Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 799-809
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.19595.2089

Gronwall type inequalities: new fractional integral results with some applications on hybrid differential equations

Soumia Ferraoun[a], Zoubir Dahmani[a,*]

[a] *Laboratory LPAM, UMAB, University of Mostaganem, Algeria*

(Communicated by Abdolrahman Razani)

Abstract

In this paper, we are concerned with a class of fractional integral inequalities of Gronwall type. New integral results with some generalizations are proved. Then, some applications on hybrid differential equations with Hadamard derivative are established.

Keywords: Hybrid differential equations, Gronwall inequality, Hadamard derivative.
2010 MSC: 34A38, 26A33.

1. Introduction

Gronwall inequality, or Gronwall-Bellman inequality as it is named in different works, is an inequality that has been used in studying differential and integral equations. Some types of these inequalities have been used for showing that a small perturbation in the order of differential equations causes only a small perturbation in their solutions, some others have been used to prove the uniqueness of solutions of systems of differential equations. Also, we find applications of these type of inequalities in finding approximations to solutions of differential equations. For more information, we refer the reader to [4, 10, 14, 17].

Hybrid differential equations of fractional order have attracted the interest of many mathematicians, like for instance [2, 3, 8, 11, 18]. They are the result of some perturbation techniques applied to a certain type of nonlinear differential equation that was hard to solve as it is.

In this paper, we are concerned with a class of nonlinear inequalities of Gronwall-Bellman type. The proposed main results generalize some interesting results in the papers [5] and [15]. Our results

*Corresponding author
Email addresses: soumia.ferraoun@gmail.com (Soumia Ferraoun), zzdahmani@yahoo.fr (Zoubir Dahmani)

Received: January 2020 *Accepted:* March 2020

have some relationship with the good paper [16]. Also, our proposed integral results can be used as an effective tool to study linear and/or nonlinear fractional differential equations and fractional integro-differential equations. So in this sense, some applications on hybrid fractional differential equations are also discussed in this paper.

2. Preliminaries

The integral version of the famous Gronwall-Belman inequality states that if a function x for $t \in [t_0, T), T \leq \infty$ satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \tag{2.1}$$

for any continuous function h on $[t_0, T)$, and any positive function k on the same interval, then we have:

Lemma 2.1. *[6] If the above assumptions on (2.1) are valid, then, we have*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)exp\Big[\int_{s}^{t} k(u)du\Big]ds, \qquad t \in [t_0, T). \tag{2.2}$$

The following Jensen Lemma is needed in this work.

Lemma 2.2. *[15] Let $n \in \mathbb{N}$, and let $a_1, ..., a_n$ be nonnegative real numbers. Then, for $r > 1$,*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r \tag{2.3}$$

We are also concerned with the following interesting results:

Lemma 2.3. *[15] Let $I = [t_0, T) \in \mathbb{R}$, $k, b, p \in C(I, \mathbb{R}^+), T \leq \infty)$. Suppose that $u \in C(I, R^+)$, and*

$$u(t) \leq k(t) + \int_{t_0}^{t} b(s)u(s)ds + \int_{t_0}^{t} p(s)u^{\gamma}(s)ds, \ t \in I, \tag{2.4}$$

where $0 \leq \gamma < 1$. Then, for $t \in I$, we have

$$u(t) \leq \left[A^{1-\gamma}(t) + (1-\gamma)\int_{t_0}^{t} \exp\left((\gamma - 1)\int_{t_0}^{s} b(\sigma)d\sigma\right)p(s)ds\right]^{1/(1-\gamma)} \times \exp\left(\int_{t_0}^{t} b(s)ds\right), \tag{2.5}$$

where $A(t) = \max_{t_0 \leq s \leq t} k(t)$.

