Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 751-765
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5127

Some types of fibrewise fuzzy topological spaces

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The aim of this paper is to introduce and study the notion type of fibrewise topological spaces, namely fibrewise fuzzy j-topological spaces, Also, we introduce the concepts of fibrewise j-closed fuzzy topological spaces, fibrewise j-open fuzzy topological spaces, fibrewise locally sliceable fuzzy j-topological spaces and fibrewise locally sectionable fuzzy j-topological spaces. Furthermore, we state and prove several Theorems concerning these concepts, where $j = \{\delta, \theta, \alpha, p, s, b, \beta\}$.

Keywords: FWF j-TS's, FWCF j-TS's, FWOF j-TS's, FW-LSL-F j-TS's and FW-LSE-F jTS's.
2010 MSC: 55R70, 54A40, 54C08, 54C10.

1. Introduction and Preliminaries

To begin with our work in the type of fibrewise sets over a given set, named the base set. If the base set is denoted by B then a fibrewise set over B consists of a set M together with a function $p : M \to B$, named the projection. For all point b of B the fibre over b is the subset $M_b = p^{-1}(b)$ of M; fibres may be empty because we do not require p to be surjective, in addition for all subset B^* of B we consider $M_{B^*} = p^{-1}(B^*)$ as a fibrewise set over B^* with the projection determined by p. The concept of fuzzy sets was introduced by Zadeh [21]. The idea of fuzzy topological spaces was introduced by Chang [2]. Several types of fuzzy continuous functions. Different aspects of such spaces have been developed, by several investigators. We studied the connected between fibrewise topological spaces and fuzzy j-topological space also some related concepts such as fibrewise f. j-open, fibrewise f. j-closed, fibrewise locally sliceable and fibrewise locally sectionable fuzzy topological spaces. The purpose of this paper is introduced a new class of fibrewise topology called fibrewise fuzzy j-topological space are introduced and few of their properties are investigated, we built on some of the result in [15, 16, 17, 18, 19, 20, 21], where $j \in \{\delta, \theta, \alpha, p, s, b, \beta\}$.

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Received: February 2021 *Accepted:* April 2021

Definition 1.1. [7] *A mapping $\vartheta : M \to N$, where M and N are FW sets over B, with proj.'s $p_M : M \to B$ and $p_N : N \to B$, is said to be FW mapping (written as* $\mathrm{FW-M}$ *) if $p_N \circ \vartheta = p_M$, or $\vartheta(M_b) \subseteq N_b$, for all point $b \in B$.*

Observe that a FW-M $\vartheta : M \to N$ over B limited by restriction, a FW-M $\vartheta : M_{B^*} \to N_{B^*}$ over B^* for all subset $B^* \subseteq B$.

Definition 1.2. [7] *The fibrewise topology (written as FWT) on a FW set M over a topological space (B, σ) signify any topology on M for which the proj. p is continuous (written as FWTS).*

Definition 1.3. [7] *Let M and* n *be FWTS's over B, the FW-M $\vartheta : M \to N$ is said tobe:*

(a) continuous if $b \in B$ and for all point $m \in M_b$, the pre image of all open set of $\vartheta(m)$ is an open set of m.

(b) open if $b \in B$ and for all point $m \in M_b$, the image of all open set of m is an open set of $\vartheta(m)$.

Definition 1.4. [7] *The FWTS (M, τ) over (B, σ) is said to be:*

(a) FW closed (written as FWC) if the proj. p is closed mapping.

(b) FW open (written as FWO) if the proj. p is open mapping.

Definition 1.5. [22] *Let X be a nonempty set, a fuzzy set A in X is characterized by a function $\mu_A : X \longrightarrow I$ where I is the closed unite interval $[0,1]$ which is written as:*

$$A = \{(x, \mu_A(x)) : x \in X, 0 \leq \mu_A(x) \leq 1\}$$

The collection of all fuzzy subsets in X will be denoted by I^X that is $I^X = \{A : A$ is fuzzy subset of $X\}$ and μ_A is called the membership function.

Example 1.6. [10] *We will suppose a possible membership function for the fuzzy set of real numbers close to zero as follows, $\mu_A : \mathbb{R} \longrightarrow [0,1]$, where*

$$\mu_A(x) = \frac{1}{1 + (x - 10^2)}, \forall x \in \mathbb{R}$$

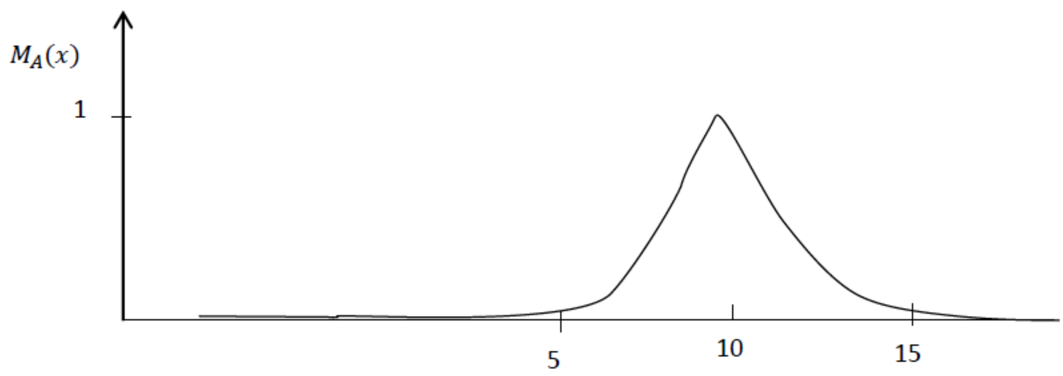

Figure 1: Diagraph of Example 1.6.

Definition 1.7. [22] *A fuzzy set in X is empty denoted by $\overline{0}_x$, if its membership function is identically the zero function, i.e.,*

$$\overline{0}_X : X \to [0,1] \quad s.t\, \overline{0}_X(x) = 0 \quad \forall x \in X$$

Definition 1.8. [22] *A universal fuzzy set in X, denoted by $\overline{1}_X$, is a fuzzy set defined as*

$$\overline{1}_X(x) = 1 \quad \forall x \in X$$

Definition 1.9. [22] *Let $\mu, \lambda \in I^X$. A fuzzy set μ is a subset of a fuzzy set λ, denoted by $\mu \leq \lambda$ iff $\mu(x) \leq \lambda(x), \forall x \in X$.*

Two fuzzy sets μ and λ are said to be equal $(\lambda = \mu)$ if $\lambda(x) = \mu(x), \forall x \in X$

Definition 1.10. [22] *Let λ and μ be fuzzy sets in X. Then, for all $x \in X$,*

$$\begin{aligned}
&\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = \max\{\lambda(x), \mu(x)\} \\
&\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = \min\{\lambda(x), \mu(x)\} \\
&\eta = \lambda^c \Leftrightarrow \eta(x) = 1 - \lambda(x)
\end{aligned}$$

More generally, for a family $\Lambda = \{\lambda_i \mid i \in I\}$ of fuzzy sets in X, the union $\psi = \mathrm{V}_i \lambda_i$ and intersection $\delta = \Lambda_i \lambda_i$ are defined by

$$\begin{aligned}
\psi(x) &= \sup_i \{\lambda_i(x) \mid x \in X\}, \\
\delta(x) &= \inf_i \{\delta_i(x) \mid x \in X\}
\end{aligned}$$

Definition 1.11. [3] *A fuzzy topology is a family τ of fuzzy sets in X, which satisfies the following conditions:*

(a) $\overline{0}, \overline{1} \in \tau$

(b) If $\lambda, \mu \in \tau$, then $\lambda \wedge \mu \in \tau$;

(c) If $\lambda_i \in \tau$ for each $i \in I$, then $\bigcup_i \lambda_i \in \tau$.

