perp x_1$. If $x_0 = x_1$ then x_0 is a fixed point of T. Hence we assume that $x_0 \neq x_1$. and $d(x_0, x_1) > 0$. Since $Tx_0, Tx_1 \in CB(X)$, there exists a point $x_2 \in Tx_1$ such that,

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + (\tfrac{1}{\sqrt{\delta(d(x_0,x_1))}} - 1)H(Tx_0, Tx_1) \\ &= \tfrac{1}{\sqrt{\delta(d(x_0,x_1))}} H(Tx_0, Tx_1) \end{aligned}$$

and so from (2.1) we have,

$$d(x_1, x_2) \leq \tfrac{1}{\sqrt{\delta(d(x_0,x_1))}} H(Tx_0, Tx_1) \leq \sqrt{\delta(d(x_0, x_1))} d(x_0, x_1).$$

Since T is an $\perp^\star-$preserving then, $x_0 \perp x_1$ implies $u \perp v$ for all $u \in Tx_0$ and $v \in Tx_1$. This implies $x_1 \perp x_2$. If $x_1 = x_2$ then x_1 is a fixed point of T. Hence we assume that $x_0 \neq x_1$. If $d(x_1, x_2) = 0$, then $x_1 = x_2$, which is a contradiction, so we assume that $d(x_1, x_2) > 0$. Again, since $Tx_1, Tx_2 \in CB(X)$, there exists a point $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + (\tfrac{1}{\sqrt{\delta(d(x_1,x_2))}} - 1)H(Tx_1, Tx_2) \\ &= \tfrac{1}{\sqrt{\delta(d(x_1,x_2))}} H(Tx_1, Tx_2), \end{aligned}$$

and so from (2.1) we have,

$$d(x_2, x_3) \leq \tfrac{1}{\sqrt{\delta(d(x_1,x_2))}} H(Tx_1, Tx_2) \leq \sqrt{\delta(d(x_1, x_2))} d(x_1, x_2).$$

Continuing in this fashion, we obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \perp x_{n-1}$, $x_n \neq x_{n-1}$, $d(x_{n-1}, x_n) > 0$ and

$$d(x_n, x_{n+1}) \leq \sqrt{\delta(d(x_{n-1}, x_n))} d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing and so convergent. Since, $\beta \in \Delta_\beta$, then there exist $b \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $\delta(d(x_{n-1}, x_n)) < b$ for all $n \geq n_0$. Now, we obtain, for all $n \geq n_0$,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \sqrt{\delta(d(x_{n-1}, x_n))} d(x_{n-1}, x_n) \\ &\leq \sqrt{\delta(d(x_{n-2}, x_{n-1}))} \sqrt{\alpha(d(x_{n-1}, x_n))} d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \sqrt{\delta(d(x_0, x_1))} \dots \sqrt{\delta(d(x_{n_0-1}, x_{n_0}))} \sqrt{\delta(d(x_{n_0}, x_{n_0+1}))} \dots \\ &\sqrt{\delta(d(x_{n-2}, x_{n-1}))} \sqrt{\delta(d(x_{n-1}, x_n))} d(x_0, x_1) \\ &\leq \sqrt{\delta(d(x_{n_0-1}, x_{n_0}))} \sqrt{\delta(d(x_{n_0}, x_{n_0+1}))} \dots \\ &\sqrt{\delta(d(x_{n-2}, x_{n-1}))} \sqrt{\delta(d(x_{n-1}, x_n))} d(x_0, x_1) \\ &\leq (\sqrt{b})^{(n-n_0)} d(x_0, x_1). \end{aligned}$$

That is,

$$d(x_n, x_{n+1}) \leq r^{(n-n_0)} d(x_0, x_1).$$

for all $n \geq n_0$, where $0 < r := \sqrt{b} < 1$. Now for all $m > n \geq n_0$ we have,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} r^{(i-n_0)} d(x_0, x_1) < \sum_{i=n}^{\infty} r^{(i-n_0)} d(x_0, x_1)$$

Since the series $\sum_{i=1}^{\infty} r^{(i-n_0)} d(x_0, x_1)$, is convergence, this implies

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Hence we proved that $\{x_n\}$ is a Cauchy sequence. Since X is an complete, then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is $\perp^\star-$continuous, then

$$\lim_{n\to\infty} H(Tx_{n-1}, Tz) = 0.$$

For each $x_n \in Tx_{n-1}$ $(n \in \mathbb{N})$ there exists $y_n \in Tz$ such that

$$d(x_n, y_n) < H(Tx_{n-1}, Tz) + \frac{1}{n}.$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Therefore

$$d(z, y_n) \leq d(z, x_n) + d(x_n, y_n).$$

By taking limit as $n \to \infty$ in the above inequality we get $\lim_{n\to\infty} d(z, y_n) = 0$. That is the sequence $\{y_n\}$ converges to z. Since Tz is closed then $z \in Tz$. □

The following Corollary is Theorem of Nadler (Theorem 1.1) in the setting of orthogonal metric spaces.

Corollary 2.2. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^\star-$preserving multifunction. There exists $0 < k < 1$ such that*

$$\begin{cases} x \neq y, \\ x \perp y, \end{cases} \implies H(Tx, Ty) \leq k d(x, y).$$

Also, T is $\perp^\star-$continuous. Then T has a fixed point.

If in Theorem 2.1 we take $\delta(t) = \frac{1}{1+t} (t > 0)$ then we obtain the following Corollary.

Corollary 2.3. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^\star-$preserving multifunction. Assume that for $x \neq y$ and $x \perp y$,*

$$H(Tx, Ty) \leq \frac{d(x, y)}{1 + d(x, y)}.$$

Also, T is $\perp^\star-$continuous. Then T has a fixed point.

For $\perp-$Mizoguchi-Takahash that is not $\perp-$continuous we have the following theorem.

Theorem 2.4. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^\star-$preserving multifunction. There exists $\delta \in \Delta_\delta$,*

$$\begin{cases} x \neq y, \\ x \perp y, \end{cases} \implies H(Tx, Ty) \leq \delta(d(x, y))d(x, y). \tag{2.2}$$

Let $\{x_n\}$ be an $O-$sequence in X with $x_n \to x$ as $n \to \infty$, then $x_n \perp x$ hold for all $n \in \mathbb{N}$. Then T has a fixed point.