Theorem 2.4. *[5] Let $u, a, b, h_i, (i = 1, ..., n)$ be real valued nonnegative continuous functions and there exists a series of positive real numbers $p_1, p_2, ..., p_n$ and $u(t)$ that satisfies the integral inequality*

$$u^p(t) \leq a(t) + b(t)\int_{0}^{t}\sum_{i=1}^{i=n} h_i(s)u^{p_i}(s)ds, \tag{2.6}$$

for $t \in \mathbb{R}^+$.
Then

$$u(t) \leq \left\{a(t) + b(t)\int_{0}^{t}\sum_{i=1}^{n} h_i(s)\left(\frac{p_i}{p}a(s) + \frac{p - p_i}{p}\right) \times exp\left(\int_{s}^{t} b(\sigma)\sum_{i=1}^{n}\frac{p_i}{p}h_i(\sigma)\right)ds\right\}^{1/p} \tag{2.7}$$

for $p \geq p^* = \max p_i, i = 1, ..., n$.

3. Main Results

We propose the following main result that generalizes Theorem 4 of [15]. We have

Theorem 3.1. *Let $I=[t_0,T]$, $t_0\geq 1$, $\alpha>0, 0<\gamma<1$ and $a,b,p\in C(I,\mathbb{R}^+)$. If $u\in C(I,\mathbb{R}^+)$ and*

$$u(t) \leq a(t)+\int_{t_0}^{t}(log(\frac{t}{s}))^{\alpha-1}b(s)u(s)s^{-1}ds+\int_{t_0}^{t}(log(\frac{t}{s}))^{\alpha-1}p(s)u^{\gamma}(s)s^{-1}ds, \tag{3.1}$$

then, the following two cases are valid:

(i) *If $\alpha>1/2$, then*

$$\begin{aligned} u(t) \leq & \Big[A_1^{1-\gamma}(t)+(1-\gamma)G_1 \\ & \times\int_{t_0}^{t}\exp\left((\gamma-1)G_1\int_{t_0}^{s}b^2(\sigma)\sigma^{-1}d\sigma\right)p^2(s)s^{3\gamma-4)}ds\Big]^{\frac{1}{2(1-\gamma)}} \\ & \times t^{3/2}\exp\left((G_1/2)\int_{t_0}^{t}b^2(s)s^{-1}ds\right), \qquad t\in I, \end{aligned} \tag{3.2}$$

where $A_1(t)=\max_{t_0\leq s\leq t}3s^{-3}a^2(s)$, and $G_1=\Gamma(2\alpha-1)/9^{\alpha-1}$.

(ii) *Suppose that $\alpha\in(0,1/2]$, $q=(1+\alpha)/\alpha$, and $p=1+\alpha$. Then, we have*

$$\begin{aligned} u(t) \leq & \big[A_2^{1-\gamma}(t)+(1-\gamma)G_2 \\ & \times\int_{t_0}^{t}p^q(s)s^{q(\gamma(\frac{p+1}{p})-2)}\exp\left((\gamma-1)G_2\int_{t_0}^{s}b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right)ds\Big]^{\frac{1}{q(1-\gamma)}} \\ & \times t^{\frac{p+1}{p}}\exp\left((G_2/q)\int_{t_0}^{t}b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right), \end{aligned} \tag{3.3}$$

where $A_2(t)=\max_{t_0\leq s\leq t}3^{q-1}s^{-q(\frac{p+1}{p})}a^q(s)$, and $G_2=3^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{\frac{q}{p}}$.

Proof . Let $t\in I$. We have:

$$u(t) \leq a(t)+\int_{t_0}^{t}(log(\frac{t}{s}))^{\alpha-1}b(s)s^{-2}su(s)ds+\int_{t_0}^{t}(log(\frac{t}{s}))^{\alpha-1}p(s)s^{-2}su^{\gamma}(s)ds \tag{3.4}$$

(i) Applying Cauchy-Schwartz inequality, we get:

$$\begin{aligned} u(t) \leq & a(t)+\Big(\int_{t_0}^{t}(log(\frac{t}{s}))^{2(\alpha-1)}s^2ds\Big)^{1/2}\Big(\int_{t_0}^{t}b^2(s)s^{-4}u^2(s)ds\Big)^{1/2} \\ & +\Big(\int_{t_0}^{t}(log(\frac{t}{s}))^{2(\alpha-1)}s^2ds\Big)^{1/2}\Big(\int_{t_0}^{t}p^2(s)s^{-4}u^{2\gamma}(s))ds\Big)^{1/2} \\ \leq & a(t)+\Big(\frac{3t^3\Gamma(2\alpha-1)}{9^{\alpha}}\Big)^{1/2}\Big(\int_{t_0}^{t}b^2(s)s^{-4}u^2(s)ds\Big)^{1/2} \\ & +\Big(\frac{3t^3\Gamma(2\alpha-1)}{9^{\alpha}}\Big)^{1/2}\Big(\int_{t_0}^{t}p^2(s)s^{-4}u^{2\gamma}(s))ds\Big)^{1/2} \end{aligned} \tag{3.5}$$

where $\alpha > 1/2$.