τ is called a fuzzy topology for X, and the pair (X, τ) is an fts. Every member of τ is called τ -open fuzzy set (or simply an open fuzzy set). A fuzzy set is τ -closed if and only if its complement is τ -open.

As in general topology, the indiscrete fuzzy topology contains only $\overline{0}$ and $\overline{1}$, while the discrete fuzzy topology contains all fuzzy sets.

Definition 1.12. [3] *Let (X, τ) be a fuzzy topological space and $A \in F(X)$. The fuzzy closure (resp., fuzzy interior) of A is denoted by $\mathrm{cl}(A)$ (resp., int (A)) is defined by:*

$$\begin{aligned}
\mathrm{cl}(A) &= \wedge\{F^c \in \tau, A \leq F\} \\
\mathrm{int}(A) &= \mathrm{V}\{O \in \tau; O \leq A\}
\end{aligned}$$

Evidently, cl(A) (resp., int (A)) is the smallest fuzzy closed (resp., largest f. open) subset of X which contains (resp., contained in) A. Note that A is f. closed (resp., f. open) if and only if $A = \mathrm{Cl}(A)($ resp., $\mathrm{int}(A))$.

Definition 1.13 (3). *Let $f : X \to Y$ be a mapping. Let β be a fuzzy set in Y with membership function $\beta(y)$. Then the inverse of β, written as $f^{-1}(\beta)$, is a fuzzy set in X whose membership function is defined by $f^{-1}(\beta)(x) = \beta(f(x)), \forall x \in X.(1)$ Conversely, let λ be a fuzzy set in X with membership function $\lambda(x)$. The image of written as $f(\lambda)$, is a fuzzy set in Y whose membership function is given by*

$$f(\lambda)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \{(\lambda(x)\}, \ \textit{if } f^1(y) & \textit{is nonempty,} \\ 0, & \textit{otherwise,} \end{cases}$$

for all $y \in Y$.

Definition 1.14. [8] *A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\lambda(0 < \lambda \leq 1)$ we denote this fuzzy point by x_λ, where the point x is called its support.*

Definition 1.15. [8] *A fuzzy point x_λ is said to be quasi-coincident with A denoted by $x_\lambda qA$ iff $\lambda + A(x) > 1$*

Definition 1.16. [8] *A fuzzy set A in (X, τ) is called a "Q-neighborhood of x_λ " iff $\exists B \in \tau$ such that $x_\lambda qB < A$ The family of all Q-nbhds of x_λ is called the system of Q- nbhd's of x_λ.*

Definition 1.17. [9] *A fuzzy set A is fuzzy θ -closed [5] if $A = cl_\theta(A) = \{p$ fuzzy point in $(X, \tau) : (cl(U))qA, U$ is fuzzy open q -nbd. of $p\}$. The complement of fuzzy θ -closed called fuzzy θ -open set.*

Definition 1.18. [13] *A fuzzy set A is fuzzy δ -closed* [4] *if $A = cl_\delta(A) = \{p$ fuzzy point in(X, τ) : int $(cl(U))qA, U$ is fuzzy open q -nbd. of $p\}$. The complement of fuzzy $\delta-$ closed called fuzzy δ -open set.*

Definition 1.19. *A fuzzy set of a fuzzy topological space (X, τ) is called :*

(a) *f. α -open* (f.α *-closed*) *set if $A \leq$ int cl int $A(A \geq$ cl int* clA), [2].

(b) f. preopen (f. preclosed) set if $A \leq$ int cl$A$ ($A \geq$ cl int A), [2].

(c) f. *simeopen* (f. *simeclosed)set if $A \leq$* cl *int $A(A \geq$ int* cl$A$)[1]

(d) f. β -open (f. β -closed) set if $A \leq$ cl int cl$A$ ($A \geq$ int cl int A), [14].

(e) f. *b- open* (f.b *- closed*) *set if $A \leq$ cl int $A\vee$ int cl $A(A \geq$ cl int $A\vee$ int cl$A$),* [5].

Definition 1.20. [3] *A mapping $f : (X, \tau_x) \to (Y, \tau_y)$ is said to be*

(a) fuzzy continuous (briefly f. *continuous) if the inverse image of every fuzzy open set of Y is a* f. *open set in X.*

(b) fuzzy open (briefly f. *open) map if the image of every fuzzy open set of X is a* f. *open set in Y.*

(c) fuzzy close (briefly f. close) map if the image of every fuzzy close set of X is a f. close set in Y.

Definition 1.21. [11, 12] *A function $f : (X, \tau) \to (Y, \sigma)$, is said to be*

(a) *Fuzzy θ -continuous (f.θ.c, for short) if for each fuzzy point $\tilde{p}$ in (X, τ) and each fuzzy open q-nbd. v of $f(\tilde{p})$, there exists fuzzy open q-nbd. u of $\tilde{p}$ such that $f(cl(u)) \subseteq \mathrm{cl}(v)$*

(b) *Fuzzy δ -continuous (f. δ.c, for short) if for each fuzzy point $\tilde{p}$ in (X, τ) and each fuzzy open q-nbd. v of $f(\tilde{p})$, there exists fuzzy open q-nbd. u of $\tilde{p}$ such that $f(\mathrm{int}(cl(u))) \subseteq \mathrm{int}(\mathrm{cl}(v))$*

Definition 1.22. $[1, 2, 5, 14]$ *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be* f. j *- continuous if the inverse image of every fuzzy open set of Y is a* f. *j-open set in X, where $j \in \{\alpha, P, S, b, \beta\}$.*

2. Fibrewise Fuzzy j-topological spaces $j \in \{\delta, \theta, \alpha, p, s, b, \beta\}$

In this section, topology we give a definition of fibrewise fuzzy j-topology and its related properties, where $j = \{\delta, \theta, \alpha, p, s, b, \beta\}$.