Proof . As in the proof of Theorem 2.1 we deduce an O-sequence $\{x_n\}$ starting at x_0 is Cauchy and so converges to a point $z \in X$. Then, we have

$$x_n \perp z.$$

Now if $x_n = z$ for some $n \in \mathbb{N}$, then clearly,

$$H(Tx_n, Tz) \leq d(x_n, z).$$

Also, if $x_n \neq z$ for some $n \in \mathbb{N}$, then from (2.2) we obtain,

$$H(Tx_n, Tz) \leq \delta(d(x_n, z))d(x_n, z) \leq d(x_n, z).$$

That is, for all $n \in \mathbb{N}$, we have,

$$H(Tx_n, Tz) \leq d(x_n, z).$$

Now we can write,

$$\begin{aligned} d(z, Tz) &\leq d(z, Tx_n) + H_1(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + H_1(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + d(x_n, z). \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get, $d(z, Tz) = 0$. Now there exists a sequence $\{y_n\} \subseteq Tz$ such that $\lim_{n\to\infty} d(z, y_n) = 0$. Since Tz is closed then $z \in Tz$. □

Corollary 2.5. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^\star-$preserving multifunction. There exists $0 < k < 1$ such that*

$$\begin{cases} x \neq y, \\ x \perp y, \end{cases} \implies H(Tx, Ty) \leq kd(x, y).$$

Let $\{x_n\}$ be an $O-$sequence in X with $x_n \to x$ as $n \to \infty$, then $x_n \perp x$ hold for all $n \in \mathbb{N}$. Then T has a fixed point.

Corollary 2.6. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^\star-$preserving multifunction. Assume that for $x \neq y$ and $x \perp y$,*

$$H(Tx, Ty) \leq \frac{d(x, y)}{1 + d(x, y)}.$$

Let $\{x_n\}$ be an $O-$sequence in X with $x_n \to x$ as $n \to \infty$, then $x_n \perp x$ hold for all $n \in \mathbb{N}$. Then T has a fixed point.

Corollary 2.7. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to X$ be an $\perp-$preserving self-mapping. There exists $\delta \in \Delta_\delta$,*

$$\begin{cases} x \neq y, \\ x \perp y, \end{cases} \implies d(Tx, Ty) \leq \delta(d(x, y))d(x, y).$$

Let $\{x_n\}$ be an $O-$sequence in X with $x_n \to x$ as $n \to \infty$, then $x_n \perp x$ hold for all $n \in \mathbb{N}$. Then T has a fixed point.

Corollary 2.8. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to CB(X)$ be an $\perp^\star-$preserving multifunction. Assume that for $x \neq y$ and $x \perp y$,*

$$H(Tx, Ty) \leq \frac{d(x, y)}{1 + d(x, y)}.$$

Let $\{x_n\}$ be an $O-$sequence in X with $x_n \to x$ as $n \to \infty$, then $x_n \perp x$ hold for all $n \in \mathbb{N}$. Then T has a fixed point.

Corollary 2.9. *Let $(X, d, \perp)$ be an orthogonal complete metric space. Let $T : X \to X$ be an $\perp-$preserving self-mapping. There exists $\delta \in \Delta_\delta$,*

$$\begin{cases} x \neq y, \\ x \perp y, \end{cases} \implies d(Tx, Ty) \leq \delta(d(x, y))d(x, y).$$

Let $\{x_n\}$ be an $O-$sequence in X with $x_n \to x$ as $n \to \infty$, then $x_n \perp x$ hold for all $n \in \mathbb{N}$. Then T has a fixed point.

3. Fractional Differential Equation

In this section, we study the existence of solutions for the following fractional differential equation

$$cD^\alpha u(t) = f(t, u(t), cD^{\beta_1}u(t), cD^{\beta_2}u(t), ...cD^{\beta_n}u(t)),\ 0 \leq t \leq 1,\ 0 < \beta_i < 1,\ i = 1, 2, ...n \tag{3.1}$$

with boundary values

$$\begin{cases} a_1 u(0) + a_2(cD^\gamma u(0)) = \int_0^1 g_0(s, u(s))ds, \\ b_1 u(1) + b_2(cD^\gamma u(1)) = \int_0^1 g_1(s, u(s))ds \\ b_1 u(\eta) + b_2(cD^\gamma u(\eta)) = \int_0^1 g_2(s, u(s))ds \end{cases} \tag{3.2}$$

where, $2 < \alpha \leq 3$, $0 < \gamma < 1$, $0 < \eta$, $a_1 \neq 0$, $a_2, b_1, b_2 \in \mathbb{R}$ and $g_0, g_1, g_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $f : [0, 1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous.

In this way we need the following Lemma.

Lemma 3.1. *[6] Assume that $\alpha > 0$ and $n = [\alpha] + 1$. Then,*

$$I^\alpha cD^\alpha u(t) = u(t) + l_0 + l_1 t + l_2 t^2 + ... + l_{n-1}t^{n-1},$$

where $l_0, l_1, l_2, ..., l_{n-1}$ are some real numbers.

First, we prove the following Lemma.

Lemma 3.2. *Let $u \in C(J,\mathbb{R})$ and*

$$R(\eta) = (b_1 + \frac{b_2}{\Gamma(2-\gamma)})(b_1\eta^2 + \frac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)}) - (b_1 + \frac{b_2}{\Gamma(3-\gamma)})(b_1\eta + \frac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)}) > 0.$$

Then the problem

$$cD^{\alpha}u(t) = y(t) \tag{3.3}$$

with boundary values (3.2) has the unique solution

$$\begin{aligned} u(t) &= \Theta(t) + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &+\omega_1(t)\big[\tfrac{b_1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}y(s)ds + \tfrac{b_2}{\Gamma(\alpha-\gamma)}\int_0^{\eta}(\eta-s)^{\alpha-\gamma-1}y(s)ds\big] \\ &+\omega_2(t)\big[\tfrac{b_1}{\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}y(s)ds + \tfrac{b_2}{\Gamma(\alpha-\gamma)}\int_0^{1}(1-s)^{\alpha-\gamma-1}y(s)ds\big] \end{aligned}$$

where

$$\Theta(t) = \omega_0(t)\int_0^1 g_0(s,x(s))ds - \omega_2(t)\int_0^1 g_1(s,x(s))ds - \omega_1(t)\int_0^1 g_2(s,x(s))ds,$$

$$\begin{aligned} \omega_0(t) = \tfrac{1}{a_1} &+\tfrac{b_1 t}{a_1 R(\eta)}\big(b_1(\eta^2-1) + b_2(\tfrac{\eta^{2-\gamma}}{\Gamma(3-\gamma)} - \tfrac{1}{\Gamma(2-\gamma)})\big) \\ &+\tfrac{b_1 t^2}{a_1 R(\eta)}\big(b_1(1-\eta) + b_2(\tfrac{1}{\Gamma(3-\gamma)} - \tfrac{\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big), \end{aligned}$$

$$\omega_1(t) = \frac{t^2}{R(\eta)}(b_1 + \frac{b_2}{\Gamma(3-\gamma)}) - \frac{t}{R(\eta)}(b_1 + \frac{b_2}{\Gamma(2-\gamma)})$$

and

$$\omega_2(t) = \frac{t}{R(\eta)}(b_1\eta^2 + \frac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)}) - \frac{t^2}{R(\eta)}(b_1\eta + \frac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)}).$$