Using Lemma 2.2 for $r = 2$, the above inequality becomes

$$\begin{aligned} u^2(t) &\leq 3a^2(t) + \left(\frac{t^3\Gamma(2\alpha-1)}{9^{\alpha-1}}\right)\left(\int_{t_0}^{t} b^2(s)s^{-4}u^2(s)ds\right) \\ &+ \left(\frac{t^3\Gamma(2\alpha-1)}{9^{\alpha-1}}\right)\left(\int_{t_0}^{t} p^2(s)s^{-4}u^{2\gamma}(s))ds\right) \end{aligned} \tag{3.6}$$

Let us introduce the function $w(t) := [u^2(t)t^{-3}]$. Then, (3.6) gives:

$$w(t) \leq A_1(t) + G_1\left(\int_{t_0}^{t} b^2 s^{-1}(s)w(s)ds\right) + G_1\left(\int_{t_0}^{t} p^2(s)s^{3\gamma-4}w^{\gamma}(s))ds\right) \tag{3.7}$$

Since $A_1(t)$ is nondecreasing, then by Lemma 2.3, it yields that:

$$\begin{aligned} w(t) &\leq \Big[A_1^{1-\gamma}(t) + (1-\gamma)G_1 \\ &\times \int_{t_0}^{t} p^2 s^{3\gamma-4}(s)\exp\left((\gamma-1)G_1\int_{t_0}^{s} b^2(\sigma)\sigma^{-1}d\sigma\right)ds\Big]^{1/(1-\gamma)} \\ &\times \exp\left(G_1\int_{t_0}^{t} b^2(s)s^{-1}ds\right), \end{aligned} \tag{3.8}$$

Replacing w by its quantity, we get (3.2).

(ii) Taking $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, and $p = 1+\alpha$, then we get $(1/p) + (1/q) = 1$. Thanks to Holder inequality, we obtain

$$\begin{aligned} u(t) &\leq a(t) + \left(\int_{t_0}^{t}\left(log\frac{t}{s}\right)^{p(\alpha-1)} s^p ds\right)^{\frac{1}{p}}\left(\int_{t_0}^{t} b^q(s)u^q(s)s^{-2q}ds\right)^{\frac{1}{q}} \\ &+ \left(\int_{t_0}^{t}\left(log\frac{t}{s}\right)^{p(\alpha-1)} s^p ds\right)^{\frac{1}{p}}\left(\int_{t_0}^{t} p^q(s)u^{q\gamma}(s)s^{-2q}ds\right)^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

Consequently,

$$\begin{aligned} u(t) &\leq a(t) + \left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{\frac{1}{p}}\left(\int_{t_0}^{t} b^q(s)u^q(s)s^{-2q}ds\right)^{\frac{1}{q}} \\ &+ \left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{\frac{1}{p}}\left(\int_{t_0}^{t} p^q(s)u^{q\gamma}(s)s^{-2q}ds\right)^{\frac{1}{q}} \end{aligned} \tag{3.10}$$

Using Lemma 2.2 with $r = q$, we can write

$$\begin{aligned} u^q(t) &\leq 3^{q-1}a^q(t) + 3^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{\frac{q}{p}} \\ &\times \left(\int_{t_0}^{t} b^q(s)u^q(s)s^{-2q}ds + \int_{t_0}^{t} p^q(s)u^{q\gamma}(s)s^{-2q}ds\right). \end{aligned} \tag{3.11}$$

Considering the function $w(t) := (u(t)t^{-(p+1)/p})^q$, it yields that

$$w(t) \leq A_2(t) + G_2 \int_{t_0}^{t} b^q(s) s^{-q\left(\frac{p-1}{p}\right)} w(s) ds + G_2 \int_{t_0}^{t} p^q(s) w^\gamma(s) s^{q(\gamma\left(\frac{p+1}{p}\right)-2)} ds. \tag{3.12}$$