Definition 2.1. *The fibrewise fuzzy j-topology (briefly, FWF j-TS) on a FW set M over FTS (B, Λ) signify any fuzzy topology on M for which the proj. p is* f. j *-continuous (briefly, f. j-continuous), where $j = \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

For example, we can assume (B, Λ) like a FWF j *-TS over itself by the identity as proj. Also, the fuzzy topological product (see [6]) $B \times T$, for all FTS T, can be regarded like a FWF j-TS's over B, by the first proj. and in the same way for every fuzzy subspace (see [4]) of $B \times T$*

Remark 2.2. *In FWF topological space we work over at fuzzy topological base space B, say. When B is a point-space the theory reduces to that of ordinary fuzzy topology. A FWF topological (resp., f. j-topological) spaces over B is just a fuzzy topological (resp., f. j-topological) space M together with a fuzzy continuous (resp., f. j-continuous) projection function $p : (M, \tau) \to (B, \sigma)$. So the implication between FWF topological spaces and the families of FWF j-topological spaces are given in the following diagram where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*

$$
\begin{array}{lll}
FWF\ \delta - \text{topological space} & & \\
\Downarrow & & \\
FWF\ \theta - \text{topological space} & & \\
\Uparrow & & \\
FWF\ \text{topological space} & & \\
\Downarrow & & \\
FWF\alpha - \text{topological space} & \Rightarrow & FWFS - \text{topological space} \\
\Downarrow & & \Downarrow \\
FWFP - \text{topological space} & \Rightarrow & FWFb - \text{topological space} \\
 & & \Downarrow \\
 & & FWF\beta - \text{topological space.}
\end{array}
$$

Figure 2.1.1: Implication between fibrewise fuzzy topology and fibrewise fuzzy j -topology, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

The following examples show that these implications are not reversible.

Example 2.3. *Let $M = B = [0,1]$. Let $\tau = \{\overline{0}, \overline{1}, \mu\}$ and $\sigma = \{\overline{0}, \overline{1}, \lambda\}$, where*

$$\mu(x) = \begin{cases} \frac{1}{3}, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \lambda(x) = \begin{cases} \frac{1}{4}, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Let (M,τ) be a FWF topological space over (B,σ) and let the projection function $p : (M,\tau) \to (B,\sigma)$ be the fuzzy function as the identity maps. Then (M,τ) is FWF θ- topological space but not FWF δ - topological space.

Example 2.4. *Let $M = B = [0,1]$. Let $\tau = \{\overline{0}, \overline{1}, \mu\}$ and $\sigma = \{\overline{0}, \overline{1}, \lambda\}$, where*

$$\mu(x) = \begin{cases} \frac{1}{4}, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \lambda(x) = \begin{cases} \frac{1}{5}, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Let (M,τ) be a FWF topological space over (B,σ). and let the projection function $p : (M,\tau) \to (B,\sigma)$ be the fuzzy function as the identity maps. Then (M,τ) is FWF θ topological space but not FWF topological space.

Example 2.5. *Let $= \{a,b\}, B = \{x,y\}, \tau = \left\{\overline{0}, \overline{1}, \mu_1, \mu_2, \mu_3\right\}$ where*
$\mu_1 = \{(a,0.2),(b,0.3)\}$
$\mu_2 = \{(a,0.5),(b,0.6)\}$
$\mu_3 = \{(a,0.7),(b,0.7)\}$
And let $\sigma = \{\overline{0}, \overline{1}, \lambda\}$, where $\lambda = \{(x,0.6),(y,0.7)\}$ be the fuzzy topologies on set M and B respectively and let the projection function $p : (M,\tau) \to (B,\sigma)$ be the fuzzy function as the identity maps. Then (M,τ) FWF α -topological space but not FWF topological space.

Example 2.6. *Let $M = \{a,b\}, B = \{x,y\}, \tau = \left\{\overline{0}, \overline{1}, \mu_1, \mu_2\right\}$ where*
$\mu_1 = \{(a,0.1),(b,0.2)\}$,
$\mu_2 = \{(a,0.3),(b,0.4)\}$.
And let $\sigma = \{\overline{0}, \overline{1}, \lambda\}$, where $\lambda = \{(x,0.6),(y,0.6)\}$ be the fuzzy topologies on set M and B respectively and let the projection function $p : (M,\tau) \to (B,\sigma)$ be the fuzzy function as the identity maps. Then (M,τ) FWF S-topological space but not FWF α- topological space.

Example 2.7. *Let $M = \{a,b\}, B = \{x,y\}, \tau = \left\{\overline{0}, \overline{1}, \mu_1, \mu_2\right\}$ where*
$\mu_1 = \{(a,0.8),(b,0.9)\}$
$\mu_2 = \{(a,0.6),(b,0.7)\}$
And let $\sigma = \{\overline{0}, \overline{1}, \lambda\}$, where $\lambda = \{(x,0.5),(y,0.5)\}$ be the fuzzy topologies on set M and B respectively and let the projection function $p : (M,\tau) \to (B,\sigma)$ be the fuzzy function as the identity maps. Then (M,τ) FWF P-topological space but not FWF α- topological space.

Example 2.8. *Let $M = \{a,b,c\}, B = \{x,y,z\}, \tau = \left\{\overline{0}, \overline{1}, \mu_1, \mu_2, \mu_3, \mu_4\right\}$ where*
$\mu_1 = \{(a,0.5),(b,0.2),(c,0.6)\}$
$\mu_2 = \{(a,0.3),(b,0.4),(c,0.3)\}$
$\mu_2 = \{(a,0.3),(b,0.2),(c,0.3)\}$
$\mu_3 = \{(a,0.5),(b,0.4),(c,0.6)\}$
And let $\sigma = \{\overline{0}, \overline{1}, \lambda\}$, where $\lambda = \{(x,0.5),(y,0.5),(z,0.5)\}$ be the fuzzy topologies on set M and B respectively and let the projection function $p : (M,\tau) \to (B,\sigma)$ be the fuzzy function as the identity maps. Then (M,τ) FWF b-topological space but not FWF P-topological space and not FWF S-topological space.

Example 2.9. *Let* $M = \{a, b, c\}, B = \{x, y, z\}, \tau = \{\overline{0}, \overline{1}, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ *where*
$\mu_1 = \{(a, 0.7), (b, 0.8), (c, 0.9)\}$
$\mu_2 = \{(a, 0.5), (b, 0.3), (c, 0.6)\}$
$\mu_3 = \{(a, 0.3), (b, 0.4), (c, 0.3)\}$
$\mu_4 = \{(a, 0.3), (b, 0.3), (c, 0.3)\}$
$\mu_5 = \{(a, 0.5), (b, 0.4), (c, 0.6)\}$
And let $\sigma = \{\overline{0}, \overline{1}, \lambda\}$*, where* $\lambda = \{(x, 0.2), (y, 0.6), (z, 0.2)\}$ *be the fuzzy topologies on set* M *and* B *respectively and let the projection function* $p : (M, \tau) \to (B, \sigma)$ *be the fuzzy function as the identity maps. Then* (M, τ) *FWF* β *-topological space but not FWF b-topological space.*

Proposition 2.10. *A fibrewise fuzzy set is FWF* α *-topological space iff it is FWF S-topological space and FWF P-topological space.*

Proof . ($\Leftarrow$) Let (M, τ) be a FWF S-topological space and FWF P-topological space over (B, σ) then the projection function $p : (M, \tau) \to (B, \sigma)$ exists. To show that p is f.α- continuous . Since (M, τ) is FWF S-topological space and FWF P-topological space over (B, σ), then p is f. S-continuous and f. P-continuous then p is f. α -continuous by proposition (1.3.18). Thus, (M, τ) is FWF α -topological space over (B, σ).
($\Rightarrow$) It obvious. □

