Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 699-715
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5122

Mixed Topp-Leone-Kumaraswamy distribution

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this article, a new generalization of the Topp-Leone distribution with a unit interval, namely Mixed Topp-Leone-Kumaraswamy distribution is defined and studied. The mathematical properties of this mixing distribution are described. Moments, quantile function, R?nyi entropy, incomplete moments and moments of residual are obtained for the new Mixed Topp-Leone - Kumaraswamy distribution. The maximum likelihood (MLE), Crans (CM) , Percentile (PM) and Particle Swarm Optimization(PSO) estimators of the parameters are derived. The percentile Method is more efficient method as compred to the others. Two real data sets are used to illustrate an application and superiority of the proposed distribution.

Keywords: Topp-Leone distribution, Kumaraswamy distribution, mixing Transformation, Moments, Renyi's entropy ,maximum likelihood method,Cran method, Percentile Method, Particle Swarm method.
2010 MSC: Primary 62E10; Secondary 60E05.

1. Introduction

The Topp-Leone (T-L) distribution is introduced by Topp and Leone,[25] in (1955), with one parameter the T-L distribution has J–shaped density and U-shaped or bathtub failure function which is capable to model high, constant and again high patterns failure times. The distribution has simple closed form of cumulative distribution function and bounded support of $(0, 1)$, so it is more attractive as compared with beta distribution and useful in reliability and survival analysis. On the other hand, the Kumaraswamy distribution is introduced by Kumaraswamy,[13] in (1980) as a better alternative to the beta distribution, because the last does not accurately fit hydro-logical data such as daily

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Received: March 2021 *Accepted:* April 2021

rainfall, daily stream flow, etc. and its distribution function is an incomplete beta function ratio and its quantile function the inverse thereof,[15].

Over the last few years, many generalized distributions have been studied inspired by the increasing demand of probability distributions in many applications, one of them are proposed by mixing of two or more distributions in a mathematical based way to model a wide variety of data. In many situations, observed data may be assumed to have come from a mixture population of two or more distributions, so these generalized distributions are effective and flexible models to analyze and interpret data that come from a possibly heterogeneous population. So many research works have been done such as Alzaatreh et al.,[1], Haq et al.,[11], Hashmi et al.,[10], Elgarhy et al.,[8] and ZeinEldin et al.,[26] for more see Cordeiroa and Castro (2010)[4], Aryal and Tsokos (2011)[2], Silva et al.,[22] and Mohammed and Mohammed,[19].

James,[12] (1978) considered the problem of estimating the mixing proportion in a mixture of two normal distributions, the parameters of which were assumed known, he noticed that very large samples were needed if reasonably precise estimates were to be obtained. Mohammed et al,[18] (2014) derived a parametric mixture model of three different distributions, consisting of Exponential, Gamma and Weibull distributions to model heterogeneous survival time data. Yilmaz and Buyum,[17] (2015) derived different methods to estimate parameters for two component mixed exponential distributions. Zhang,[28] (2015) applied expectation conditional maximization (ECM) algorithm to estimate three exponential distributions.

Zhai et al,[27] (2018) used the Weibull-normal mixture distribution.They employed lowest Akaike information criterion (AIC) value to determine the components number of the mixture where the parameter estimation method based on maximizing the log likelihood function using an intelligent optimization algorithm and genetic algorithm. Szulczewski1 and Jakubowski1,[24] (2018) applied a mixture of a three-parameter generalized extreme values distribution and a two-parameter gamma distribution in hydro-logical field.Hasan et al,[9] (2020) Employed The Log-normal, Gamma and Weibull distributions, as well as their mixed to fitting non-zero six-minutes rainfall data.

The main aim of this paper is to mix the one parameter Topp-Leone distribution with one parameter Kumaraswamy distribution called mixed Topp-Leone- Kumaraswamy distribution (MTLK) and describe the properties and estimation methods of its parameters.

The probability density function of Kumaraswamy distribution $k(\alpha, \beta)$ is:

$$f_1 = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}; 0 < x < 1 \quad , \quad \alpha, \beta > 0 \tag{1.1}$$

where α, β are the shape parameters.[15] The distribution function is:

$$F_1 = 1 - (1-x^{\alpha})^{\beta} \tag{1.2}$$

The probability density function of Topp-Leone distribution TL(θ) is:

$$f_2 = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}; 0 < x < 1 \quad , \quad \theta > 0 \tag{1.3}$$

where θ are the shape parameters.[25] The distribution function is:

$$F_2 = x^{\theta}(2-x)^{\theta} \tag{1.4}$$

By mixing $k(1, \beta)$ with TL(θ), we get Mixed Topp-Leone-Kumaraswamy Distribution $MTLK(\beta, \theta)$ as follow:

$$f(x) = wf_1(x) + (1-w) f_2(x)$$

$$f\left(x\right)=\frac{\left(1-x\right)}{\left(\beta+\theta\right)}\left[\beta^{2}{\left(1-x\right)}^{\beta-2}+2\theta^{2}x^{\theta-1}{\left(2-x\right)}^{\theta-1}\right];0<x<1\ ,\quad \beta,\theta>0 \tag{1.5}$$

Where: $w=\frac{\beta}{\beta+\theta}$, β,θ are shape parameters.
The equation (1.5) is probability density function, where:

$$\int_{0}^{1}f\left(x\right)dx=\int_{0}^{1}\frac{\left(1-x\right)}{\left(\beta+\theta\right)}\left[\beta^{2}{\left(1-x\right)}^{\beta-2}+2\theta^{2}x^{\theta-1}{\left(2-x\right)}^{\theta-1}\right]dx$$

$$=\int_{0}^{1}\frac{\beta^{2}}{\beta+\theta}{\left(1-x\right)}^{\beta-1}dx+\int_{0}^{1}\frac{2\theta^{2}}{\beta+\theta}x^{\theta-1}\left(1-x\right)\left(2-x\right)^{\theta-1}dx$$

$$=\frac{\beta}{\beta+\theta}+\frac{\theta^{2}}{\beta+\theta}\int_{0}^{1}\left(2-2x\right)\left(2x-x^{2}\right)^{\theta-1}dx$$

$$=\frac{\beta}{\beta+\theta}+\frac{\theta}{\beta+\theta}=1$$

We observe from pdf curves for different values of parameters (see Fig.1) that as θ values are small the MTKL distribution curve's are concave and like bathtub shape as β values are small and decreasing curves as θ values are large. As θ values are large the MTKL distribution curve's are decreasing curves as β values are small and convex in shape as θ values are large.

