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An approximate solution of integral equation using Bezier control points

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Abstract

The integral equations are computed numerically using a Bezier curve. We have written the linear Fredholm integral equation into a matrix formulation by using a Bezier curves as a piecewise polynomials of degree n and we use (n+1) unknown control points on unit interval to determine Bezier curve. two examples have been discussed in details.

Keywords: Bezier curve, Bernstien polynomials, Fredholm integral equation.
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1. Introduction

In many practical applications, the solution to an integral equation is very important, so many different approaches exist to solving these problems numerically [7, 11] are available. Integral equations are used in biology as well as in the modeling of physical processes [6] and engineering [1, 3]. Due to the piecewise polynomial being differentiable and integrable, the Bezier curve [10, 8] is defined over any finite interval. Now, polynomials are positive, and the sum is unity. In this paper we introduce an approximate approach to solve a second type linear fredholm integral equations by using Bezier curves method by convert an integral equation into algebraic equations involving unknown control points of Bezier curves . There are a multitude of books and papers that discuss Bezier curves and surface techniques. Used the Bezier curve fitting technique to built up approximations in [5, 9]. Zheng, et al. [12] proposed using the Bernstein-Bezier formula for solving differential equations numerically, and also in [2] approach has been used for solving singular-perturbed two-point boundary value problems.

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2. Bezier curve

Bezier curves are commonly used in computer graphic tools to produce curved objects that appear relatively smooth over a broad range of sizes (as opposed to polygonal lines, which will not scale nicely). Hermite interpolation can be obtained by making special cubic (whereas polygonal lines use linear interpolation). The Bezier curve is the representation that is most utilized in computer graphics and geometric modeling. Depends on a sets of control points $\{\omega_0 \ \ , \omega_1 \ \ , \ldots, \omega_n \ \ \}$ where n is given the order of a Bezier curve for example if $n = 1$ we obtain a Linear form given by:

$$\mathfrak{B}(\tau) = \omega_0 + t(\omega_1 - \omega_0) = (1-\tau)\omega_0 + \tau\,\omega_1, \qquad \tau \in [0,1]$$

and for $n = 2$, we get a quadratic Bezier described by

$$\mathfrak{B}(\tau) = (1-\tau)\left[(1-\tau)\omega_0 + \tau\omega_1\right] + \tau\left[(1-\tau)\omega_1 + \tau\omega_2\right], \qquad \tau \in [0,1]$$

After rearranging the previous equation, we get:

$$\mathfrak{B}(\tau) = (1-\tau)^2\omega_0 + 2(1-\tau)\tau\omega_2, \qquad \tau \in [0,1]$$

Then

$$\begin{aligned} \mathfrak{B}' &= 2(1-\tau)(\omega_1 - \omega_0) + 2\tau(\omega_2 - \omega_1) \\ \mathfrak{B}' &= 2\tau(\omega_2 - 2\omega_1 + \omega_0) - 2(\omega_1 - \omega_0) \end{aligned}$$

The Bezier curve's second derivative in terms of τ is :

$$\mathfrak{B}''(\tau) = 2(\omega_2 - 2\omega_1 + \omega_0)$$

Finally, for a cubic Bezier curve represented by four points $\omega_0, \omega_1, \omega_2$ and ω_3 the equation become:

$$\mathfrak{B}(\tau) = (1-\tau)^3\omega_0 - 3(1-\tau)^2\tau\omega_1 + 3(1-\tau)\tau^2\omega_2 + \omega^3\omega_3, \qquad \tau \in [0,1]$$

Generally, by expanding the analytic definition of the curve to include its Bernstian polynomial coefficients, the Bezier curve may be written in a matrix form with the polynomial power series,

$$\mathfrak{B}\left(\tau\right) = \sum_{i=0}^{n} \omega_i \mathfrak{B}_i^n\left(\tau\right) \qquad\qquad 0 \le \tau \le 1$$

3. Bernstien polynomials

The n^{th} degree Bernstien polynomials [1, 10] over the intervals [0,1] and [a,b] given by:

$$\begin{aligned} \mathfrak{B}_i^n\tau &= \binom{n}{i}\tau^i(1-\tau)^{n-i}, && 0 \le \tau \le 1, \quad i = 0, \ldots, n \\ \mathfrak{B}_i^n\tau &= \binom{n}{i}\frac{(\tau-a)^i(b-\tau)^{n-i}}{(b-a)^n}, && a \le \tau \le b \quad , i = 0, \ldots, n. \end{aligned}$$

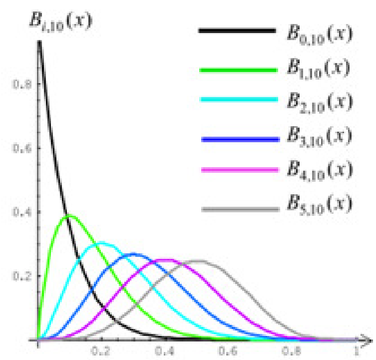
Each $(n+1)$ polynomial has degree n and satisfies a variety of properties:

1. $\mathfrak{B}_i^n(\tau) = 0 \qquad if\ i < 0\ \ or\ i > n$
2. $\sum_{i=0}^{n} \mathfrak{B}_i^n(\tau) = 1$
3. $\mathfrak{B}_i^n(a) = \mathfrak{B}_i^n(b) = 0 \qquad , 1 \leq i \leq n-1$

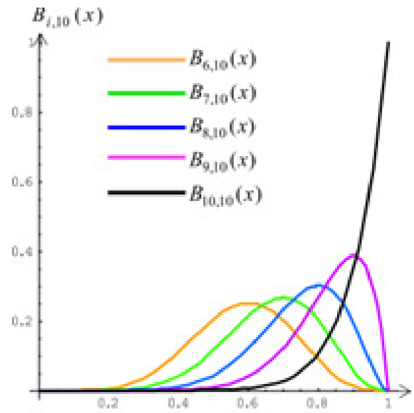
Using Mathematic code, over the interval [a,b] the first eleven Bernstien polynomials of degree ten are given:

$$\mathfrak{B}_0^{10} = \frac{(b-\tau)^{10}}{(b-a)^{10}}$$
$$\mathfrak{B}_1^{10} = \frac{10(b-\tau)^{9}(\tau-a)}{(b-a)^{10}}$$
$$\mathfrak{B}_2^{10} = \frac{45(b-\tau)^{8}(\tau-a)^2}{(b-a)^{10}}$$
$$\mathfrak{B}_3^{10} = \frac{120(b-\tau)^{7}(\tau-a)^3}{(b-a)^{10}}$$
$$\mathfrak{B}_4^{10} = \frac{210(b-\tau)^{6}(\tau-a)^4}{(b-a)^{10}}$$
$$\mathfrak{B}_5^{10} = \frac{252(b-\tau)^{5}(\tau-a)^5}{(b-a)^{10}}$$
$$\mathfrak{B}_{10}^{10} = \frac{(\tau-a)^{10}}{(b-a)^{10}}$$

Fig 1(a) shows ,the first six polynomials over [0,1], while Fig. 1(b) shows the remaining five polynomials.



(a)



(b)

Figure 1: Some tearm of Bernstein Polynomials over [0,1]

4. Solution of integral equation using Bezier curve

Take the case of a linear equation. The 2^{nd} kind [7, 11] Fredholm integral equation (FIE) is given by

$$\mathcal{F}(\tau) = g(\tau) + \int_a^b \mathrm{K}(\tau, y) f(y)\, dy \qquad a \le \tau \le b \tag{4.1}$$

where $g(\tau)$ is an continuous known function, $\mathrm{K}(\tau, y)$ is the known kernel which is a continuous function. A Bezier curve have been used to find the approximate solution for a linear Fredholm integral equations as follows:
Let,

$$\mathcal{F}(\tau) = \sum_{i=0}^{n} \omega_i \mathfrak{B}_i^n(\tau) \qquad , 0 \le \tau \le 1 \tag{4.2}$$

Substitute (4.2) in (4.1) we obtains

$$\sum_{i=0}^{n} \omega_i \mathfrak{B}_i^n(\tau) = \; g(\tau) + \int_a^b \mathrm{K}(\tau, y) \sum_{i=0}^{n} \omega_i \mathfrak{B}_i^n(\tau)\; dy$$

$$\omega_0 \mathfrak{B}_0^n + \cdots + \omega_n \mathfrak{B}_n^n = g(\tau) + \int_a^b \mathrm{K}(\tau, y) \left[\omega_0 \mathfrak{B}_0^n + \cdots + \omega_0 \mathfrak{B}_0^n\right] dy$$

$$\begin{bmatrix} \mathfrak{B}_0^n - \int_a^b \mathrm{K}(\tau, y) \mathfrak{B}_0^n(y)\, dy \\ \vdots \\ \mathfrak{B}_n^n - \int_a^b \mathrm{K}(\tau, y) \mathfrak{B}_n^n(y)\, dy \end{bmatrix} \begin{bmatrix} \omega_0 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} g(\tau_0) \\ \vdots \\ g(\tau_n) \end{bmatrix} \tag{4.3}$$

The left side of the integral of y from a to b.Thus, the outer integrand is transformed into a function of t only and for each $i = 0, 1, 2, \ldots, n$, we have $(n + 1)$ equation with unknown control points $\omega_i \quad (i = 0, 1, \ldots, n)$ is the system of $(n + 1)$ linear equation in $(n + 1)$ unknown given by:

$$\sum_{i=0}^{n} \omega_i \mathcal{C}_{ij} = \mathcal{F}_j \qquad , j = 0, 1, \ldots, n \tag{4.4}$$

where

$$\mathcal{C}_{ij} = \mathfrak{B}_i^n(\tau) - \int_a^b \mathrm{K}(\tau, y) \mathfrak{B}_i^n(y)\, dy$$

$$\mathcal{F}_j = g_j(\tau)$$

Now, we solve the system of equations (4.3) to find the unknown parameters (control points) ω_i, and then we substitute these values into (4.2) to obtain the approximate solution of $\mathcal{F}(\tau)$ of the integral equation (4.1).