Remark 2.11. *(a) In FWF j-TS we carry out over FTS* (B, σ) *as base space. If* B *is a point-space the theory changes to that of ordinary fuzzy topology, where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

(b) A FWF j-TS's over B is just a FTS (M, τ) *with a f. j-continuous proj. mapping* $p : (M, \tau) \to (B, \sigma)$*, where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

(c) The coarsest such fuzzy topology got by p, in which the f. j *-open set of* (M, τ) *are the exactly the pre image of the fuzzy open set of* (B, σ)*, where* $j \in \{\delta, \theta, \alpha, P, S,, b, \beta\}$*.*

(d) The FWF j *-TS over* (B, σ) *is defined to be a FW set over* B *with FWF j-TS.*

(e) We consider the fuzzy topological product (written as FTP) $B \times T$*, for every FTS* T*, like a FWF j-TS's over b by the first proj., where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

Definition 2.12. *The FW-M* ϑ*:* $M \to N$ *where* (M, τ) *and* (N, Λ) *are FWF j-TS's over* (B, σ) *is said to be:*

(a) f. j-continuous if $b \in B$ *and for all point* $m \in M_b$*, the pre image of all fuzzy open set of* $\vartheta(m)$ *is a f. j-open set of* m*, where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

(b) f. *j-open if* $b \in B$ *and for all point* $m \in M_b$*, the image of all fuzzy open set of* m *is a f. j-open set of* $\vartheta(m)$*, where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

(c) f. j-closed if $b \in B$ *and for all point* $m \in M_b$*, the image of all fuzzy closed set of* m *is a f. j-closed set of* $\vartheta(m)$*, where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

If $\vartheta : M \to N$ *is a FW-M where* M *is a FW set and* (N, Λ) *is a FWF* j *-TS over* (B, σ)*. We can give* M *the induced fuzzy topology (see* [4]*), in the ordinary sense and this is necessarily a FWF-topology. We may refer to it, therefore, like the induced FWFtopology and note the next characterizations, where* $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*.*

Theorem 2.13. *Let $\vartheta : M \to N$ be a FW-M, where (N, Λ) a FWF j -TS over (B, σ) and M has the induced FWF-topology. Then for all FWF* j *-TS (K, ϱ) a FW-M $\phi : K \to (M, \tau)$ f. j-continuous iff the composition $\vartheta \circ \phi : K \to N$ is f. j-continuous, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . $(\Rightarrow)$ Suppose that ϕ is f. j-continuous. Let $\in K_b; b \in B$ and let v be fuzzy open set of $(\vartheta \circ \phi)(k) = n \in N_b$ in N. Since ϑ is f. continuous, then $\vartheta^{-1}(v)$ is fuzzy open set containing $\phi(k) = m \in M_b$ in M. Since ϕ is f. j-continuous, then $\phi^{-1}(\vartheta^{-1}(v))$ is a f. j-open set containing $k \in K_b$ in K and $\phi^{-1}(\vartheta^{-1}(v)) = (\vartheta \circ \phi)^{-1}(v)$ is a f. j-open set containing $k \in K_b$ in K, $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. $(\Leftarrow)$ Suppose that $\vartheta \circ \phi$ is f. j-continuous. Let $k \in K_b; b \in B$ and $\mathcal{U}$ be a fuzzy open set of $\phi(k) = m \in M_b$ in M. Since ϑ is fuzzy open then, $\vartheta(\mathcal{U})$ is a fuzzy open set containing $\vartheta(m) = \vartheta(\phi(k)) = n \in N_b$ in N. Since $\vartheta \circ \phi$ is f. j-continuous, then $(\vartheta \circ \phi)^{-1}(\vartheta(\mathcal{U})) = \phi^{-1}(\mathcal{U})$ is a f. j-open set containing $k \in K_b$ in K, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 2.14. *Let $\vartheta : M \to N$ be a FW-M where, (N, Λ) a FWFTS over (B, σ) and M has the induced FWF-topology. If for every FWF j-TS (K, ϱ) a subjective FW-M $\phi : K \to (M, \tau)$ is* f. *j-open iff the composition $\vartheta \circ \phi : K \to N$ is* f. *j-open, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . $(\Rightarrow)$ Suppose that ϕ is f. j-open. Let $\in K_b; b \in B$ and let $\mathcal{U}$ be fuzzy open set of k in K. Since ϕ is f. j-open, $\phi(\mathcal{U})$ is f. j-open set containing $\phi(k) = m \in M_b$ in M. Since ϑ is f. j-open, then $\vartheta(\phi(\mathcal{U}))$ is a f. j-open set containing $\vartheta(m) = \vartheta(\phi(q)) = (\vartheta \circ \phi)(q) = n \in N_b$ in N and $\vartheta(\phi(\mathcal{U})) = (\vartheta \circ \phi)(\mathcal{U})$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$ $(\Leftarrow)$ Suppose that $\vartheta \circ \phi$ is f. j -open. Let $k \in K_b; b \in B$ and $\mathcal{U}$ be a fuzzy open set of $k \in K_b$ in K. Since $\vartheta \circ \phi$ is f. j-open then, $(\vartheta \circ \phi)(U)$ is f. j-open set containing $(\vartheta \circ \phi(q)) = n \in N_b$ in N. Since M has the induced FWF-topology then $\vartheta^{-1}(\vartheta \circ\ \phi)(\mathcal{U}) = \phi(\mathcal{U})$ is f. j-open containing $\vartheta^{-1}(\vartheta \circ \phi)(q)) = \phi(q) = m \in M_b$ in M, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

Let us pass of general cases of Theorems (2.13) and (2.14) as follows: Similarly in case of families $\{\vartheta_r\}$ of FW-M's, where $\vartheta_r : M \to N_r$ with (N_r, Λ_r) FWF j-TS over B for every r. Specially, given a family $\{(M_r, \tau_r)\}$ of FWF j -TS over B, the FWF-topological product $\prod_n M_r$ is defined to be the FW-product with the FWFtopology generated by the family of proj's $\pi_r : \prod_n M_r \to M_r$. Then for every FWF j-TS (K, ϱ) over B a FW-M $\xi : K \to \prod_n M_r$ is f. j-continuous (resp. f. j-open). For example when $M_r = M$ for all index r we see that the diagonal $\Delta : M \to \prod_n M_r$ is f. j-continuous (resp. f. j-open) iff the composition $\pi_r \circ \Delta = id_M$ is f. j-continuous (resp. f. j-open), where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

Again if $\{(M_r, \tau_r)\}$ is a family of FWF j -TS's over B and $\phi : \coprod_n M_r \to M$ is a FW-M where (M, τ) a FWF-topology over B and $\coprod_n M_r$ is FWF-topological coproduct at the set-theoretic level with the ordinary coproduct fuzzy topology, also for every FWF-topology (M_r, τ_r) with the family of FW insertions $\Lambda_r : M_r \to \coprod_n M_r$ is f. jcontinuous (resp. f. j-open) iff the composition $\phi_r = \phi \circ \Lambda_r : M_r \to M$ is f. jcontinuous (resp. f. j-open). For example when $M_r = M$ for every index r we see that the codiagonal $\nabla : \coprod_n M_r \to M$ is f. j-continuous (resp. f. j-open), where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

3. Fibrewise j-closed and fibrewise j-open fuzzy topological spaces $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$

We present the ideas of fibrewise j-closed and fibrewise j-open FTS's fuzzy topological spaces over B, several property on the obtained concepts are studies, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

Definition 3.1. *The FWF* j *-TS* (M,τ) *over* (B,σ) *is said to be fibrewise* f. *j-closed (written as FWCF* j *-TS) if the proj., p is* f. *j-closed, where* $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.