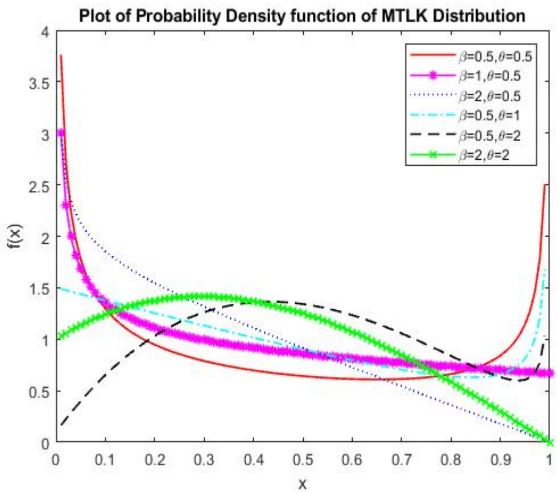

Figure 1: Probability Density Function Plot of MTKL Distribution

The cumulative distribution function will be:

$$F\left(\mathrm{x}\right)=pr\left(X\leq x\right)=\int_{0}^{x}f\left(u\right)du$$

$$=\frac{1}{\beta+\theta}\left[\beta-\beta{\left(1-x\right)}^{\beta}+\theta x^{\theta}{\left(2-x\right)}^{\theta}\right] \tag{1.6}$$

We observe from curves of cdf for different values of parameters (see Fig.2) that curves are near main diagonal as β,θ are small , above it as β larger than θ and verse vice.

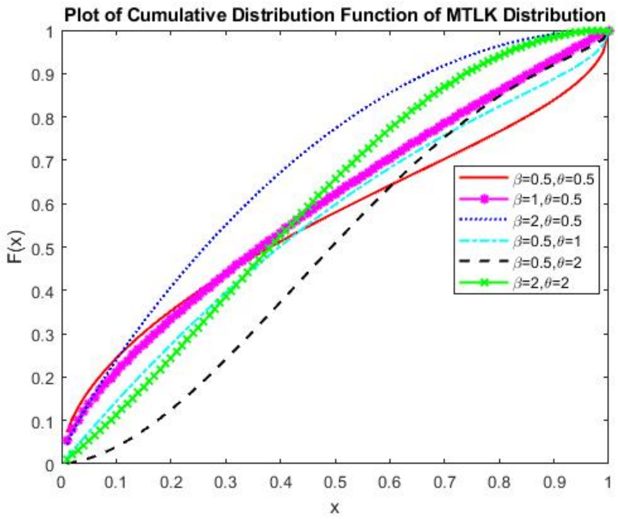

Figure 2: Plot of Cumulative Distribution Function of MTKL Distribution

The reliability function will be:

$$R\,(\mathrm{t}) = 1 - F\,(t) = 1 - \frac{1}{\beta + \theta}\left[\beta - \beta(1-x)^{\beta} + \theta x^{\theta}(2-x)^{\theta}\right] \tag{1.7}$$

We observe from curves of reliability function for different values of parameters (see Fig.3) that curves are near diagonal as β, θ are small, above it as β smaller than θ and verse vice.

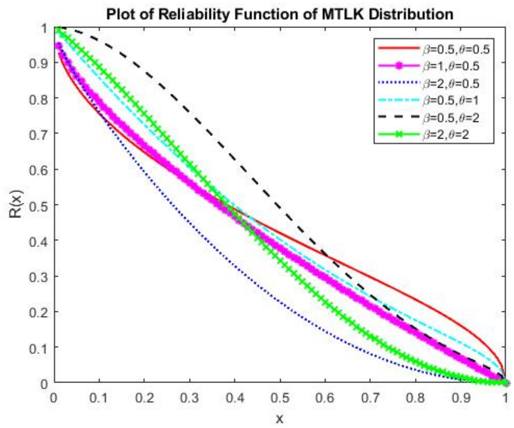

Figure 3: Reliability Function Plot of MTLK Distribution.

The hazard function will be:

$$h\,(x) = \frac{f(x)}{R(x)} = \frac{\frac{(1-x)}{(\beta+\theta)}\left[\beta^2(1-x)^{\beta-2} + 2\theta^2 x^{\theta-1}(2-x)^{\theta-1}\right]}{1 - \frac{1}{\beta+\theta}\left[\beta - \beta(1-x)^{\beta} + \theta x^{\theta}(2-x)^{\theta}\right]} \tag{1.8}$$

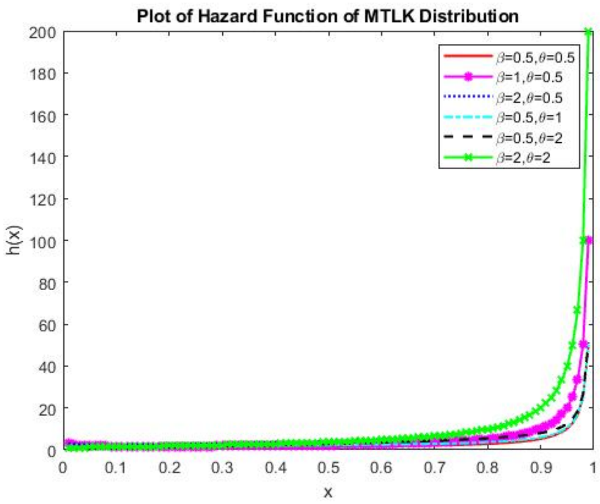

Figure 4: Hazard Function Plot of MTLK Distribution.