Thanks to Lemma 2.3, we observe that

$$\begin{aligned} w(t) \leq & \Big[A_2^{1-\gamma}(t) + (1-\gamma)G_2 \\ & \times \int_{t_0}^{t} p^q(s) s^{q(\gamma\left(\frac{p+1}{p}\right)-2)} \exp\left((\gamma-1)G_2 \int_{t_0}^{t} b^q(\sigma)\sigma^{-q\left(\frac{p-1}{p}\right)} d\sigma\right) ds\Big]^{1/(1-\gamma)} \\ & \times \exp\left(G_2 \int_{t_0}^{t} b^q(\sigma)\sigma^{-q\left(\frac{p-1}{p}\right)} d\sigma\right). \end{aligned} \tag{3.13}$$

By replacing w with its value, we get the inequality (3.3). The proof is thus achieved. □

Example:

Let $t_0 = 1$, $T = e$, and $a(t) = \exp(t)$, $b(t) = \sqrt{3}t^{-2}$, $p(t) = t^{\frac{-3}{2}\gamma}$. It is obvious that a, b, and p are in $C(I, \mathbb{R}^+)$. So, for $\gamma = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we have:

$$u(t) \leq \exp(t) + \int_{1}^{t} (log(\frac{t}{s}))^{\frac{-1}{4}} \sqrt{3} s^{-2} u(s) s^{-1} ds + \int_{1}^{t} (log(\frac{t}{s}))^{\frac{-1}{4}} s^{\frac{-3}{4}} u^{\frac{1}{2}}(s) s^{-1} ds, \tag{3.14}$$

Since, $\alpha > 1/2$ and thanks to (3.2), we get:

$$\begin{aligned} u(t) \leq & \left[(3t^{-3}\exp(2t))^{1/2} + \frac{1}{3}\exp\left(-\frac{\sqrt{\pi}}{2\sqrt{3}}(t^{-3}-1)\right) - \frac{1}{3}\right]^{-1} \\ & \times t^{3/2} \exp\left(-\frac{\sqrt{\pi}}{2\sqrt{3}}(t^{-3}-1)\right). \end{aligned} \tag{3.15}$$

We propose this second main result that generalizes Theorem 2.3 of [5].

Theorem 3.2. *Let u, a, k_i real nonnegative functions defined on $t \in [t_0, T]$ where $t_0 \geq 1$, $\delta_i < 1$ for $i = 1, ..., n$. If*

$$u(t) \leq a(t) + \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} \sum_{i=1}^{i=n} k_i(s) u^{\delta_i}(s) s^{-1} ds, \tag{3.16}$$

then, we have the following results:

(i) *If $\alpha > 1/2$, then*

$$\begin{aligned} u(t) \leq & \Bigg\{2a^2(t) + \frac{6t^3}{9^\alpha}\Gamma(2\alpha-1)\int_{t_0}^{t}\sum_{i=1}^{n} nk_i^2(s)s^{-4}\left(\delta_i 2a^2(s) + 1 - \delta_i\right) \\ & \times exp\left(\int_{s}^{t}\frac{6\sigma^3}{9^\alpha}\Gamma(2\alpha-1)\sum_{i=1}^{n} n\delta_i k_i^2(\sigma)(\sigma)^{-4}d\sigma\right)ds\Bigg\}^{1/2} \end{aligned} \tag{3.17}$$

(ii) *Suppose that $\alpha \in (0,1/2]$, $q=(1+\alpha)/\alpha$, and $p=1+\alpha$. Then, we have*

$$\begin{aligned} u(t) \leq & \Bigg\{ 2^{q-1}a^q(t) + 2^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p} \\ & \times \int_{t_0}^{t}\sum_{i=1}^{n} n^{q-1}k_i^q(s)s^{-2q}\left(\delta_i 2^{q-1}a^q(s)+(1-\delta_i)\right) \\ & \times exp\left(\int_s^t 2^{q-1}\left(\frac{\sigma^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}\sum_{i=1}^{n} n^{q-1}\delta_i k_i^q(\sigma)\sigma^{-2q}\right)ds\Bigg\}^{1/q} \end{aligned} \tag{3.18}$$