Proof . According to Lemma 3.1, the general solution of (3.3) is

$$u(t) = I^{\alpha}y(t) + l_0 + l_1 t + l_2 t^2,$$

i.e.,

$$u(t) = \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + l_0 + l_1 t + l_2 t^2$$

where l_0, l_1, l_2 are real arbitrary constants. Then,

$${}^{c}D^{\gamma}u(t) = \tfrac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}y(s)ds + \tfrac{l_1 t^{1-\gamma}}{\Gamma(2-\gamma)} + \tfrac{l_2 t^{2-\gamma}}{\Gamma(3-\gamma)}.$$

So we can write,

$$a_1u(0) + a_2({}^{c}D^{\gamma}u(0)) = a_1 l_0 = \int_0^1 g_0(s,u(s))ds, \tag{3.4}$$

$$\begin{aligned} &b_1u(1) + b_2({}^{c}D^{\gamma}u(1)) = b_1 l_0 + (b_1 + \tfrac{b_2}{\Gamma(2-\gamma)})l_1 + (b_1 + \tfrac{b_2}{\Gamma(3-\gamma)})l_2 \\ &+\tfrac{b_1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds + \tfrac{b_2}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}y(s)ds = \int_0^1 g_1(s,u(s))ds \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &b_1u(\eta) + b_2({}^{c}D^{\gamma}u(\eta)) = b_1 l_0 + (b_1\eta + \tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})l_1 + (b_1\eta^2 + \tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})l_2 \\ &+\tfrac{b_1}{\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1}y(s)ds + \tfrac{b_2}{\Gamma(\alpha-\gamma)}\int_0^{\eta} (\eta-s)^{\alpha-\gamma-1}y(s)ds = \int_0^1 g_2(s,u(s))ds. \end{aligned} \tag{3.6}$$

Assume that $\Lambda(t) := \frac{b_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}y(s)ds$. Now by applying (3.4), (3.5) and (3.6) we get

$$\begin{aligned}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})l_1 + (b_1+\tfrac{b_2}{\Gamma(3-\gamma)})l_2 &= -\tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds + \int_0^1 g_1(s,u(s))ds\\ &-\tfrac{b_1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)ds\\ &-\tfrac{b_2}{\Gamma(\alpha-\gamma)}\int_0^1(1-s)^{\alpha-\gamma-1}y(s)ds\\ &= \int_0^1 g_1(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(1)\end{aligned}$$

and

$$\begin{aligned}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})l_1 + (b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})l_2 &= -\tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds + \int_0^1 g_2(s,u(s))ds\\ &-\tfrac{b_1}{\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}y(s)ds\\ &-\tfrac{b_2}{\Gamma(\alpha-\gamma)}\int_0^\eta(\eta-s)^{\alpha-\gamma-1}y(s)ds\\ &= \int_0^1 g_2(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(\eta).\end{aligned}$$

Hence,

$$\begin{aligned}l_1 &= \tfrac{1}{R(\eta)}\Big[(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big(\int_0^1 g_2(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(\eta)\big)\\ &-(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\big(\int_0^1 g_1(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(1)\big)\Big]\end{aligned}$$

and

$$\begin{aligned}l_2 &= -\tfrac{1}{R(\eta)}\Big[(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})\big(\int_0^1 g_2(s,u(s))ds - -\tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(\eta)\big)\\ &-(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big(\int_0^1 g_1(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(1)\big),\Big]\end{aligned}$$

and so,

$$\begin{aligned}u(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \tfrac{1}{a_1}\int_0^1 g_0(s,u(s))ds\\ &+\tfrac{t}{R(\eta)}\Big[(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big(\int_0^1 g_2(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(\eta)\big)\\ &-(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\big(\int_0^1 g_1(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(1)\big)\Big]\\ &-\tfrac{t^2}{R(\eta)}\Big[(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})\big(\int_0^1 g_2(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(\eta)\big)\\ &-(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big(\int_0^1 g_1(s,u(s))ds - \tfrac{b_1}{a_1}\int_0^1 g_0(s,u(s))ds - \Lambda(1)\big)\Big]\\ &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \Big[\tfrac{1}{a_1}+\tfrac{b_1 t}{a_1R(\eta)}\big(b_1(\eta^2-1)+b_2(\tfrac{\eta^{2-\gamma}}{\Gamma(3-\gamma)}-\tfrac{1}{\Gamma(2-\gamma)})\big)\\ &+\tfrac{b_1t^2}{a_1R(\eta)}\big(b_1(1-\eta)+b_2(\tfrac{1}{\Gamma(3-\gamma)}-\tfrac{\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big)\Big]\int_0^1 g_0(s,u(s))ds\\ &+\big[\tfrac{t^2}{R(\eta)}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)}) - \tfrac{t}{R(\eta)}(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\big]\int_0^1 g_1(s,u(s))ds\\ &+\big[\tfrac{t}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)}) - \tfrac{t^2}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})\big]\int_0^1 g_2(s,u(s))ds\\ &+\big[\tfrac{t^2}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(3-\gamma)}) - \tfrac{t}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big]\Lambda(\eta)\\ &+\big[\tfrac{t}{R(\eta)}(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)}) - \tfrac{t^2}{R(\eta)}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big]\Lambda(1).\end{aligned}$$

This completes the proof. □

If in the above Lemma we take $b_1 = 0$ and $a_1 = a_2 = b_2 = 1$ then we have the following Remark.