The hazard function's curves (see Fig.4) increase heavily as the parameter values increase.

2. Main Properties of the MTLK Distribution

Here, we provide some main properties of the MTLK distribution including, quantile function, median, mode, moments, incomplete moments, residual life function and Renyi entropy.

2.1. Quantile Function and Median

The quantile function of EITL distribution say $Q(u, \beta, \theta)$ is derived from equation (1.6), by solving $F(x) = u$ for x , as follow:

$$u = \frac{1}{\beta + \theta}\left[\beta - \beta(1-x)^{\beta} + \theta x^{\theta}(2-x)^{\theta}\right] \tag{2.1}$$

By simple rearrangement , we get:

$$\beta(1-x)^{\beta} - \theta(x\,(2-x))^{\theta} - u\,(\beta+\theta) - \beta = 0 \tag{2.2}$$

Equation (2.2) is nonlinear, so we use numerical methods for solving this equation for x value that represent the the u^{th} quantile. Where $0 < u < 1$ In particular, the median can be derived from (2.2) by setting $u = 0.5$. Also, the first and third quartiles are computed by setting u= 0.25 and 0.75 respectively in (2.2).

2.2. Moments

The moments can be used in any statistical analysis especially in applied work. Some of the most important features and shapes of a distribution , such as spread, dispersion, peakedness and symmetry can be measured by mean ,variance, kurtosis and skewness respectively.
Let X be a random variable with density (1.5). The general r^{th} moment about the origin of MTLK distribution is obtained from pdf (1.5) as follows:

$$\mu_r = E\left(x^r\right) = \int_0^1 x^r f\left(x\right) dx \tag{2.3}$$

Substituting (1.5) into (2.3), yields:

$$\mu_r = \frac{\beta}{\beta+\theta}\int_0^1 x^r\beta(1-x)^{\beta-1}dx + \frac{\theta}{\beta+\theta}\int_0^1 x^r 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx$$

$$= \frac{\beta^2}{\beta+\theta}Beta\,(r+1,\beta) + \frac{2\theta^2}{\beta+\theta}\int_0^1 x^{\theta+r-1}(1-x)2^{\theta-1}\Big(1-\frac{x}{2}\Big)^{\theta-1}dx \tag{2.4}$$

The generalized binomial expansion, for $\theta > 0$ is real non integer and $|x| < 1$ is

$$(1-|x|)^{\theta} = \sum_{i=0}^{\infty} C_i^{\theta}(-x)^i \tag{2.5}$$

Employing, the binomial theorem (2.5) in (2.4), where θ is real non integer, we have

$$\mu_r = \frac{\beta^2}{\beta+\theta}B\,(r+1,\beta) + \frac{\theta^2}{\beta+\theta}\sum_{i=0}^{\infty} C_i^{\theta-1}(-1)^i 2^{\theta-i}B(\theta+r+i,2) \tag{2.6}$$

Where $B(\cdot,\cdot)$ is the beta function. Furthermore, the r^{th} central moment of MTLK is given by:

$$\mu_r' = E(t-\mu_1)^r = \sum_{i=0}^{r} C_i^r(-\mu_i)^i\mu_{r-i} \tag{2.7}$$

The measure of symmetry is the skewness which describes the symmetry of the distribution and defined as:

$$Skewness = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{\left(\sqrt{\mu_2-\mu_1^2}\right)^3}; \mu_2 - \mu_1^2 > 0 \tag{2.8}$$

The measure of peakedness is the kurtosis which describes the peakedness of the distribution and defined as follows:

$$Kurtosis = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4}{(\mu_2-\mu_1^2)^2}; \mu_2 - \mu_1^2 > 0 \tag{2.9}$$

2.3. Incomplete Moments

They are mostly utilized to explain or measure inequality of the distribution. Particularly, the main application of the first incomplete moment refers to the Bonferroni and Lorenz curves that have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The r^{th} incomplete moment, say, μ_r(x) is obtained as follows:

$$w_r(\mathrm{t}) = E\,(x^r) = \int_0^{\mathrm{t}} x^r f\,(x)\,dx \tag{2.10}$$

Substituting (1.5) into (2.10) yields

$$w_r(\mathrm{t}) = \frac{\beta}{\beta+\theta}\int_0^t x^r\beta(1-x)^{\beta-1}dx + \frac{\theta}{\beta+\theta}\int_0^t x^r 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx \tag{2.11}$$

Employing, the binomial theorem (2.5) in (2.11), where θ is real non integer, we have

$$w_r(\mathrm{t}) = \frac{\beta^2}{\beta+\theta}IB\,(r+1,\beta,t) + \frac{\theta^2}{\beta+\theta}\sum_{i=0}^{\infty} C_i^{\theta-1}(-1)^i 2^{\theta-i}\int_0^t x^{\theta+r+i-1}(1-x)dx$$

$$w_r(\mathrm{t}) = \frac{\beta^2}{\beta+\theta} IB\left(r+1, \beta, t\right) + \frac{\theta^2}{\beta+\theta} \sum_{i=0}^{\infty} C_i^{\theta-1} (-1)^i 2^{\theta-i} IB(\theta+r+i, 2, t) \tag{2.12}$$