For example, trivial FWF j *-TS with fuzzy compact fibre (see [4]) is FWCF j-TS, where* $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.

Theorem 3.2. *Let* $\vartheta : M \to N$ *be f. j-closed FW-M where* (M,τ) *and* (N,Λ) *are FWCF j-TS's over* (B,σ). *Then M is FWCF* j *-TS if N is FWCF* j *-TS, where* $j \in \{\delta,\theta,\ \alpha,P,S,b,\beta\}$.

Proof . Assume that $\vartheta : M \to N$ is closed FW-M and N is FWCF j-TS i.e. the proj. $p_N : N \to B$ is f. j-closed. To prove that m is FWCF j-TS i.e. the proj. $p_M : M \to B$ is f. j-closed. Now, let $m \in M_b; b \in B$, and let $\mathcal{F}$ be a fuzzy closed set of m. Since ϑ is f. jclosed so that $\vartheta(\mathcal{F})$ is f. j-closed set of $\vartheta(m) = n \in N_b$ in N. Since p_N is f. j-closed so $p_N(\vartheta(\mathcal{F}))$ f. j-closed set in B. But, $p_N(\vartheta(\mathcal{F})) = p_N \circ \vartheta(\mathcal{F}) = p_M(\mathcal{F})$ is f. j-closed set of B. Thus. p_M is f. j-closed and M is FWCF j-TS. where $j \in \{\delta,\theta,\alpha,P,S,b,B\}$. □

Theorem 3.3. *If* (M,τ) *is a FWF j-TS over* (B,σ). *Assume that* M_i *is FWCF j-TS for every member* M_i *of a finite covering of* M. *Then* M *is FWCF* j *-TS, where* $j \in \{\delta,\theta,\alpha\ ,\ P,S,b,\beta\}$.

Proof . Assume that M is a FWF j -TS over B, then the proj. $p_M : M \to B$ exist. To prove that p is f. j-closed. Since M_i is FWCF j -TS, then the proj. $p_{M_i} : M_i \to B$ is f. j-closed for every member M_i of a finite covering of M. Let $\mathcal{F} \subseteq M$ be a fuzzy closed set. Then $p(\mathcal{F}) = \bigcup p_i\,(M_i \cap \mathcal{F})$ which is finite union of f. j-closed sets and so p is f. j -closed, so that M is FWCF j-TS, where $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$. □

Theorem 3.4. *Let* (M,τ) *be a FWF* j *-TS over* (B,σ). *Then* (M,τ) *is a FWCF* j *-TS iff for every fibre* M_b *of* M *and every fuzzy open set* $\mathcal{U}$ *of* $M_b \subseteq M$, *there is a* f. j *-open set* $\mathcal{O}$ *of* b *where* $M_{\mathcal{O}} \subseteq \mathcal{U}$, *where* $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.

Proof . (⇒) Assume that M is FWF j -TS i.e. the proj. $p : M \to B$ is f.j -closed. Now, let $b \in B$ and $\mathcal{U}$ be fuzzy open set of M_b, so we have $M - \mathcal{U}$ is fuzzy closed set and $p(M - U)$ is f. j-closed set. Let $\mathcal{O} = B - p(M - \mathcal{U})$ is f.j -open set of b. Hence, $M_{\mathcal{O}} = p^{-1}(B - p(M - \mathcal{U}))$ which is a subset of $\mathcal{U}$. Thus $M_{\mathcal{O}} \subseteq \mathcal{U}$, where $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.
(⇐) Suppose that the assumption is hold, to show that M is FWCF j -TS. Let $\mathcal{F}$ be fuzzy closed set in M. Let $b \in B - p(\mathcal{F})$ and every fuzzy open set $\mathcal{U}$ of $M_b \subseteq M$. By assumption there is f. j-open set $\mathcal{O}$ of b such that $M_{\mathcal{O}} \subseteq \mathcal{U}$. It's easy to show that $\mathcal{O} \subseteq B - p(\mathcal{F})$. So that $B - p(\mathcal{F})$ is f. j-open set in B. Hence $p(\mathcal{F})$ is a fuzzy j -closed in B, p is f. j-closed and M is FWCF j -TS, where $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$. □

Definition 3.5. *The FWF* j *-TS* (M,τ) *over* (B,σ) *is said to be fibrewise* f. j *-open (written as FWOF j-TS) if the proj. p is* f. *j-open, where* $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.

For example, trivial FWF j-TS's are always FWOF j-TS, where $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.

Theorem 3.6. *Let* $\vartheta : M \to N$ *be a f. j-open FW-M where* $(M,\tau),(N,\Lambda)$ *are FWF j-TS over* (B,σ). *If N is FWOF* j *-TS, then M is FWOF j-TS, where* $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$.

Proof . Since N is FWOF j -TS, we have $p_N : N \to B$ is f.j -open. To prove that p_N is f. j-open, i.e. the proj. $p_M : M \to B$ is f. j-open. Let $m \in M_b; b \in B$, and let $\mathcal{U}$ be a fuzzy open set of $m, \vartheta(\mathcal{U})$ is f.j -open set of $\vartheta(m) = n \in N_b \subseteq N$ since ϑ is f. j-open. Also, since N is FWOF j -TS, then the proj. $p_N : N \to B$ is f. j-open and $p_N(\vartheta(\mathcal{U}))$ is f. j -open set in B, but $p_N(\vartheta(\mathcal{U})) = p_N \circ \vartheta(\mathcal{U}) = p_M(\mathcal{U})$. So that p_M is f.j -open and M is FWOF j-TS, where $j \in \{\delta,\theta,\alpha,P,S,b,\beta\}$. □

Theorem 3.7. *Let $\vartheta : M \to N$ be a FW-M, where $(M,\tau),(N,\Lambda)$ are FWF* j *-TS's over (B,σ). Assume that the product:*

$$id_M \times \vartheta : (M \times_B M, \tau \times \tau) \to (M \times_B N, \tau \times \sigma)$$

is f. j-open and M is FWOF j-TS. Then ϑ itself f. j *-open, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . Consider the following figure: □