Remark 3.3. *Let $u \in C(J,\mathbb{R})$ and $R(\eta) = \frac{1}{\Gamma(2-\gamma)\Gamma(3-\gamma)}(\eta^{2-\gamma} - \eta^{1-\gamma}) > 0$. Then the problem*

$$cD^{\alpha}u(t) = y(t)$$

with boundary values

$$\begin{cases} u(0) + cD^{\gamma}u(0) = \int_0^1 g_0(s,u(s))ds, \\ cD^{\gamma}u(1) = \int_0^1 g_1(s,u(s))ds \\ cD^{\gamma}u(\eta) = \int_0^1 g_2(s,u(s))ds \end{cases}$$

has the unique solution

$$\begin{aligned} u(t) = \Theta(t) \quad & + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ & + \tfrac{\omega_1(t)}{\Gamma(\alpha-\gamma)}\int_0^{\eta}(\eta-s)^{\alpha-\gamma-1}y(s)ds \\ & + \tfrac{\omega_2(t)}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}y(s)ds \end{aligned}$$

where

$$\Theta(t) = \int_0^1 g_0(s,x(s))ds - \omega_2(t)\int_0^1 g_1(s,x(s))ds - \omega_1(t)\int_0^1 g_2(s,x(s))ds,$$

$$\omega_1(t) = \frac{t^2}{R(\eta)\Gamma(3-\gamma)} - \frac{t}{R(\eta)\Gamma(2-\gamma)}$$

and

$$\omega_2(t) = \frac{\eta^{2-\gamma}t}{R(\eta)\Gamma(3-\gamma)} - \frac{\eta^{1-\gamma}t^2}{R(\eta)\Gamma(2-\gamma)}.$$

Similarly, we have the following Remark,

Remark 3.4. *Let $u \in C(J,\mathbb{R})$ and $R(\eta) = \eta^2 - \eta > 0$. Then the problem*

$$cD^{\alpha}u(t) = y(t)$$

with boundary values

$$\begin{cases} u(0) = \int_0^1 g_0(s,u(s))ds, \\ u(1) = \int_0^1 g_1(s,u(s))ds \\ u(\eta) = \int_0^1 g_2(s,u(s))ds \end{cases}$$

has the unique solution

$$\begin{aligned} u(t) \quad & = \Theta(t) + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ & + \omega_1(t)\big[\tfrac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}y(s)ds\big] \\ & + \omega_2(t)\big[\tfrac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds\big] \end{aligned}$$

where

$$\Theta(t) = \omega_0(t)\int_0^1 g_0(s,x(s))ds - \omega_2(t)\int_0^1 g_1(s,x(s))ds - \omega_1(t)\int_0^1 g_2(s,x(s))ds,$$

$\omega_0(t) = 1 + \frac{t}{R(\eta)}\big(\eta^2 - 1\big) + \frac{t^2}{R(\eta)}(1-\eta)$, $\omega_1(t) = \frac{t^2}{R(\eta)} - \frac{t}{R(\eta)}$ *and* $\omega_2(t) = \frac{\eta^2 t}{R(\eta)} - \frac{\eta t^2}{R(\eta)}$.

Assume that $C(I)$ be the space of all continuous real-valued functions on $I = [0, 1]$. Consider the Banach space

$$X = \{u|\ u \in C(I) \text{ and } cD^{\beta_i}u \in C(I),\ 0 < \beta_i < 1,\ i = 1, 2, ...n\}$$

endowed with the norm

$$\|u\| = \sup_{t\in I} |u| + \Sigma_{i=1}^{n} \sup_{t\in I} |cD^{\beta_i}u(t)|.$$

Define the mapping $F : X \to X$ by

$$\begin{aligned}(Fu)(t) &= \Theta(t) + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F^{\beta}(s,u)ds \\ &+\omega_1(t)\big[\tfrac{b_1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-s)^{\alpha-1} F^{\beta}(s,u)ds + \tfrac{b_2}{\Gamma(\alpha-\gamma)} \int_0^{\eta} (\eta-s)^{\alpha-\gamma-1} F^{\beta}(s,u)ds\big] \\ &+\omega_2(t)\big[\tfrac{b_1}{\Gamma(\alpha)} \int_0^{1} (1-s)^{\alpha-1} F^{\beta}(s,u)ds + \tfrac{b_2}{\Gamma(\alpha-\gamma)} \int_0^{1} (1-s)^{\alpha-\gamma-1} F^{\beta}(s,u)ds\big]\end{aligned}$$

where

$$F^{\beta}(s,u) = f\big(s, u(s), cD^{\beta_1}u(s), cD^{\beta_2}u(s), ..., cD^{\beta_n}u(s)\big).$$

Put

$$\begin{aligned}\Omega_1 = \tfrac{1}{\Gamma(\alpha+1)} &+\tfrac{1}{R(\eta)}(b_1 + \tfrac{b_2}{\Gamma(3-\gamma)})\big(\tfrac{b_1\eta^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{b_2\eta^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\big) \\ &+\tfrac{1}{R(\eta)}(b_1\eta^2 + \tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})(\tfrac{b_1}{\Gamma(\alpha+1)} + \tfrac{b_2}{\Gamma(\alpha-\gamma+1)})\end{aligned}$$

and

$$\begin{aligned}\Omega_2 = &\tfrac{\Gamma(\alpha+1)}{\alpha\Gamma(\alpha-1)} \textstyle\sum_{i=1}^{n} \tfrac{1}{\Gamma(\alpha-\beta_i+2)} \\ &+\tfrac{2}{R(\eta)}\big[(b_1 + \tfrac{b_2}{\Gamma(3-\gamma)})\big(\tfrac{b_1\eta^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{b_2\eta^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\big) \\ &-(b_1\eta + \tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)} + \tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big)\big] \textstyle\sum_{i=1}^{n} \tfrac{1}{\Gamma(3-\beta_i)} \\ &+\tfrac{1}{R(\eta)}\big[\big(b_1\eta^2 + \tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)}\big)\big(\tfrac{b_1}{\Gamma(\alpha+1)} + \tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big) \\ &-(b_1 + \tfrac{b_2}{\Gamma(2-\gamma)})\big(\tfrac{b_1\eta^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{b_2\eta^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\big)\big] \textstyle\sum_{i=1}^{n} \tfrac{1}{\Gamma(2-\beta_i)}.\end{aligned}$$

We denote by Δ_ν the set of all functions $\nu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

(ν1) $\nu(0, x) \leq 0$ for all $x \in \mathbb{R}$.

Now, we are ready to state and prove main result of this section.

Theorem 3.5. *Assume that there exist $\delta \in \mathfrak{S}$ and $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0, 1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$\begin{aligned}|F^{\beta}(t,u) - F^{\beta}(t,v)| \leq \frac{1}{\Omega_1 + \Omega_2}\delta(\|u-v\|)\Big(&|u(t) - v(t)| \\ +\ \ |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| &+ |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)|\Big).\end{aligned}$$

Also the following conditions hold:

(i) $\nu((Fx)(t), (Fy)(t)) \leq \nu((x)(t), (y)(t))$ for all $x, y \in X$ and $t \in [0, 1]$,

(ii) Let $\{x_n\}$ be an sequence in X such that $\nu(x_n(t), x_{n+1}(t)) \leq 0$ for all $n \in \mathbb{N}$ and $t \in [0, 1]$ with $x_n(t) \to x(t)$ as $n \to \infty$, then $\nu(x_n(t), x(t)) \leq 0$ hold for all $n \in \mathbb{N}$ and $t \in [0, 1]$.