Where $IB(\cdot,\cdot,\cdot)$ is the incomplete beta function.
In addition, for lifetime distributions, the r^{th} conditional moment of the MTLK distribution is obtained as follows:

$$E\left(x^r | X > t\right) = \mu_r - \frac{\beta^2}{\beta+\theta} IB\left(r+1, \beta, t\right) + \frac{\theta^2}{\beta+\theta} \sum_{i=0}^{\infty} C_i^{\theta-1}$$

$$(-1)^i 2^{\theta-i} IB(\theta+r+i, 2, t) \tag{2.13}$$

2.4. Moments of Residual Life

The residual life of a unit with age t is the period beyond until the time of failure, and defined by the conditional random variable $X - t|X > t$. Therefore, the r^{th} moment of the residual lifetime, of the MTLK distribution is given by:

$$R_r\left(t\right) = \frac{1}{s(t)} \int_t^{\infty} \left(x-t\right)^r f\left(x\right) dx \tag{2.14}$$

$$R_r\left(t\right) = \frac{1}{s(t)} \int_t^{\infty} \left(x-t\right)^r \frac{(1-x)}{(\beta+\theta)} \left[\beta^2 (1-x)^{\beta-2} + 2\theta^2 x^{\theta-1} (2-x)^{\theta-1}\right] dx$$

By applying binomial expansion and binomial theorem (2.5), we have:

$$R_r\left(t\right) = \frac{1}{s(t)} \left(\frac{\beta^2}{\beta+\theta} I_1 + \frac{2\theta^2}{\beta+\theta} I_2\right) \tag{2.15}$$

Where:

$$I_1 = 1 - \sum_{i=0}^{r} C_i^r (-t)^{r-i} IB(i+1, \beta, t).$$

$$I_2 = 1 - \sum_{j=0}^{r} \sum_{k=0}^{\infty} C_k^{\theta-1} C_j^r (-t)^{r-j} (-1)^k (2)^{\theta-1-k} IB(\theta+j+k, 2, t)$$

Where $IB(\cdot,\cdot,\cdot)$ is the incomplete beta function.

2.5. Renyi Entropy

Entropy is a measure of variation or uncertainty of a random variable and it has many application in numerous fields like physics and communication. The Renyi entropy of order ρ, where $\rho > 0$ and $\rho \neq 1$, for the MTLK distribution is derived as follows:

$$R_\delta\left(x\right) = (1-\delta)^{-1} log \left(\int_0^1 f(x)^\delta dx\right) \tag{2.16}$$

$$= (1-\delta)^{-1} \log \left(\int_0^1 \left(\frac{(1-x)}{(\beta+\theta)} \left[\beta^2 (1-x)^{\beta-2} + 2\theta^2 x^{\theta-1} (2-x)^{\theta-1}\right]\right)^\delta dx\right)$$

By applying binomial expansion and binomial theorem (2.5), we have:

$$R_\delta\left(x\right) = (1-\delta)^{-1} \log(\frac{1}{(\beta+\theta)^\delta} \int_0^1 \sum_{i=0}^{\delta} C_i^\delta 2^i \theta^{2i} [x\left(2-x\right)]^{i(\theta-1)}$$

$$(1-x)^{(\beta-1)(\delta-i)+i}\beta^{2(\delta-i)}dx)$$

$$R_\delta\left(x\right)=(1-\delta)^{-1}log(\frac{1}{(\beta+\theta)^\delta}\sum_{i=0}^{\delta}\sum_{j=0}^{\infty}(-1)^j C_i^\delta C_j^{i(\theta-1)}2^{i+i(\theta-1)+j}\theta^{2i}\beta^{2(\delta-i)}$$

$$B\left(i\left(\theta-1\right)+j+1,\left(\beta-1\right)\left(\delta-i\right)+i+1\right)) \tag{2.17}$$

Where $B(\cdot,\cdot)$ is the complete beta function.

3. Estimation methods

3.1. Maximum Likelihood Estimator (MLE),[4]

Assume that $t_1, t_2, \ldots, t_n$ be a complete random sample of MTLK distribution with parameters (β,θ) then , the likelihood function of the sample can be obtained below:

$$\mathrm{L}\left(\beta,\theta\right)=\prod_{i=1}^{n}f\left(t_i\right)$$

$$=\left(\beta+\theta\right)^{-n}\prod_{i=1}^{n}\left(1-x_i\right)\prod_{i=1}^{n}\left[\beta^2(1-x_i)^{\beta-2}+2\theta^2x_i^{\theta-1}(2-x_i)^{\theta-1}\right] \tag{3.1}$$

To obtain MLE, firstly we differentiate the log-likelihood equation w.r.t. the parameters and equate it to zero. Thus, The logarithm of the likelihood function will be:

$$LogL=-nlog\left(\beta+\theta\right)+\sum_{i=1}^{n}\left(1-x_i\right)+$$

$$\sum_{i=1}^{n}log\left[\beta^2(1-x_i)^{\beta-2}+2\theta^2x_i^{\theta-1}(2-x_i)^{\theta-1}\right] \tag{3.2}$$

the derivative of equation (3.2) for β,θ respectively will be:

$$\frac{\partial LogL}{\partial\beta}=\frac{-n}{\widehat{\beta}+\widehat{\theta}}+\sum_{i=1}^{n}\frac{\widehat{\beta}^2(1-x_i)^{\widehat{\beta}-2}log\left(1-x_i\right)+2\widehat{\beta}(1-x_i)^{\widehat{\beta}-2}}{\widehat{\beta}^2(1-x_i)^{\widehat{\beta}-2}+2\widehat{\theta}^2x_i^{\widehat{\theta}-1}(2-x_i)^{\widehat{\theta}-1}}=0 \tag{3.3}$$

$$\frac{\partial LogL}{\partial\theta}=\frac{-n}{\widehat{\beta}+\widehat{\theta}}+2\widehat{\theta}$$

$$\sum_{i=1}^{n}\frac{x_i^{\widehat{\theta}-1}\left[\widehat{\theta}(2-x_i)^{\widehat{\theta}-1}log\left(2-x_i\right)+(2-x_i)^{\widehat{\theta}-1}\left(\widehat{\theta}logx_i+2\right)\right]}{\widehat{\beta}^2(1-x_i)^{\widehat{\beta}-2}+2\widehat{\theta}^2x_i^{\widehat{\theta}-1}(2-x_i)^{\widehat{\theta}-1}}=0 \tag{3.4}$$

The equations (3.3) and (3.4) are nonlinear equation in $\widehat{\beta}$ and $\widehat{\theta}$, so an analytical solution is not possible ,so we will use of Newton-Raphson (N-R) algorithm to get the numerical solution that represent the value of maximum likelihood estimator of β and θ.