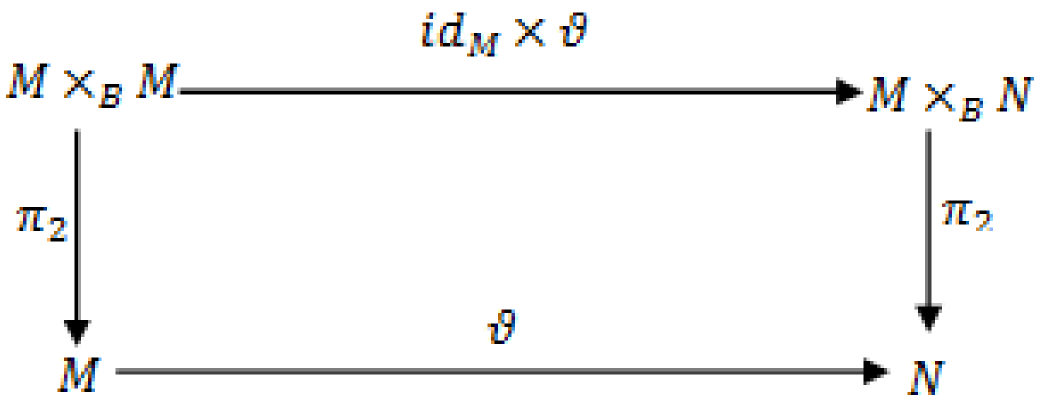

Figure 2: Diagraph of Theorem3.7.

The projection on the right is surjective. While the projection on the left is f. j-open since M is FWOF j -TS. Thus $\pi_2 \circ id_M \times \vartheta = \vartheta \circ \pi_2$ is f.j -open and so ϑ is f.j - open, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

Our next three results apply equally to FWCF j-TS's and FWOF j-TS's.

Theorem 3.8. *Let $\vartheta : M \to N$ be a surjection FW fuzzy continuous where $(M,\tau),(N,\Lambda)$ are FWF* j *-TS's over (B,σ). Then N is FWCF j-TS (resp. FWOF* j *-TS) if M is* FWCFj *TS (resp. FWOF j-TS), where $j \in \{\delta, \theta, \alpha, S, P, b, \beta\}$.*

Proof . Suppose that M is FWCF j-TS (resp. FWOF j -TS). Then the proj. $p_M : M \to B$ is f. j-closed (resp. f. j-open). To prove that N is FWCF j-TS (resp. FWOF j-TS) over B i.e. the proj. $p_N : (N,\Lambda) \to (B,\sigma)$ is f. j-closed (resp. f. j-open). Suppose that $n \in N_b$; $b \in B$. Let $\mathcal{V}$ be fuzzy closed (resp. fuzzy open) set of n. Since ϑ is fuzzy continuous so $\vartheta^{-1}(\mathcal{V})$ is fuzzy closed (resp. fuzzy open) set of $\vartheta^{-1}(n) = m \in M_b \subseteq M$. Since p_M is f. j-closed (resp. f. j-open), then $p_M\left(\vartheta^{-1}(\mathcal{V})\right)$ is f. j-closed (resp. f. j-open) in B, but $p_M\left(\vartheta^{-1}(\mathcal{V})\right) = p_M \circ \vartheta^{-1}(\mathcal{V}) = p_N(\mathcal{V})$. Thus p_N is f. j-closed (resp. f. j-open), and N is FWCF j-TS (resp. FWOF j-TS), where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 3.9. *If (M,τ) is a FWF* j *-TS over (B,σ). Assume that M is FWCF j-TS (resp. FWOF j-TS) over B. Then M_{B^*} is a FWCF j-TS (resp. FWOF j-TS) over B^* for every fuzzy subspace $B^* \subseteq B$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . Assume that M is FWCF j-TS (resp. FWOF j-TS), so that the proj. $p : M \to B$ is f. j-closed (resp. f. j-open). To prove that M_{B^*} is f. j-closed (resp. f. j-open), i.e. the proj. $p_{B^*} : M_{B^*} \to B^*$ is f. j-closed (resp. f. j-open). Let $m \in M|_{B^*}$, and $(\mathfrak{G}$ be fuzzy closed (resp. fuzzy open) set of m, we have $(5 \cap M_{B^*}$ is fuzzy closed (resp. fuzzy open) set of $M_{B^*} \cdot p_{B^*}\left(\mathfrak{G} \cap M_{B^*}\right) = p\left(\mathfrak{G} \cap M_{B^*}\right) = p(\mathfrak{G}) \cap p\left(M_{B^*}\right) = p(\mathfrak{G}) \cap B^*$ which is f. j-closed (resp. f. j-open) set in B^*. p_{B^*} is f. j-closed (resp. f. j-open). So that M_{B^*} is FWCF j -TS (resp. FWOF j-TS), where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 3.10. *Let (M, τ) be a FWF* j *-TS over (B, σ). Assume that (M_{B_i}, τ) is a FWCF j-TS's (resp. FWOF j-TS's) over (B_i, σ_{B_i}) for every member of a $\sigma_{B_i} -$* f.j *-open covering of B. So M is FWCF j-TS (resp. FWOF* j *-TS) over B, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . Assume that M is FWF j -TS over B so, the proj. $p : M \to B$ is exist .To prove that p is f. j-closed (resp. f. j-open). Since M_B is FWCF j-TS (resp. FWOF j -TS) over B_i for every member of a $\sigma_{B_i} -$ f. j -open covering of B, then the proj. $p_{M_i} : M_{B_i} \to B_i$ is f.j closed (resp. f. j-open). Now, let $\mathcal{F}$ be fuzzy closed (resp. fuzzy open) set of $M_b; b \in B$, $p(\mathcal{F}) = \cup p_{B_i}(\mathcal{F} \cap M_{B_i})$ which is finite union of f. j -closed (resp. f. j -open) sets of B. Thus p is f. j-closed (resp. fuzzy j - open and M is FWCF j-TS (resp. FWOF j-TS), where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.
Actually, the past Theorem is true to locally finite closed covering and there are several subclasses of the class of FWOF j-TS's which induced many important examples and have interesting properties, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

4. Fibrewise locally sliceable and fibrewise locally sectionable fuzzy jtopological spaces $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$

We present the ideas of fibrewise locally sliceable and fibrewise locally sectionable fuzzy j-topological spaces over (B, σ), several properties on the obtained concepts are studied, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$

Definition 4.1. *The FWF* j *-TS (M, τ) over (B, σ) is said to be locally sliceable (written as FW-LSL-F* j *-TS) if for all point $m \in M_b; b \in B$, there is a* f.j *-open set $\mathbb{W}$ of b and a section $s : \mathbb{W} \to M_{\mathbb{W}}$ and $\mathfrak{s}(b) = m$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

The condition lead to p is f. j *-open for if $\mathcal{U}$ is a fuzzy open set of $m \in M$, then $\mathfrak{s}^{-1}(M_w \cap \mathcal{U}) \subseteq p(\mathcal{U})$ is a* f. *j-open set of $b \in \mathbb{W}$ and hence in B. The class of B*FW− *LSL-F* j *-TS is finitely multiplicative stated in, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Theorem 4.2. *Let $\{(M_r, \tau_r)\}_{r=1}^{k}$ be a finite family of FW-LSL-F j-TS's over (B, σ). Then the FWF* j *-topological product $(M = \prod_B M_r, \tau)$ is FW-LSL-F* j *-TS, where $j \in \{\delta\ , \theta, \alpha, P, S, b, \beta\}$.*