Then the problem (3.1) with boundary values (3.2) has a solution.

Proof . We define a binary relation $\perp$ (in X) by $x(t)\perp y(t) \Leftrightarrow \nu(x(t),y(t)) \leq 0$ for all $t \in [0,1]$. Clearly, by putting $x_0 = 0$, $(X,\perp)$ is an O-set. Let $x\perp y$. So $\nu(x,y) \leq 0$. Hence, from (i) we have, $\nu((Fx)(t),(Fy)(t)) \leq \nu((x)(t),(y)(t)) \leq 0$. i.e., $(Fx)(t)\perp(Fy)(t)$. That is F is an $\perp-$preserving self-mapping. Let $\{x_n\}$ be an $O-$sequence in X with $x_n(t) \to x(t)$ as $n \to \infty$ for all $t \in [0,1]$. Then $\nu(x_n(t),x_{n+1}(t)) \leq 0$ for all $n \in \mathbb{N}$ and $t \in [0,1]$. Therefore, from (ii), we have $\nu(x_n(t),x(t)) \leq 0$ for all $n \in \mathbb{N}$ and $t \in [0,1]$. i.e., $x_n(t)\perp x(t)$ for all $n \in \mathbb{N}$ and $t \in [0,1]$. Let $\nu(u(t),v(t)) \leq 0$ and $u \neq v$. Then we have,

$$
\begin{aligned}
|(Fu)(t)-(Fv)(t)| &= \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(F^\beta(s,u)-F^\beta(s,v)\big)ds \\
&+ \omega_1(t)\Bigg[\frac{b_1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\big(F^\beta(s,u)-F^\beta(s,v)\big)ds \\
&+ \frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^\eta (\eta-s)^{\alpha-\gamma-1}\big(F^\beta(s,u)-F^\beta(s,v)ds\Bigg] \\
&+ \omega_2(t)\Bigg[\frac{b_1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\big(F^\beta(s,u)-F^\beta(s,v)ds \\
&+ \frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\big(F^\beta(s,u)-F^\beta(s,v)\big)ds\Bigg]\Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|F^\beta(s,u)-F^\beta(s,v)\big|ds \\
&+ \omega_1(t)\Bigg[\frac{b_1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\big|F^\beta(s,u)-F^\beta(s,v)\big|ds \\
&+ \frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^\eta (\eta-s)^{\alpha-\gamma-1}\big|F^\beta(s,u)-F^\beta(s,v)\big|ds\Bigg] \\
&+ \omega_2(t)\Bigg[\frac{b_1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\big|F^\beta(s,u)-F^\beta(s,v)\big|ds \\
&+ \frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}\big|F^\beta(s,u)-F^\beta(s,v)\big|ds\Bigg] \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{1}{\Omega_1+\Omega_2}\delta(\|u-v\|)\int_0^t (t-s)^{\alpha-1} \\
&\quad \Big(|u(s)-v(s)|+|cD^{\beta_1}u(s)-cD^{\beta_1}v(s)|+|cD^{\beta_2}u(s)-cD^{\beta_2}v(s)|+... \\
&+ |cD^{\beta_n}u(s)-cD^{\beta_n}v(s)|\Big)ds \\
&+ \omega_1(t)\Bigg[\frac{b_1}{\Gamma(\alpha)}\frac{1}{\Omega_1+\Omega_2}\delta(\|u-v\|)\int_0^\eta (\eta-s)^{\alpha-1} \\
&\quad \Big(|u(s)-v(s)|+|cD^{\beta_1}u(s)-cD^{\beta_1}v(s)|+|cD^{\beta_2}u(s)-cD^{\beta_2}v(s)|+... \\
&+ |cD^{\beta_n}u(s)-cD^{\beta_n}v(s)|\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{b_2}{\Gamma(\alpha-\gamma)}\frac{1}{\Omega_1+\Omega_2}\delta(\|u-v\|)\int_0^{\eta}(\eta-s)^{\alpha-\gamma-1}\\
&\Big(|u(s)-v(s)|+|cD^{\beta_1}u(s)-cD^{\beta_1}v(s)|+|cD^{\beta_2}u(s)-cD^{\beta_2}v(s)|+...\\
&+ |cD^{\beta_n}u(s)-cD^{\beta_n}v(s)|\Big)ds\Big]\\
&+ \omega_2(t)\Big[\frac{b_1}{\Gamma(\alpha)}\frac{1}{\Omega_1+\Omega_2}\delta(\|u-v\|)\int_0^{1}(1-s)^{\alpha-1}\\
&\Big(|u(s)-v(s)|+|cD^{\beta_1}u(s)-cD^{\beta_1}v(s)|+|cD^{\beta_2}u(s)-cD^{\beta_2}v(s)|+...\\
&+ |cD^{\beta_n}u(s)-cD^{\beta_n}v(s)|\Big)ds\\
&+ \frac{b_2}{\Gamma(\alpha-\gamma)}\frac{1}{\Omega_1+\Omega_2}\delta(\|u-v\|)\int_0^{1}(1-s)^{\alpha-\gamma-1}\\
&\Big(|u(s)-v(s)|+|cD^{\beta_1}u(s)-cD^{\beta_1}v(s)|+|cD^{\beta_2}u(s)-cD^{\beta_2}v(s)|+...\\
&+ |cD^{\beta_n}u(s)-cD^{\beta_n}v(s)|\Big)ds\Big]\\
&\leq \frac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Big[\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}ds\\
&+ \omega_1(t)\big[\frac{b_1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}ds+\frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^{\eta}(\eta-s)^{\alpha-\gamma-1}ds\big]\\
&+ \omega_2(t)\big[\frac{b_1}{\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}ds+\frac{b_2}{\Gamma(\alpha-\gamma)}\int_0^{1}(1-s)^{\alpha-\gamma-1}ds\big]\Big]\\
&\leq \frac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Big[\frac{1}{\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}+\omega_1(t)\big[\frac{b_1}{\Gamma(\alpha)}\frac{\eta^{\alpha}}{\alpha}+\frac{b_2}{\Gamma(\alpha-\gamma)}\frac{\eta^{\alpha-\gamma}}{\alpha-\gamma}\big]\\
&+ \omega_2(t)\big[\frac{b_1}{\Gamma(\alpha)}\frac{1}{\alpha}+\frac{b_2}{\Gamma(\alpha-\gamma)}\frac{1}{\alpha-\gamma}\big]\Big]\\
&= \frac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\big[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\omega_1(t)\big[\frac{b_1\eta^{\alpha}}{\Gamma(\alpha+1)}+\frac{b_2\eta^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\big]\\
&+ \omega_2(t)\big[\frac{b_1}{\Gamma(\alpha+1)}+\frac{b_2}{\Gamma(\alpha-\gamma+1)}\big]\big].
\end{aligned}
$$