3.2. Cran's method (CM)[5]

This method has derive by Cran,[5] in 1988 as an enhancement of the moment method, and used to estimate the parameters of the 3- parameter Weibull distribution. It uses sample moments as a functions of differences of the observations rather than powers and in some distributions the parameter estimates are explicit formulas of low-order sample moments. It works through solving equations that given by assume equality among sample and population moments.

Let $x_1 \leq x_2 \leq \ldots \leq x_n$, be an ordered random sample from MTLK distribution. The cumulative distribution function C.D.F is estimated by:

$$F_n(x) = \begin{cases} 0 & x < x_{(1)} \\ \frac{r}{n} & x_{(r)} \leq x < x_{(r+1)}\ , \quad {\scriptstyle r=1,\ldots,n-1} \\ 1 & x > x_{(n)} \end{cases} \tag{3.5}$$

The population moment μ_k^3 is:

$$\alpha \tag{3.6}$$

The counterpart sample moment is:

$$\alpha \tag{3.7}$$

By equating equations (3.6) and (3.7) for $k = 1, 2$, the Cran's estimates of the parameters can be obtained by solving the equations through Bi- variate Newton-Raphson Iterative method.

3.3. Percentile method(PM)[3],[6]

The principle of percentile method is based on equating nonparametric estimator of cumulative distribution function with corresponding theoretical percentiles and then simultaneously solving resulting equations for unknown parameters.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from MTLK distribution. The cumulative distribution function of MTLK distribution is:

$$F\left(x_i\right) = \frac{1}{\beta+\theta}\left[\beta - \beta(1-x_i)^{\beta} + \theta x_i^{\theta}(2-x_i)^{\theta}\right] \tag{3.8}$$

Thus, using nonparametric estimator of cdf as: $\hat{F}\left(x_i\right) = \frac{i}{n+1}$, the sum of square will be:

$$\sum_{i=1}^{n}\left(\hat{F}\left(x_i\right) - \frac{1}{\beta+\theta}\left[\beta - \beta(1-x_i)^{\beta} + \theta x_i^{\theta}(2-x_i)^{\theta}\right]\right)^2 = 0 \tag{3.9}$$

The percentile estimate of the parameters are represent the values that minimize of equation (3.9). Solving Equations by using derivative free optimization method for unknown parameters, we get the percentile estimators for β and θ.

3.4. Particle Swarm Optimization Method (PSOM)[15]

Particle swarm optimization algorithm is a population based stochastic optimization technique discovered by Kennedy and Eberhart in 1995.[15] It is inspired by biological specimen like birds, insects and fish. It search for the best solution in a population of solutions for the given problem. In PSO each particle has both position and velocity, the particle position represents the parameters to be estimated. The particle velocity and position vector are updated iteratively according to the following equations:

$$V_i\left(t\right) = W(t)V_i\left(t-1\right) + c_1 r_1\left[P_{best,i} - X_i(t-1)\right] + c_2 r_2\left[G_{best} - X_i(t-1)\right] \tag{3.10}$$

$$X_i(t) = X_i(t-1) + V_i(t) \quad , i = 1, 2, \ldots, N \tag{3.11}$$

Where W is the inertia weight, c_1, c_2 are the congnitive and social learning rates respectively, it is usually assumed to be 2, r_1, r_2 are the random numbers distributed uniformly in the range 0 and 1. The value of inertia weight decreases linearly with the iteration number has been used:

$$W(t) = W_{max} - (W_{max} - W_{min} t_{max})\, t \tag{3.12}$$

Where W_{max} and W_{min} represented the initial and final values of the inertia weight and are usually assumed to be 0.9 and 0.4 respectively, t_{max} is the maximum number of iterations.
The final part in this method is the fitness function that take the form according to MTLK distribution:
Let $x_1 \leq x_2 \leq \ldots \leq x_n$, be an ordered random sample from MTLK distribution. The nonparametric estimator of cumulative distribution function C.D.F is :
$\hat{F}(x_i) = \frac{i}{n+1}$, the sum of square between them is represent the fitness function as follow:

$$\sum_{i=1}^{n} \left[\hat{F}(x_i) - \frac{1}{(\beta+\theta)} \left(\beta - \beta(1-x_i)^{\beta} + \theta {x_i}^{\theta} (2-x_i)^{\theta} \right) \right]^2 \tag{3.13}$$

By applying the PSM algorithm in MatLab , that required the x_i and $\hat{F}(x_i)$ values and the fitness function ,we will get the estimate of the parameters.

4. Simulation Study

In this section, simulation experiments has been conducted, and the mean square error of the estimated parameters are compared. The simulation process were done using sample sizes $(10, 15, 25, 50 \; and \; 100)$ that represented small, moderate and large sample sizes and different combinations for the parameters $(\beta, \theta = 0.5, 1, 2)$. A random samples for each sample size is generated by using the following formula:

$$t = \left(\frac{\beta}{\beta+\theta}\right) \left(1 - (1-u)^{\frac{1}{\beta}}\right) + \left(\frac{\theta}{\beta+\theta}\right) \left(1 + \sqrt{1 - u^{\frac{1}{\theta}}}\right) \tag{4.1}$$

Where:
U: is a uniform random variate.
built on 1000 replications, The results are tabulated in tables 1 to 9.