Proof . For $t\in[t_0,T]$, we have

$$u(t) \leq a(t) + \int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} s\sum_{i=1}^{i=n}k_i(s)u^{\delta_i}(s)s^{-2}ds. \tag{3.19}$$

(i) Using Cauchy-Shwartz inequality and Lemma 2.2, we can write:

$$u(t) \leq a(t) + \left(\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{2(\alpha-1)} s^2 ds\right)^{1/2}\left(\int_{t_0}^{t}\sum_{i=1}^{i=n}nk_i^2(s)u^{2\delta_i}(s)s^{-4}ds\right)^{1/2}. \tag{3.20}$$

Which becomes

$$u(t) \leq a(t) + \left(\frac{3t^3}{9^\alpha}\Gamma(2\alpha-1)\right)^{1/2}\left(\int_{t_0}^{t}\sum_{i=1}^{i=n}nk_i^2(s)u^{2\delta_i}(s)s^{-4}ds\right)^{1/2} \tag{3.21}$$

where $\alpha > 1/2$.

Thanks to the inequality (3.21) and Lemma 2.2, we get:

$$u^2(t) \leq 2a^2(t) + \left(\frac{6t^3}{9^\alpha}\Gamma(2\alpha-1)\right)\left(\int_{t_0}^{t}\sum_{i=1}^{i=n}nk_i^2(s)s^{-4}u^{2\delta_i}(s)ds\right) \tag{3.22}$$

Now, if we put $\tilde{p}=2, \tilde{p}_i=2\delta_i, \tilde{h}_i(t)=nk_i^2(t)t^{-4}, \tilde{a}(t)=2a^2(t), \tilde{b}(t)=\frac{6t^3}{9^\alpha}\Gamma(2\alpha-1)$, the inequality becomes:

$$u^{\tilde{p}}(t) \leq \tilde{a}(t) + \tilde{b}(t)\left(\int_{t_0}^{t}\sum_{i=1}^{i=n}\tilde{h}_i(s)u^{\tilde{p}_i}(s)ds\right) \tag{3.23}$$

which, thanks to Theorem 2.4, gives

$$u(t) \leq \left\{\tilde{a}(t) + \tilde{b}(t)\int_0^t\sum_{i=1}^{n}\tilde{h}_i(s)\left(\delta_i\tilde{a}(s)+1-\delta_i\right) \times exp\left(\int_s^t\tilde{b}(\sigma)\sum_{i=1}^{n}\delta_i\tilde{h}_i(\sigma)d\sigma\right)ds\right\}^{1/2}, \tag{3.24}$$

from which we conclude (3.17).

(ii) Let $\alpha\in(0,1/2]$, $q=(1+\alpha)/\alpha$, and $p=1+\alpha$, then we get $(1/p)+(1/q)=1$.

Using Holder inequality and Lemma 2.2, the inequality (3.19) becomes

$$u(t) \le a(t) + \left(\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{p(\alpha-1)} s^p ds\right)^{1/p}\left(\int_{t_0}^{t}\sum_{i=1}^{i=n} n^{q-1}k_i^q(s)u^{q\delta_i}(s)s^{-2q}ds\right)^{1/q} \tag{3.25}$$

So we get

$$u(t) \le a(t) + \left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{1/p}\left(\int_{t_0}^{t}\sum_{i=1}^{i=n} n^{q-1}k_i^q(s)u^{q\delta_i}(s)s^{-2q}ds\right)^{1/q} \tag{3.26}$$

Thanks to (3.26) and using Lemma 2.2, we obtain

$$u^q(t) \le 2^{q-1}a^q(t) + 2^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}\left(\int_{t_0}^{t}\sum_{i=1}^{i=n} n^{q-1}k_i^q(s)u^{q\delta_i}(s)s^{-2q}ds\right) \tag{3.27}$$

If we put $\hat{p} = q$, $\hat{p}_i = q\delta_i$, $\hat{a}(t) = 2^{q-1}a^q(t)$, $\hat{b}(t) = 2^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}$, $\hat{h}_i(t) = n^{q-1}k_i^q(t)t^{-2q}$, then,

$$u^{\hat{p}}(t) \le \hat{a}(t) + \hat{b}(t)\left(\int_{t_0}^{t}\sum_{i=1}^{i=n}\hat{h}_i(s)u^{\hat{p}_i}(s)ds\right). \tag{3.28}$$