Proof . Let $m = (m_r)$ be a point of $M_b; b \in B$, so that $m_r = \pi_r(m)$ for every index r. Since M_r is FW-LSL-F j-TS, there is a f. j-open set $\mathbb{W}_r$ of b and a section $\mathfrak{s}_r : \mathbb{W}_r \to M_r \mid \mathbb{W}_r$, where $\mathfrak{s}_r(b) = m_r$. Then the intersection $\mathbb{W} = \mathbb{W}_1 \cap \ldots \cap \mathbb{W}_n$ is a f. j -open set of b and a section $\mathfrak{s} : \mathbb{W} \to M_{\mathbb{W}}$ is given by $(\vartheta \circ \mathfrak{s})(w) = \mathfrak{s}_r(w)$ for every index r and every point $w \in \mathbb{W}$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 4.3. *Let $\vartheta : M \to N$ fuzzy continuous, surjection FW-M, where (M, τ) and (N, Λ) are FWF j-TS's over (B, σ). If M is FW-LSL-F j-TS, then N is so, where $j \in \{\ \delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . Let $n \in N_b; b \in B$. Then $n = \vartheta(m)$, for some $m \in M_b$. If M is FW-LSL-F j TS, then there is a f. j-open set $\mathbb{W}$ of b and a section $\mathfrak{s} : \mathbb{W} \to M_{\mathbb{W}}$ where $\mathfrak{s}(b) = m$. Then $\vartheta \circ \mathfrak{s} : \mathbb{W} \to M_{\mathbb{W}}$ is a section such that $\mathfrak{s}(b) = n$ as required, where $j \in \{\delta,, \theta, \alpha, P, S, b, \beta\}$. □

Definition 4.4. *The FWF* j *-TS (M, τ) over (B, σ) is said to be fibrewise discrete (written as FW-D-F j-TS) if the proj. p is a local fuzzy j-homeomorphism (i.e. f. j-continuous, f. j-open, one to one, onto), where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

This means, we recall, that for every point $b \in B$ and every point $m \in M_b$; $b \in B$ there is a f. j-open set $\mathcal{V}$ of m in M and a f. *j-open set $\mathbb{W}$ of b in B where p maps $\mathcal{V}$ fuzzy j-homeomorphically onto $\mathbb{W}$, in that case we say that $\mathbb{W}$ is evenly covered by $\mathcal{V}$. It is clear that FW-D-F j-TS's are FW-LSL-F j-TS there for FWOF j-TS, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

The class of FW-D-F j-TS's are finitely multiplicative, where $j \in \{\delta, \theta, \alpha, P, S, b\ \beta\}$.

Theorem 4.5. *Let $\{(M_r, \tau_r)\}_{r=1}^{k}$ be a finite family of FW-D-F* j *-TS over (B, σ). Then the FWF T-product $(M = \prod_B M_r, \tau)$ is FW-D-F j-TS, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . Given a point $m \in M_b; b \in B$, there is for every index r a fuzzy open set $\mathcal{U}_r$ of $\pi_r(m)$ in M_r, where the proj. $p_r = p \circ \pi_r^{-1}$ maps $\mathcal{U}_r$ fuzzy j-homeomorphically onto the f. j-open $p_r(\mathcal{U}_r) = \mathbb{W}_r$ of b. Then the fuzzy open $\prod_B \mathcal{U}_r$ of m is mapped fuzzy j homeomorphically onto the intersection $\mathbb{W} = \cap \mathbb{W}_r$ which is a f. j -open of b, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$

An attractive characterization of FW-D-F j-TS's are given by the following, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 4.6. *If (M, τ) is FWF j-TS over (B, σ). Then, M is FW-D-F j-TS iff*

(a) M is FWOF j *-TS*

(b) The diagonal embedding $\Delta : M \to M \times_B M$ m *is* f. *j-open, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . $(\Leftarrow)$ Suppose that (a) and (b) are satisfied. Let $m \in M_b; b \in B$, then $\Delta(m) = (m, m)$ admits a f. j-open set in $M \times_B M$ which is entirely contained in $\Delta(m)$. Without real lacking in general we may suppose the f. j-open set is of the form $\mathcal{U} \times_B \mathcal{U}$, where U is a fuzzy open set of m in M. Then $p \mid \mathcal{U}$ is a fuzzy j-homeomorphism. Therefore, M is FW-D-F j-TS, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.
$(\Rightarrow)$ Assume that M is FW-D-F j-TS. We have already seen that M is FWOF j-TS. To prove that Δ is f. j-open it is sufficient to prove that $\Delta(M)$ is f. j-open in $M \times_B M$. So let $m \in M_b; b \in B$, and let $\mathcal{U}$ be a fuzzy open set of m in M, where $\mathbb{W} = p(\mathcal{U})$ is a f. j open set of b in B and p maps $\mathcal{U}$ fuzzy j-homeomorphically onto $\mathbb{W}$. Then $\mathcal{U} \times_B \mathcal{U}$ is contained in $\Delta(M)$ since if not then there exist distinct $\lambda, \lambda^* \in M_{\mathrm{W}}$, where $\mathrm{w} \in \mathbb{W}$ and $\lambda, \lambda^* \in \mathcal{U}$, which is absurd, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

F. j-open subset of FW-D-F j-TS's are also FW-D-F j-TS, Actually, we have, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 4.7. *Let $\vartheta : M \to N$ be a f. j-continuous, injection FW-M, where (M, τ) and (N, Λ) are FWOF j-TS's over (B, σ). If N is FW-D-F j-TS, then M is so, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$*

Proof . Consider the diagram shown below. □

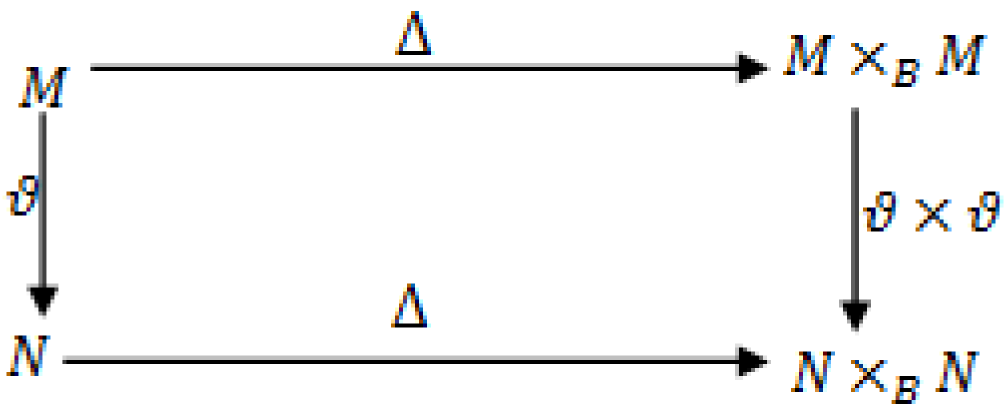

Figure 3: Diagraph of Theorem 4.7.