Table 1: Mean Square Error of Parameters Estimates where $(\beta = 0.5, \theta = 0.5)$

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	0.829389	1.639992	1.957144	1.193573	MLE
	θ	0.472296	4.606259	6.608298	12.0863	MLE
15	β	0.864341	1.612607	0.772712	0.443415	PSO
	θ	0.178426	4.597595	2.800742	4.120755	MLE
25	β	0.836714	1.44578	1.406526	0.64206	PM
	θ	0.271055	2.397363	2.015194	6.565719	PM
50	β	0.29436	1.014199	0.519944	0.281692	PSO
	θ	0.25671	2.128235	1.373637	9.603016	PM
100	β	0.327086	0.585593	0.354065	0.273689	CM
	θ	0.177721	1.679122	1.523505	11.45173	PM
Percentage of Cases		30%	10%	40%	20%	PM

Table 2: Mean Square Error of Parameters Estimates where ($\beta = 1, \theta = 0.5$)

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	0.511108	0.581375	0.668101	22.06049	MLE
	θ	0.9692	1.557991	0.653070	43.52773	PM
15	β	0.710704	0.608579	0.371710	3.59338	PM
	θ	2.430043	1.095174	0.63260	15.92369	PM
25	β	0.825954	1.01645	0.837654	0.836886	MLE
	θ	0.687861	1.132319	0.592660	3.118761	PM
50	β	0.17151	0.077661	0.067231	0.803276	PM
	θ	0.803808	0.571819	0.439413	6.414302	PM
100	β	0.290948	0.240631	0.26333	0.715743	CM
	θ	1.575672	1.123346	0.821967	5.030755	PM
Percentage of Cases		20%	10%	70%	0%	PM

Table 3: Mean Square Error of Parameters Estimates where ($\beta = 2, \theta = 0.5$)

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	0.735909	0.392414	0.392990	4.142019	PM
	θ	1.142244	0.386828	0.057758	14.8575	PM
15	β	1.491485	1.380994	1.235211	3.169709	PM
	θ	0.973588	1.046528	0.570849	1.925281	PM
25	β	0.551895	0.406538	0.256164	3.426928	PM
	θ	0.728819	0.443252	0.188225	1.605154	PM
50	β	0.391326	0.1141	0.081533	6.363612	PM
	θ	0.939086	0.380042	0.111260	1.8451	PM
100	β	0.519931	0.167257	0.157887	3.743454	PM
	θ	0.650051	0.133263	0.063148	1.202645	PM
Percentage of Cases		0%	0%	100%	0%	PM

Table 4: Mean Square Error of Parameters Estimates where ($\beta = 0.5, \theta = 1$)

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	1.250054	2.084563	1.684932	0.321904	Pso
	θ	0.925311	1.145583	0.681376	20.44344	PM
15	β	2.316984	1.165103	1.678662	1.51605	CM
	θ	3.798652	2.629498	3.763431	4.80588	CM
25	β	2.732695	1.355842	1.077498	2.48964	PM
	θ	3.162426	0.982491	0.903345	3.581775	PM
50	β	0.944251	0.404551	0.188285	0.334595	PM
	θ	1.889209	1.0344	0.675611	3.472001	PM
100	β	2.582081	0.853324	1.134473	2.974342	CM
	θ	1.653908	0.900637	1.386903	2.658246	CM
Percentage of Cases		0%	40%	50%	10%	PM

Table 5: Mean Square Error of Parameters Estimates where ($\beta = 1, \theta = 1$)

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	1.292832	0.785593	0.549556	1.012763	CM
	θ	1.902187	1.454232	1.126532	4.394473	MLE
15	β	0.731259	0.335356	0.898407	0.757952	CM
	θ	1.962047	1.69314	1.107330	3.205082	PM
25	β	1.031416	0.692375	0.637705	0.703294	PM
	θ	1.758544	0.698382	1.431964	3.223342	CM
50	β	0.895463	0.272567	0.190925	1.512228	PM
	θ	1.49336	0.272104	0.18841	2.022032	PM
100	β	0.58534	0.112957	0.069165	1.044062	PM
	θ	0.708144	0.236281	0.164842	1.507096	PM
Percentage of Cases		10%	30%	60%	0%	PM

Table 6: Mean Square Error of Parameters Estimates where $(\beta = 2, \theta = 1)$

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	0.479359	0.663743	1.514083	9.366327	PM
	θ	2.148881	1.073026	0.590327	25.69284	PM
15	β	0.910256	0.297134	0.284237	3.78382	PM
	θ	0.93729	0.589129	0.656620	5.578449	CM
25	β	0.599774	0.450948	0.231546	2.960751	PM
	θ	0.400932	0.245783	0.543280	1.467767	CM
50	β	0.465239	0.18023	0.032371	2.907101	PM
	θ	0.858175	0.454208	0.954448	1.728961	CM
100	β	0.336362	0.122122	0.083644	3.002358	PM
	θ	0.333518	0.206056	0.782904	1.740268	CM
Percentage of Cases		0%	40%	60%	0%	PM

Table 7: Mean Square Error of Parameters Estimates where $(\beta = 0.5, \theta = 2)$

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	1.802542	0.278488	2.803906	8.798045	CM
	θ	11.41464	3.118551	16.63739	16.26787	CM
15	β	0.913129	3.875323	0.526970	0.320021	PSO
	θ	1.748418	31.90441	2.330522	14.35047	Mle
25	β	2.278332	0.522594	0.777128	2.110519	CM
	θ	4.495001	0.321015	1.621317	11.1504	CM
50	β	1.721563	0.247079	0.685865	0.256425	CM
	θ	2.941061	0.556773	2.194488	8.097175	CM
100	β	0.827968	0.257422	0.095409	0.563598	PM
	θ	1.449151	0.37964	0.259698	4.617894	PM
Percentage of Cases		10%	60%	20%	10%	CM

Table 8: Mean Square Error of Parameters Estimates where $(\beta = 1, \theta = 2)$