According to Theorem 2.4, we have

$$u(t) \le \left\{\hat{a}(t) + \hat{b}(t)\int_0^t\sum_{i=1}^{n}\hat{h}_i(s)\left(\delta_i\hat{a}(s)+(1-\delta_i)\right)\times exp\left(\int_s^t\hat{b}(\sigma)\sum_{i=1}^{n}\delta_i\hat{h}_i(\sigma)\right)ds\right\}^{1/q} \tag{3.29}$$

Therefore, we have (3.18) which completes the proof. □

4. Applications

In this section, we are concerned with the following hybrid differential problem:

$$\begin{cases} {}_HD^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) &= g(t,x(t)) + h(t)x(t), \quad 1 \le t \le T,\ 0 < \alpha \le 1, \\ {}_HI^{1-\alpha}x(t)|_{t=1} &= \eta, \end{cases} \tag{4.1}$$

where ${}_HD^{\alpha}$ is the Hadamard fractional derivative, $f \in C([1,T]\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $g \in C([1,T]\times\mathbb{R},\mathbb{R})$, $h \in C([1,T],\mathbb{R})$, ${}_HI^{1-\alpha}$ is the Hadamard fractional integral, and $\eta \in \mathbb{R}$.

It is to note that in the case where h is identically zero, the associated problem has been discussed by B. Ahmed et al., see [1].

The following definitions and lemmas are important for better understanding Hadamard fractional operators.

Definition 4.1. *[9] Let $t \in \mathbb{R}^+$, and $\Re(\alpha) > 0$. The Hadamard fractional integral of order α, applied to the function $y \in L^p[a,b]$, $1 \le p < +\infty$, $0 < a < b < \infty$, for $t \in [a,b]$, is defined as*

$$_HI^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t(log\frac{t}{\tau})^{\alpha-1}y(\tau)\frac{d\tau}{\tau}. \tag{4.2}$$

Definition 4.2. *[9] Let $\delta = t\frac{d}{dt}$, $\Re(\alpha) > 0$, and $n = [\alpha]+1$, where $[\alpha]$ is the integer part of α. The Hadamard fractional derivative of order α applied to the function $y \in AC^n_\delta[a,b]$, $0 \le a < b < \infty$, is defined as*

$$ {}_HD^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n \int_a^t (log\frac{t}{\tau})^{n-\alpha-1} y(\tau)\frac{d\tau}{\tau} = \delta^n({}_HI^{(n-\alpha)}y)(t) \tag{4.3}$$

where

$$AC^n_\delta[a,b] = \{y : t \in [a,b] \to \mathbb{R} \text{ such that } (\delta^{n-1}y) \in AC[a,b]\}. \tag{4.4}$$

Corollary 4.3. *[9] Let $\Re(\alpha) > 0$, $n = [\Re(\alpha)]+1$, and $0 < a < b < \infty$. The equality $({}_HD^\alpha y)(x) = 0$ is valid if, and only if,*

$$y(x) = \sum_{j=1}^n c_j (log\frac{x}{a})^{\alpha-j} \tag{4.5}$$

and the following formula holds:

$${}_HI^\alpha({}_HD^\alpha y(x)) = y(x) + \sum_{j=1}^n c_j(log\frac{x}{a})^{\alpha-j}$$

where $c_j \in \mathbb{R}$, $j = 1, 2, ..., n$, and $n-1 < \alpha < n$.

Thanks to [1], the integral equation that is equivalent to (4.1) is given by:

$$x(t) = f(t,x(t))\Big(\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}(g(s,x(s)) + h(s)x(s))\frac{ds}{s}\Big), \quad t \in [1,T] \tag{4.6}$$

Introducing the following two hypotheses,

($\mathcal{H}$.1) For $t \in [1,T]$, there exist a positive constant F, such that $|f(t,x(t))| \le F$.