Theorem 4.8. *Assume that $\vartheta : M \to N$ is a f. j-open, surjection FW-M, where (M, τ) and* (N, Λ) *are FWOF j-TS's over (B, σ). If M is FW-D-F j-TS, then N is so, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . In the above figure, with these fresh hypotheses on ϑ, if M is FW-D-F j-TS, then $\Delta(M)$ is f. j-open in $M \times_B M$, by Theorem (4.6), so $\Delta(N) = \Delta(\vartheta(M)) = (\Delta\ \times_B \Delta)\,(\Delta(M))$ is f. j-open in $N \times_B N$. Thus, Theorem (4.8) follows from Theorem (4.6) again, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Theorem 4.9. *If $\vartheta, \phi : M \to N$ is a f. j-continuous FW-M, where (M, τ) is FWF* j *-TS and (N, Λ) is FW-D-F j-TS over (B, σ). Then the coincidence set $L(\vartheta, \phi)$ of ϑ and ϕ is f. j-open in M, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . The coincidence set is precisely $\Delta^{-1}(\vartheta \times \phi)^{-1}(\Delta(N))$, where: □

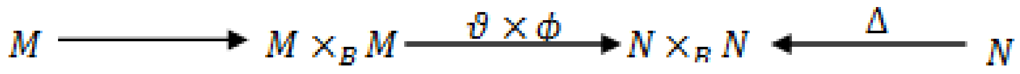

Figure 4: Diagram of Theorem 4.9.Fig. 4.2.

Hence Theorem (4.9) follows at once from Theorem (4.6). In particular take $M = N$, take $\vartheta = id_M$ and take $\phi = s \circ \mathcal{P}$ where s is a section, we conclude that s is a f. j open embedding when M is FW-D-F j-TS, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$

Theorem 4.10. *If $\vartheta : M \to N$ is a f. j-continuous FW-M, where (M, τ) is FWOF j-TS and (N, Λ) is* $\mathrm{FW} - \mathrm{D} - \mathrm{Fj}$ *-TS over (B, σ). Then the FW-graph: $\Gamma : M \to M \times_B N$ of ϑ is a f. j-open embedding, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . The FW-graph is defined in the same way as the ordinary graph, but with values in the FWF j-T-product, therefore the diagram shown below is commutative. □

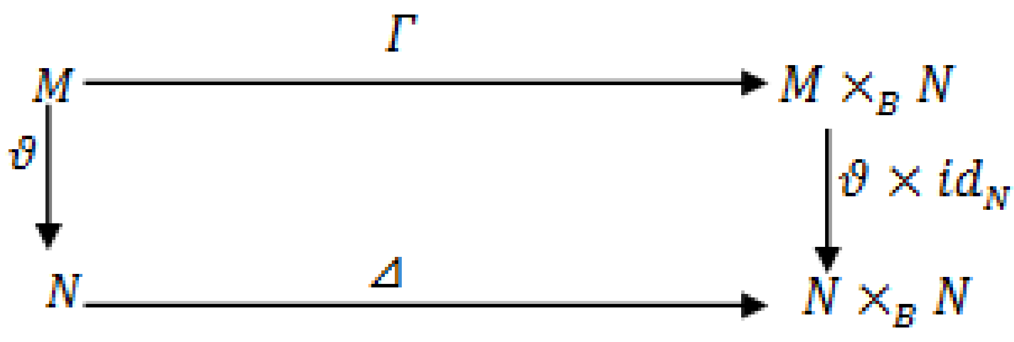

Figure 5: Diagram of Theorem 4.10.

Since $\Delta(N)$ is f. j-open in $N \times_B N$, by Theorem (4.6), so $\Gamma(M) = (\vartheta \times id_N)^{-1}\,(\Delta(N))$ is f. j-open in $M \times_B N$ as asserted, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.

Remark 4.11. *If (M, τ) is FW-D-F* j *-TS over (B, σ) then for every point $m \in M_b; b \in B$, there is a* f. *j-open set w of b a unique section $\mathfrak{s} : \mathbb{W} \to M_{\mathbb{W}}$ exist satisfying $\mathfrak{s}(b) = m$, we may refer to* s *as the section through m, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Definition 4.12. *The FWF j-TS (M, τ) over (B, σ) is said to be locally sectionable (written as FW-LSE-F j-TS) if for every point $b \in B$, admits a f. j-open set $\mathbb{W}$ and a section $\mathfrak{s} : \mathbb{W} \to M_{\mathbb{W}}$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Remark 4.13. *The non-empty FW-LSL-F j-TS's are FW-LSE-F j-TS's, but the converse is false. In fact, FW-LSE-F j-TS's are not necessarily FWOF j-TS, for example take $M = (-1, 1] \subseteq \mathbb{R}$ with (M, τ), the natural projection onto $B = \mathbb{R} \mid \mathbb{Z}; (B, \sigma)$, where $j \in \{\delta, \theta\ , \alpha, P, S, b, \beta\}$.*

The class of FW-LSE-F j-TS's is finitely multiplicative, where $j \in \{\delta, \theta\ , \alpha, P, S, b, \beta\}$

Theorem 4.14. *If $\{(M_r, \tau_r)\}_{r=1}^{k}$ is a finite family of FW-LSE-F j-TS's over (B, σ). Then the FWFT-product $(M = \prod_B M_r, \tau)$ is FW-LSE-F j-TS, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Proof . Given a point b of B, there exist f. j-open set $\mathbb{W}_r$ of b and a section $\mathfrak{s}_r : \mathbb{W}_r \to M_r \mid \mathbb{W}_r$ for every index r. Since there are finite number of indices the intersection $\mathbb{W}$ of the f. j -open sets $\mathbb{W}_r$ is also a f. j-open set of b, and a section $\mathfrak{s} : \mathbb{W} \to (\prod_n M_r)_{\mathbb{W}}$ is given by $(\pi_r \circ \mathfrak{s})(w) = \mathfrak{s}_r(w)$, for $w \in \mathbb{W}$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$. □

Our last two result apply equally well to every of the above three Theorems.

Theorem 4.15. *If (M, τ) is a FW-D-F j -TS over (B, σ). Suppose that (M, τ) is FW-LSLF j-TS, FW-D-F j-TS or FW-LSE-F j-TS's over (B, σ). Then so is M_{B^*} over B^* for every f. j-open set $B^* \subseteq B$, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Theorem 4.16. *Let (M, τ) be FWF* j *-TS over (B, σ). Assume that M_{B_i} is FW-LSL-F* j *TS, FW-D-F j-TS or FW-LSE-F j-TS over B_i for every member B_i of a f. j-open covering of B. So is M over B, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

Remark 4.17. *It is not difficult to give example of different FW-D-F j-TS's on the same FW-set which are in equivalent, as FWF j-TS's. For this reason we must be careful not to say the FW-D-F j-TS, where $j \in \{\delta, \theta, \alpha, P, S, b, \beta\}$.*

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