Since, $0\leq t\leq 1$, then $\omega_1(t)\leq\frac{1}{R(\eta)}(b_1+\frac{b_2}{\Gamma(3-\gamma)})$ and $\omega_2(t)\leq\frac{1}{R(\eta)}(b_1\eta^2+\frac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})$. So we get,

$$
\begin{aligned}
|(Fu)(t)-(Fv)(t)| &\leq \frac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\big[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{R(\eta)}(b_1+\frac{b_2}{\Gamma(3-\gamma)})\big(\frac{b_1\eta^{\alpha}}{\Gamma(\alpha+1)}\\
&+ \frac{b_2\eta^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\big)+\frac{1}{R(\eta)}(b_1\eta^2+\frac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\big(\frac{b_1}{\Gamma(\alpha+1)}+\frac{b_2}{\Gamma(\alpha-\gamma+1)}\big)\big]\\
&= \frac{\Omega_1\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}.
\end{aligned}
$$

Also we have,

$$
\begin{aligned}
& \Big| cD^{\beta_i}(Fu)(t) - cD^{\beta_i}(Fv)(t) \Big| \\
= & \Big| \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} (Fu)'(s) ds - \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} (Fv)'(s) ds \Big| \\
= & \Big| \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \Big(\Theta'(s) + \frac{1}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} F^\beta(\tau,u) d\tau \\
+ & \omega_1'(s) \big[\frac{b_1}{\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{\alpha-1} F^\beta(\tau,u) d\tau + \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^\eta (\eta-\tau)^{\alpha-\gamma-1} F^\beta(\tau,u) d\tau \big] \\
+ & \omega_2'(s) \big[\frac{b_1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} F^\beta(\tau,u) d\tau + \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^1 (1-\tau)^{\alpha-\gamma-1} F^\beta(\tau,u) d\tau \big] \Big) ds \\
- & \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \Big(\Theta'(s) + \frac{1}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} F^\beta(\tau,v) d\tau \\
+ & \omega_1'(s) \Big[\frac{b_1}{\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{\alpha-1} F^\beta(\tau,v) d\tau \\
+ & \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^\eta (\eta-\tau)^{\alpha-\gamma-1} F^\beta(\tau,v) d\tau \Big] \\
+ & \omega_2'(s) \Big[\frac{b_1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} F^\beta(\tau,v) d\tau + \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^1 (1-\tau)^{\alpha-\gamma-1} F^\beta(\tau,v) d\tau \Big] \Big) ds \Big| \\
\leq & \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \Big(\frac{1}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} |F^\beta(\tau,u) - F^\beta(\tau,v)| d\tau \\
+ & \omega_1'(s) \Big[\frac{b_1}{\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{\alpha-1} |F^\beta(\tau,u) - F^\beta(\tau,v)| d\tau \\
+ & \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^\eta (\eta-\tau)^{\alpha-\gamma-1} |F^\beta(\tau,u) - F^\beta(\tau,v)| d\tau \Big] \\
+ & \omega_2'(s) \Big[\frac{b_1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} |F^\beta(\tau,u) - F^\beta(\tau,v)| d\tau \\
+ & \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^1 (1-\tau)^{\alpha-\gamma-1} |F^\beta(\tau,u)\big) - F^\beta(\tau,v)| d\tau \Big] \Big) ds \\
\\
\leq & \frac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2} \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \Big(\frac{1}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} d\tau \\
+ & \omega_1'(s) \big[\frac{b_1}{\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{\alpha-1} d\tau + \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^\eta (\eta-\tau)^{\alpha-\gamma-1} d\tau \big] \\
+ & \omega_2'(s) \big[\frac{b_1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} d\tau + \frac{b_2}{\Gamma(\alpha-\gamma)} \int_0^1 (1-\tau)^{\alpha-\gamma-1} d\tau \big] \Big) ds \\
= & \frac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2} \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} \Big(\frac{1}{\alpha\Gamma(\alpha-1)} s^\alpha + \omega_1'(s) \big[\frac{b_1}{\Gamma(\alpha+1)} \eta^\alpha \\
+ & \frac{b_2}{\Gamma(\alpha-\gamma+1)} \eta^{\alpha-\gamma} \big] + \omega_2'(s) \big[\frac{b_1}{\Gamma(\alpha+1)} + \frac{b_2}{\Gamma(\alpha-\gamma+1)} \big] \Big) ds
\end{aligned}
$$

and so

$$
\begin{aligned}
&\Big|cD^{\beta_i}(Fu)(t)-cD^{\beta_i}(Fv)(t)\Big|\\
&\leq \tfrac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Bigg(\tfrac{1}{\alpha\Gamma(1-\beta_i)\Gamma(\alpha-1)}\int_0^t(t-s)^{-\beta_i}s^{\alpha}ds+\tfrac{1}{\Gamma(1-\beta_i)}\big[\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}\\
&+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big]\int_0^t(t-s)^{-\beta_i}\omega_1'(s)ds\\
&+\tfrac{1}{\Gamma(1-\beta_i)}\big[\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big]\int_0^t(t-s)^{-\beta_i}\omega_2'(s)ds\Bigg)\\
&=\tfrac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Bigg(\tfrac{1}{\alpha\Gamma(1-\beta_i)\Gamma(\alpha-1)}\int_0^t(t-s)^{-\beta_i}s^{\alpha}ds+\tfrac{1}{\Gamma(1-\beta_i)}\big[\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}\\
&+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big]\int_0^t(t-s)^{-\beta_i}\big[\tfrac{2}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})s-\tfrac{1}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big]ds\\
&+\tfrac{1}{\Gamma(1-\beta_i)}\big[\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big]\int_0^t(t-s)^{-\beta_i}\big[\tfrac{1}{R(\eta)}(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})-\tfrac{2}{R(\eta)}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})s\big]ds\Bigg)\\
&\leq \tfrac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Bigg(\tfrac{1}{\alpha\Gamma(1-\beta_i)\Gamma(\alpha-1)}\int_0^1(1-s)^{-\beta_i}s^{\alpha}ds+\tfrac{1}{\Gamma(1-\beta_i)}\big[\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}\\
&+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big]\big[\tfrac{2}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})\int_0^1(1-s)^{-\beta_i}sds-\tfrac{1}{R(\eta)}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\int_0^1(1-s)^{-\beta_i}ds\big]\\
&+\tfrac{1}{\Gamma(1-\beta_i)}\big[\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big]\big[\tfrac{1}{R(\eta)}(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\int_0^1(1-s)^{-\beta_i}ds\\
&-\tfrac{2}{R(\eta)}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\int_0^1(1-s)^{-\beta_i}sds\big]\Bigg).
\end{aligned}
$$