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	1.937412	0.712162	1.155973	5.744878	PM
	θ	2.682623	0.721456	5.466372	2.355666	CM
15	β	0.76394	0.584032	0.672066	0.755391	CM
	θ	1.584391	1.102773	0.885995	0.872446	PSO
25	β	1.073134	0.403483	0.797122	0.862048	CM
	θ	2.169483	0.712468	0.427195	1.041964	PM
50	β	0.521207	0.155121	0.386132	0.887935	CM
	θ	1.790013	0.502671	0.193129	2.41493	PM
100	β	0.100889	0.234747	0.588339	0.891275	Mle
	θ	0.157028	0.36141	0.062413	0.188215	PM
Percentage of Cases		10%	40%	40%	10%	PM

Table 9: Mean Square Error of Parameters Estimates where ($\beta = 2, \theta = 2$)

n	Parameters	MLE	CM	PM	PSO	BEST
10	β	1.431907	2.239681	0.659226	9.569032	PM
	θ	0.555744	3.250244	1.324418	5.420397	MLE
15	β	0.634734	0.772007	0.753714	3.864217	MLE
	θ	0.347974	0.57802	1.892665	0.744498	MLE
25	β	0.089786	0.199255	0.246017	2.827029	MLE
	θ	1.001309	0.743746	1.359584	1.573664	CM
50	β	0.463547	0.494465	0.436928	2.86591	PM
	θ	0.134588	0.214372	0.866618	0.564047	MLE
100	β	0.107857	0.020739	0.047485	2.822946	CM
	θ	0.013621	0.052904	0.009183	0.217131	PM
Percentage of Cases		50%	20%	30%	0%	MLE

It is observed that the mse of estimators decrease as the sample size increases and The PM method is almost more efficient than the other methods for almost all cases used.

5. Application

In this section an analysis based on two real data sets are conducted. To show that the MTLK seems to be a very competitive model for these data than the Two Parameters Kumaraswamy distribution TPK, Two Parameters Topp-Leone distribution TPTL and Generalized Beta distribution GB. The first data set, which have also been analyzed by Silva et al. (2013) [23] is related to the study of the soil fertility in influence and the characterization of the biologic fixation of N2 for the Dimorphandra wilsonii rizz growth. They made measures of the phosphorus concentration in the leaves for 128 plants. The data set is presented in table 10. The second data set used from Nigm et al. (2003) [19] is about ordered failure of components, The data set is presented in table 12. The descriptive statistics of the two sample data sets are shown in table 11 and 13, that indicates the data are positively skewed or skewed right and a platykurtic distribution or flat-tailed distribution.

Table 10: Phosphorus concentration data.

phosphorus concentration in the leaves							
0.22	0.11	0.19	0.09	0.14	0.11	0.13	0.19
0.12	0.16	0.24	0.11	0.25	0.1	0.11	0.21
0.17	0.12	0.21	0.22	0.07	0.13	0.12	0.15
0.09	0.09	0.19	0.1	0.16	0.15	0.14	0.16
0.11	0.1	0.18	0.23	0.09	0.12	0.14	0.07
0.23	0.1	0.21	0.2	0.13	0.17	0.07	0.08
0.1	0.06	0.26	0.22	0.05	0.15	0.09	0.09
0.25	0.12	0.22	0.12	0.11	0.14	0.07	0.1
0.15	0.2	0.19	0.19	0.06	0.12	0.1	0.17
0.23	0.12	0.17	0.15	0.11	0.12	0.19	0.06
0.06	0.17	0.17	0.27	0.11	0.11	0.13	0.1
0.24	0.1	0.08	0.08	0.11	0.18	0.17	0.08
0.05	0.2	0.18	0.16	0.16	0.11	0.09	0.08
0.2	0.09	0.08	0.12	0.08	0.14	0.18	0.12
0.07	0.11	0.2	0.28	0.2	0.15	0.11	0.15
0.08	0.17	0.06	0.09	0.22	0.18	0.16	0.13

Table 11: Descriptive statistics of phosphorus concentration in the leaves.

N	Min	Max	Mean	Median	Mode	Standard Deviation	Skewness	Kurtosis Excess
128	0.05	0.28	0.1408	0.13	0.11	0.0544	0.4544	2.3552

Table 12: Ordered failure of components data.

Ordered failure of components							
0.0009	0.004	0.0142	0.0221	0.0261	0.0418	0.0473	0.0834
0.1091	0.1252	0.1404	0.1498	0.175	0.2031	0.2099	0.2168
0.2918	0.3465	0.4035	0.6143				

Table 13: Descriptive statistics of ordered failure of components data.

N	Min	Max	Mean	Median	Standard Deviation	Skewness	Kurtosis Excess
20	0.0009	0.6143	0.1613	0.1328	0.1573	1.3302	4.5145

To verify that MTLK distribution is suitable model for the data set a minus two times of negative log-likelihood value, Akaike information criteria (AIC), Bayesian information criteria (BIC), corrected Akaike information criterion (AICC), Hannan-Quinn information criterion (HQIC), consistent akaike information criterion (CAIC) and Kolmogorov - smirnov K-S distance with P_values are used.
where:

$$\mathrm{AIC} = 2\mathrm{k} - 2\mathrm{logl} \tag{5.1}$$

$$\mathrm{BIC} = \mathrm{klog(n)} - 2\mathrm{logl} \tag{5.2}$$

$$\mathrm{AICC} = \mathrm{AIC} + \frac{2\mathrm{k}(\mathrm{k}+1)}{\mathrm{n}-\mathrm{k}-1} \tag{5.3}$$

$$\mathrm{HQIC} = -2\mathrm{logl} + 2\mathrm{klog(log(n))} \tag{5.4}$$

$$\mathrm{CAIC} = -2\mathrm{logl} + \frac{2\mathrm{kn}}{\mathrm{n}-\mathrm{k}-1} \tag{5.5}$$

where $logl$ denotes the log-likelihood at MLEs, k is the number of parameters, and n is a sample size. The table 14 and 15 shows that the MTLK has lower values for $-2Logl$,(AIC), (BIC), (AICC), (HQIC), (CAIC) and K-S distance and than TPK, TPTL and GB and higher P_Values respectively. So, which indicate that the MTLK could be chosen as the best model than TPK, TPTL and GB distributions for both data sets. The empirical and estimated cdf of MTLK, TPK, TPTL and GB are displayed in figure 5 and 6.