($\mathcal{H}$.2) For $b(t), p(t) \in C([1,T], \mathbb{R}^+)$, $0 < \gamma < 1$, we have $|h(t)x(t) + g(t,x(t)| \le b(t)|x(t)| + p(t)|x(t)|^\gamma$,

we prove the following theorem.

Theorem 4.4. *Suppose that ($\mathcal{H}$.1) and ($\mathcal{H}$.2) are valid. If $x(t)$ is a solution of (4.1), then the following estimations hold:*

(i) Suppose that $\alpha > 1/2$. Then

$$\begin{aligned} |x(t)| \;\le\; & \Big[\tilde{A}_1^{1-\gamma}(t) + (1-\gamma)\frac{G_1F^2}{\Gamma^2(\alpha)} \\ & \times \int_1^t \exp\left((\gamma-1)\left(\frac{G_1F^2}{\Gamma^2(\alpha)}\right)\int_1^s b^2(\sigma)\sigma^{-1}d\sigma\right)p^2(s)s^{3\gamma-4}ds\Big]^{\frac{1}{2(1-\gamma)}} \\ & \times t^{3/2}\exp\left(\left(\frac{G_1F^2}{\Gamma^2(\alpha)}\right)\int_1^t b^2(s)s^{-1}ds\right), \qquad t \in I \end{aligned} \tag{4.7}$$

where $\tilde{A}_1(t) = \max_{1\le s\le t} 3s^{-3}\frac{\eta^2F^2}{\Gamma^2(\alpha)}(logs)^{2(\alpha-1)}$, and $G_1 = \Gamma(2\alpha-1)/9^{\alpha-1}$.

(ii) Suppose that $\alpha \in (0,1/2]$, $q=(1+\alpha)/\alpha$, and $p=1+\alpha$. Then

$$\begin{aligned} |x(t)| \leq & \Bigg[\tilde{A}_2^{1-\gamma}(t)+(1-\gamma)\frac{G_2F^q}{\Gamma^q(\alpha)} \\ & \times \int_1^t p^q(s)s^{q(\gamma(\frac{p+1}{p})-2)}\exp\left((\gamma-1)\left(\frac{G_2F^q}{\Gamma^q(\alpha)}\right)\int_1^s b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right)ds\Bigg]^{\frac{1}{q(1-\gamma)}} \\ & \times t^{\frac{p+1}{p}}\exp\left(\left(\frac{G_2F^q}{q\Gamma^q(\alpha)}\right)\int_1^t b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right) \end{aligned} \tag{4.8}$$

where $\tilde{A}_2(t)=\max\limits_{1\leq s\leq t}3^{q-1}s^{-q(\frac{p+1}{p})}\frac{|\eta|^qF^q}{\Gamma^q(\alpha)}(logs)^{q(\alpha-1)}$, *and* $G_2=3^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{q/p}$.

Proof .

For $t\in[1,T]$, we have:

$$\begin{aligned} |x(t)| \leq & |f(t,x(t)|\Big(|\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1}|+\frac{1}{\Gamma(\alpha)} \\ & \times\int_1^t(log\frac{t}{s})^{\alpha-1}|h(t)x(t)+g(s,x(s))|\frac{ds}{s}\Big) \end{aligned} \tag{4.9}$$

Thanks to hypothesis $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$, we get:

$$\begin{aligned} |x(t) \leq & F|\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1}|+\frac{F}{\Gamma(\alpha)} \\ & \times\int_1^t(log\frac{t}{s})^{\alpha-1}b(s)|x(s)|+p(t)|x(t)|^\gamma\frac{ds}{s} \end{aligned} \tag{4.10}$$

Using Theorem 3.1, we get the desired results. □

Now, let's consider the following equation:

$$\begin{cases} {}_HD^\alpha\left(\dfrac{x(t)}{f(t,x(t))}\right) = \sum\limits_{i=1}^{i=n} g_i(t,x(t)), \quad 1\leq t\leq T,\ 0<\alpha\leq 1, \\ {}_HI^{1-\alpha}x(t)|_{t=1} = \eta, \end{cases} \tag{4.11}$$

where ${}_HD^\alpha$ is the Hadamard fractional derivative, $f\in C([1,T]\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $g_i\in C([1,T]\times\mathbb{R},\mathbb{R})$ $(i=1,...,n)$, ${}_HI^{1-\alpha}$ is the Hadamard fractional integral, and $\eta\in\mathbb{R}$.