Now since, $B(a,b)=\int_0^1\xi^{a-1}(1-\xi)^{b-1}d\xi=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, we get,

$$
\begin{aligned}
&|cD^{\beta_i}(Fu)(t)-cD^{\beta_i}(Fv)(t)|\\
&\leq \tfrac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Bigg(\tfrac{\Gamma(\alpha+1)}{\alpha\Gamma(\alpha-1)\Gamma(\alpha-\beta_i+2)}+\big[\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}\\
&+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big]\big[\tfrac{2}{R(\eta)\Gamma(3-\beta_i)}(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})-\tfrac{1}{R(\eta)\Gamma(2-\beta_i)}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big]\\
&+\big[\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big]\big[\tfrac{1}{R(\eta)\Gamma(2-\beta_i)}(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\\
&-\tfrac{2}{R(\eta)\Gamma(3-\beta_i)}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big]\Bigg)\\
&=\tfrac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Bigg(\tfrac{\Gamma(\alpha+1)}{\alpha\Gamma(\alpha-1)\Gamma(\alpha-\beta_i+2)}+\tfrac{2}{R(\eta)\Gamma(3-\beta_i)}(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big)\\
&-\tfrac{1}{R(\eta)\Gamma(2-\beta_i)}(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big)\\
&+\tfrac{1}{R(\eta)\Gamma(2-\beta_i)}(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big)\\
&-\tfrac{2}{R(\eta)\Gamma(3-\beta_i)}(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big)\Bigg)\\
&=\tfrac{\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}\Bigg(\tfrac{\Gamma(\alpha+1)}{\alpha\Gamma(\alpha-1)\Gamma(\alpha-\beta_i+2)}+\tfrac{2}{R(\eta)}\Bigg[(b_1+\tfrac{b_2}{\Gamma(3-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big)\\
&-(b_1\eta+\tfrac{b_2\eta^{1-\gamma}}{\Gamma(2-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big)\Bigg]\tfrac{1}{\Gamma(3-\beta_i)}\\
&+\tfrac{1}{R(\eta)}\Bigg[(b_1\eta^2+\tfrac{b_2\eta^{2-\gamma}}{\Gamma(3-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\big)\\
&-(b_1+\tfrac{b_2}{\Gamma(2-\gamma)})\big(\tfrac{b_1}{\Gamma(\alpha+1)}\eta^{\alpha}+\tfrac{b_2}{\Gamma(\alpha-\gamma+1)}\eta^{\alpha-\gamma}\big)\Bigg]\tfrac{1}{\Gamma(2-\beta_i)}\Bigg)
\end{aligned}
$$

which implies,

$$\sum_{i=1}^{n} |cD^{\beta_i}(Fu)(t) - cD^{\beta_i}(Fv)(t)| \leq \frac{\Omega_2 \delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2}.$$

Therefore,

$$\begin{aligned} \|Fu - Fv\| &\leq \tfrac{\Omega_1 \delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2} + \tfrac{\Omega_2 \delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2} \\ &= \tfrac{(\Omega_1+\Omega_2)\delta(\|u-v\|)\|u-v\|}{\Omega_1+\Omega_2} \\ &= \delta(\|u-v\|)\|u-v\|. \end{aligned}$$

Hence all conditions of Corollary 2.9 hold and F has a fixed point, which is the solution of problem (3.1). □

If in Theorem 3.5 we take $b_1 = 0$ and $a_1 = a_2 = b_2 = 1$ then we deduce the following Corollary.

Corollary 3.6. *Assume that there exist $\delta \in \mathfrak{S}$ and $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0,1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$\begin{aligned} |F^\beta(t,u) - F^\beta(t,v)| &\leq \frac{1}{\Omega_1+\Omega_2}\delta(\|u-v\|)\Big(|u(t)-v(t)| \\ + \ |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| &+ |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)|\Big). \end{aligned}$$

Also the following condition hold:

(i) $\nu((Fx)(t),(Fy)(t)) \leq \nu((x)(t),(y)(t))$ *for all* $x, y \in X$ *and* $t \in [0,1]$,

(ii) Let $\{x_n\}$ *be an sequence in* X *such that* $\nu(x_n(t), x_{n+1}(t)) \leq 0$ *for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$ *with* $x_n(t) \to x(t)$ *as* $n \to \infty$, *then* $\nu(x_n(t), x(t)) \leq 0$ *hold for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$.

Then the problem (3.1) with boundary values

$$\begin{cases} u(0) + cD^\gamma u(0) = \int_0^1 g_0(s, u(s))ds, \\ cD^\gamma u(1) = \int_0^1 g_1(s, u(s))ds \\ cD^\gamma u(\eta) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

has a solution.

If in Corollary 3.6 we take $\delta(t) = \frac{1}{1+t}(t > 0)$ then we deduce the following Corollary.

Corollary 3.7. *Assume that there exists $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0,1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$\begin{aligned} |F^\beta(t,u) - F^\beta(t,v)| &\leq \frac{1}{(\Omega_1+\Omega_2)(1+\|u-v\|)}\Big(|u(t)-v(t)| \\ + \ |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| &+ |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)|\Big). \end{aligned}$$

Also the following condition hold:

(i) $\nu((Fx)(t),(Fy)(t)) \leq \nu((x)(t),(y)(t))$ *for all* $x, y \in X$ *and* $t \in [0,1]$,

(ii) Let $\{x_n\}$ be an sequence in X such that $\nu(x_n(t), x_{n+1}(t)) \leq 0$ for all $n \in \mathbb{N}$ and $t \in [0,1]$ with $x_n(t) \to x(t)$ as $n \to \infty$, then $\nu(x_n(t), x(t)) \leq 0$ hold for all $n \in \mathbb{N}$ and $t \in [0,1]$.

Then the problem (3.1) with boundary values

$$\begin{cases} u(0) + cD^{\gamma}u(0) = \int_0^1 g_0(s, u(s))ds, \\ cD^{\gamma}u(1) = \int_0^1 g_1(s, u(s))ds \\ cD^{\gamma}u(\eta) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

has a solution.

If in Corollary 3.6 we take $\delta(t) = k$ where $0 < k < 1$ then we deduce the following Corollary.