Table 14: Parameter estimates with different criteria of phosphorus concentration.

Model	Parameter estimate	$-2Logl$	AIC	BIC	AICC	HQIC	CAIC	K-S	P_Value
MTLK	$\widehat{\beta} = 6.86$	-	-	-	-	-	-	0.342	1.016e-13
	$\widehat{\theta} = 0.43$	256.124	252.124	246.420	252.028	249.806	252.028		
TPK	$\widehat{\alpha} = 2.81$	504.007	508.007	513.711	508.103	510.325	508.103	0.810	5.746e-75
	$\widehat{\beta} = 0.75$								
TPTL	$\widehat{\alpha} = 0.17$	-0.879	3.120	8.824	3.216	5.437	3.216	0.578	2.203e-38
	$\widehat{\beta} = 71.59$								
GB	$\widehat{\alpha} = 0.45$	44.954	54.954	69.214	55.446	60.748	55.446	0.440	1.770e-22
	$\widehat{\beta} = 49.37$								
	$\widehat{\lambda} = 3e-05$								
	$\widehat{\theta} = 0.59$								
	$\widehat{\gamma} = 4.04$								

Table 15: Parameter estimates with different criteria of ordered failure of components.

Model	Parameter estimate	$-2Logl$	AIC	BIC	AICC	HQIC	CAIC	K-S	P_Value
MTLK	$\widehat{\beta} = 0.65$	-15.8	-11.8	-9.883	-11.16	-11.4	-11.1	0.451	3e-04
	$\widehat{\theta} = 0.59$								
TPK	$\widehat{\alpha} = 0.764$	-4.8	-0.8	1.175	-0.110	-0.427	-0.110	0.546	3.89e-06
	$\widehat{\beta} = 0.75$								
TPTL	$\widehat{\alpha} = 0.14$	-12.6	-8.6	-6.67	-7.958	-8.275	-7.958	0.504	3.14e-05
	$\widehat{\beta} = 158.84$								
GB	$\widehat{\alpha} = 0.254$	13.3	23.3	28.3	27.640	24.327	27.640	0.498	3.97e-05
	$\widehat{\beta} = 106.26$								
	$\widehat{\lambda} = 6e-07$								
	$\widehat{\theta} = 0.60$								
	$\widehat{\gamma} = 1.19$								

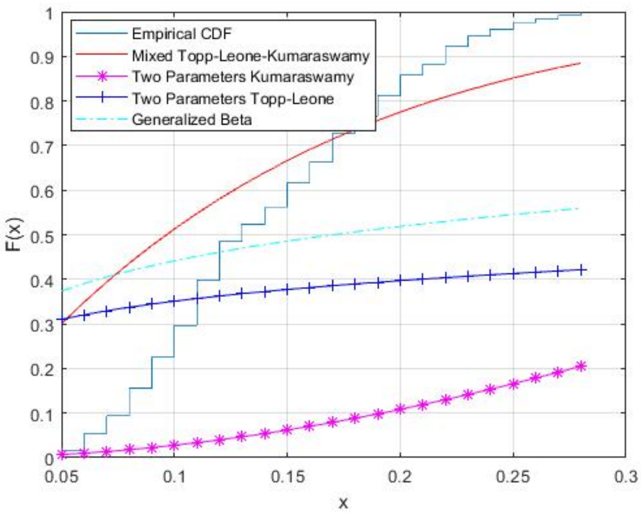

Figure 5: Empirical, fitted Mixed Topp-Leone-Kumaraswamy, Two Parameter Kumaraswamy , Two Parameter Topp-Leone and Generalized Beta cdf's of Phosphorus Concentration data.

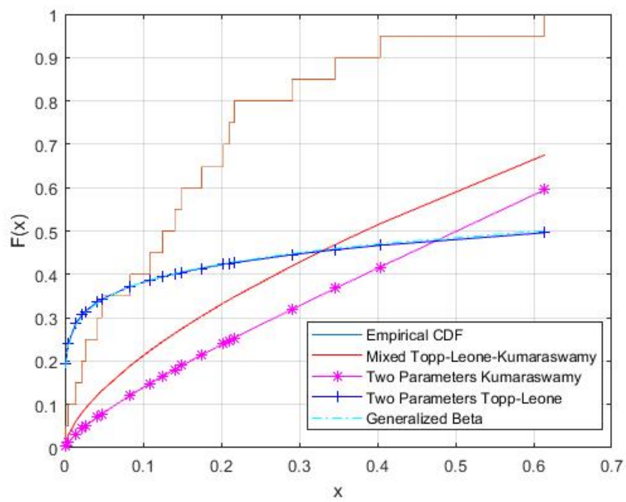

Figure 6: Empirical, fitted Mixed Topp-Leone-Kumaraswamy, Two Parameter Kumaraswamy , Two Parameter Topp-Leone and Generaliazed Beta cdf's of ordered failure of components data.

6. Conclusion

We mixed the one parameter Topp-Leone distribution with one parameter Kumaraswamy distribution, the two parameter mixed Topp-Leone- Kumaraswamy distribution has nice properties. The general moments formula and other important functions are derived. The percentile estimation method is best one according to simulation experiments used. The Topp-Leone- Kumaraswamy distribution is applied to two real data and indicated that could be chosen as the best model than the Two Parameters Kumaraswamy pistribution, two Parameters Topp - Leone distribution and Generalized Beta distribution.

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