5. Convergence of the method

Without loss of generality, we shall look at convergence of the control point based method on (4.1) with time intervals [a,b].

6. Numerical Examples

This section provides three examples for each example we find and explains how to approximate the solutions using a variety of different Bezier curves.

Example 6.1. *We'll look at the FIE of the Second Kind, where*

$$\mathrm{K}(\tau, y) = -\left(\tau y + \tau^2 y^2\right), \quad g(\tau) = 1 \ , a = -1 \ , b = 1$$

Being the ultimate solution, $\mathcal{F}(\tau) = 1 + \frac{10}{9}\chi^2$ *Using the formulation gauged in section 4, the equations* (4.3) *and* (4.4) *lead to the following results,*

$$\begin{bmatrix} \mathfrak{B}_0^n - \int_{-1}^{1} -(\tau y + \tau^2 y^2)\, \mathfrak{B}_0^n(y)\, dy \\ \vdots \\ \mathfrak{B}_n^n - \int_{-1}^{1} -(\tau y + \tau^2 y^2)\, B_n^n(y)\, dy \end{bmatrix} \begin{bmatrix} \omega_0 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Solving the above systems for $n = 3$*, the values of the control-points are:*

$$\omega_0 = \frac{19}{9} \quad , \omega_1 = \frac{17}{27}, \omega_2 = \frac{17}{27}, \omega_3 = \frac{19}{9}$$

Example 6.2. *We consider FIE status with regards to this second item where* $\mathrm{K}(\tau, y) = (y^4 - \tau^4)$, $g(\tau) = \tau$, $a = -1, b = 1$*, getting the exact solution* $\mathcal{F}(\tau) = \tau$ *and as above example 6.1, the scheme of equations becomes as*

$$\begin{bmatrix} \mathfrak{B}_0^n - \int_{-1}^{1} (y^4 - \tau^4)\, \mathfrak{B}_0^n(y)\, dy \\ \vdots \\ \mathfrak{B}_n^n - \int_{-1}^{1} (y^4 - \tau^4)\, \mathfrak{B}_n^n(y)\, dy \end{bmatrix} \begin{bmatrix} \omega_0 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} \tau \\ \vdots \\ \tau \end{bmatrix}$$

Once solving the pre-solved system for $n = 3$*, the values of the control-point* ω_i*,* $i = 0, 1, 2, 3$ *are*

$$\omega_0 = -1 \quad , \omega_1 = -0.333, \omega_2 = 0.333, \omega_3 = 1$$

Example 6.3. *The FIE of the 2nd Kind is one in which*

$$\mathrm{K}(\tau, y) = 2e^{\tau}e^{y} \qquad g(\tau) = e^{\tau} \qquad a = 0, b = 1$$

Having the exact solution $\mathcal{F}(\tau) = \frac{e^{\tau}}{2 - e^2}$
Equations (4.3) *and* (4.4)*, respectively, lead us using the formulation described in section 4,*

$$\begin{bmatrix} \mathfrak{B}_0^n - \displaystyle\int_0^1 2e^{\tau}e^{y}\mathfrak{B}_0^n(y)\, dy \\ \vdots \\ \mathfrak{B}_n^n - \displaystyle\int_0^1 2e^{\tau}e^{y}\mathfrak{B}_n^n(y)\, dy \end{bmatrix} \begin{bmatrix} \omega_0 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} e^{\tau} \\ \vdots \\ e^{\tau} \end{bmatrix}$$

Solving the above systems for $n = 4, 5, 6, 7$ *the exact and the approximate solution are, respectively:*

t	*Exact Solutions*	*Approximate resolution* $n = 4$	*Approximate resolution* $n = 5$	*Approximate resolution* $n = 5$	*Approximate resolution* $n = 7$
0	-0.1855612526	-0.185412	-0.18557103	-0.185561	-0.18556129
0.2	-0.2266450257	-0.226719	-0.2266434	-0.2266450	-0.2266450
0.4	-0.2768248595	-0.276786	-0.2768280	-0.2768249	-0.2768249
0.6	-0.3381146470	-0.338077	-0.338112	-0.3381146	-0.33811465
0.8	-0.4129741624	-0.4130518	-0.412976	-0.4129742	-0.4129742
1.0	-0.5044077810	-0.504213	-0.5043971	-0.5044077	-0.5044077

7. Conclusion

We have introduced an approximation solution of a second type Fredholm integral equations by represented the unknown function using the control point Bezier curve then we have solved a system of $(n+1)$ unknown coefficient . We have analyzed the formula and verified the steps with appropriate examples.

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