Corollary 3.8. *Assume that there exist $0 < k < 1$ and $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0,1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$|F^{\beta}(t,u) - F^{\beta}(t,v)| \leq \frac{k}{\Omega_1 + \Omega_2}\Big(|u(t) - v(t)| + |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| + |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)|\Big).$$

Also the following condition hold:

(i) $\nu((Fx)(t), (Fy)(t)) \leq \nu((x)(t), (y)(t))$ for all $x, y \in X$ and $t \in [0,1]$,

(ii) Let $\{x_n\}$ be an sequence in X such that $\nu(x_n(t), x_{n+1}(t)) \leq 0$ for all $n \in \mathbb{N}$ and $t \in [0,1]$ with $x_n(t) \to x(t)$ as $n \to \infty$, then $\nu(x_n(t), x(t)) \leq 0$ hold for all $n \in \mathbb{N}$ and $t \in [0,1]$.

Then the problem (3.1) with boundary values

$$\begin{cases} u(0) + cD^{\gamma}u(0) = \int_0^1 g_0(s, u(s))ds, \\ cD^{\gamma}u(1) = \int_0^1 g_1(s, u(s))ds \\ cD^{\gamma}u(\eta) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

has a solution.

Similarly we can deduce the following Corollaries.

Corollary 3.9. *Assume that there exist $\delta \in \mathfrak{S}$ and $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0,1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$|F^{\beta}(t,u) - F^{\beta}(t,v)| \leq \frac{1}{\Omega_1 + \Omega_2}\delta(\|u - v\|)\Big(|u(t) - v(t)| + |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| + |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)|\Big).$$

Also the following condition hold:

(i) $\nu((Fx)(t), (Fy)(t)) \leq \nu((x)(t), (y)(t))$ for all $x, y \in X$ and $t \in [0,1]$,

(ii) Let $\{x_n\}$ be an sequence in X such that $\nu(x_n(t), x_{n+1}(t)) \leq 0$ for all $n \in \mathbb{N}$ and $t \in [0,1]$ with $x_n(t) \to x(t)$ as $n \to \infty$, then $\nu(x_n(t), x(t)) \leq 0$ hold for all $n \in \mathbb{N}$ and $t \in [0,1]$.

Then the problem (3.1) with boundary values

$$\begin{cases} u(0) = \int_0^1 g_0(s, u(s))ds, \\ u(1) = \int_0^1 g_1(s, u(s))ds \\ u(\eta) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

has a solution.

Corollary 3.10. *Assume that there exists $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0,1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$|F^\beta(t,u) - F^\beta(t,v)| \leq \frac{1}{(\Omega_1 + \Omega_2)(1 + \|u - v\|)} \Big(|u(t) - v(t)| + |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| + |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)| \Big).$$

Also the following condition hold:

(i) $\nu((Fx)(t), (Fy)(t)) \leq \nu((x)(t), (y)(t))$ *for all* $x, y \in X$ *and* $t \in [0,1]$,

(ii) Let $\{x_n\}$ *be an sequence in* X *such that* $\nu(x_n(t), x_{n+1}(t)) \leq 0$ *for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$ *with* $x_n(t) \to x(t)$ *as* $n \to \infty$, *then* $\nu(x_n(t), x(t)) \leq 0$ *hold for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$.

Then the problem (3.1) with boundary values

$$\begin{cases} u(0) = \int_0^1 g_0(s, u(s))ds, \\ u(1) = \int_0^1 g_1(s, u(s))ds \\ u(\eta) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

has a solution.

Corollary 3.11. *Assume that there exist $0 < k < 1$ and $\nu \in \Delta_\nu$ for all $u, v \in X$ and $t \in [0,1]$ with $\nu(u(t), v(t)) \leq 0$ and $u \neq v$,*

$$|F^\beta(t,u) - F^\beta(t,v)| \leq \frac{k}{\Omega_1 + \Omega_2} \Big(|u(t) - v(t)| + |cD^{\beta_1}u(t) - cD^{\beta_1}v(t)| + |cD^{\beta_2}u(t) - cD^{\beta_2}v(t)| + ... + |cD^{\beta_n}u(t) - cD^{\beta_n}v(t)| \Big).$$

Also the following condition hold:

(i) $\nu((Fx)(t), (Fy)(t)) \leq \nu((x)(t), (y)(t))$ *for all* $x, y \in X$ *and* $t \in [0,1]$,

(ii) Let $\{x_n\}$ *be an sequence in* X *such that* $\nu(x_n(t), x_{n+1}(t)) \leq 0$ *for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$ *with* $x_n(t) \to x(t)$ *as* $n \to \infty$, *then* $\nu(x_n(t), x(t)) \leq 0$ *hold for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$.

Then the problem (3.1) with boundary values

$$\begin{cases} u(0) = \int_0^1 g_0(s, u(s))ds, \\ u(1) = \int_0^1 g_1(s, u(s))ds \\ u(\eta) = \int_0^1 g_2(s, u(s))ds \end{cases}$$

has a solution.

References

[1] B. Ahmad and S. Sivasundaram, *On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order*, Appl. Math. Comput. 217 (2010) 480–487.

[2] H. Baghani, M. Eshaghi Gordji and M. Ramezani, *Orthogonal sets: The axiom of choice and proof of a fixed point theorem*, J. Fixed Point Theory Appl. 18 (2016) 465–477.

[3] D. Baleanu, S. Z. Nazemi and Sh. Rezapour, *Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations*, Adv. Differ. Equ. 368 (2013).

[4] M. Eshaghi, M. Ramezani, M. D. L. Sen and Y. J. Cho, *On orthogonal sets and Banach's fxed point theorem*, Fixed Point Theory, 18 (2017) 569–578.

[5] M. A. Khamsi, W. M. Kozlowski and S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal. 14 (1990) 935–953.

[6] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[7] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. 141 (1989) 177-188.

[8] S. B. Nadler, *Multivalued contraction mappings*, Pac. J. Math. 30 (1969) 475-488.

[9] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[10] S. Reich, *Fixed point of contractive functions*, Boll Un Mat Ital. 4 (1972) 26-42.

[11] S. Reich, *Some fixed point problems*, Rend Lincei-Mat Appl. 57 (1974) 194–198.

[12] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science, Switzerland, 1993.

[13] X. Su and L. Liu, *Existence of solution for boundary value problem of nonlinear fractional differential equation*. Appl. Math. Chin. Univ. 22 (2007) 291–298.

[14] X. Su and S. Zhang, *Solutions to boundary value problems for nonlinear differential equations of fractional order*. Electron. J. Differ. Equ. 26 (2009) 1–15.

[15] Ch. Zhai and J. Ren, *The unique solution for a fractional q-difference equation with three-point boundary conditions*, Indag. Math. 29 (2018) 948–961.