The equivalent integral representation of (4.11) can be given by:

$$x(t)=f(t,x(t))\Big(\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_1^t(log\frac{t}{s})^{\alpha-1}\sum_{i=1}^{i=n}g_i(s,x(s))\frac{ds}{s}\Big),\quad t\in[1,T]. \tag{4.12}$$

Introducing the following hypothesis,

$(\mathcal{H}.3)$ For $i=1,...n$, $k_i(t)\in C([1,T],\mathbb{R}^+)$, $0<\delta_i<1$, we have $|g_i(t,x(t)|\leq k_i(t)|x(t)|^{\delta_i}$, we present to the reader the following main result.

Theorem 4.5. *Suppose that $(\mathcal{H}.1)$ and $(\mathcal{H}.3)$ are satisfied.*
Then, the following two cases are valid:
(i) *If $\alpha > 1/2$, we have*

$$
\begin{aligned}
|x(t)| \leq & \left\{ 2\hat{A}^2(t) + \frac{6t^3F^2\Gamma(2\alpha-1)}{9^\alpha\Gamma^2(\alpha)} \int_1^t \sum_{i=1}^n nk_i^2(s)s^{-4}\left(\delta_i 2\hat{A}^2(s) + 1 - \delta_i\right) \right. \\
& \left. \times exp\left(\int_s^t \frac{6\sigma^3F^2\Gamma(2\alpha-1)}{9^\alpha\Gamma^2(\alpha)} \sum_{i=1}^n n\delta_i k_i^2(\sigma)(\sigma)^{-4}d\sigma \right) ds \right\}^{1/2},
\end{aligned} \tag{4.13}
$$

where, $\hat{A}(t) = \left(F\frac{|\eta|}{\Gamma(\alpha)}|(logt)|^{\alpha-1}\right)$.

(ii) *Suppose that $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, and $p = 1+\alpha$. Then, we have*

$$
\begin{aligned}
|x(t)| \leq & \left\{ 2^{q-1}\hat{A}^q(t) + 2^{q-1}\left(\frac{F}{\Gamma(\alpha)}\right)^q \left(\frac{t^{p+1}\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{q/p} \right. \\
& \times \int_1^t \sum_{i=1}^n n^{q-1}k_i^q(s)s^{-2q}\left(\delta_i 2^{q-1}\hat{A}^q(s) + (1-\delta_i)\right) \\
& \left. \times exp\left(\int_s^t 2^{q-1}\left(\frac{F}{\Gamma(\alpha)}\right)^q \left(\frac{\sigma^{p+1}\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{q/p} \sum_{i=1}^n n^{q-1}\delta_i k_i^q(\sigma)\sigma^{-2q} \right) ds \right\}^{1/q}.
\end{aligned} \tag{4.14}
$$

Proof . Let $t \in [1, T]$. Then,

$$
|x(t)| \leq |f(t, x(t))| \Big(\frac{|\eta|}{\Gamma(\alpha)}|(logt)|^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t (log\frac{t}{s})^{\alpha-1} \sum_{i=1}^{i=n} |g_i(s, x(s))| \frac{ds}{s} \Big) \tag{4.15}
$$

Thanks to $(\mathcal{H}.1)$ and $(\mathcal{H}.3)$, we can write:

$$
|x(t)| \leq \Big(F\frac{|\eta|}{\Gamma(\alpha)}|(logt)|^{\alpha-1} + \frac{F}{\Gamma(\alpha)} \int_1^t (log\frac{t}{s})^{\alpha-1} \sum_{i=1}^{i=n} |k_i(s)||x^{\delta_i}(s)| \frac{ds}{s} \Big) \tag{4.16}
$$

Thanks to Theorem 3.2, we achieve the proof of this theorem. □

5. Conclusion

In this paper, we have proved some integral results to certain types of inequalities of Gronwall-Bellman. Some results on the papers [5, 15] have been generalized to expand their usefulness so they can accommodate the cases with Hadamard derivative. We have also provided some hybrid fractional differential equations with Hadamard derivative to show the applicability of our integral results. We think that these kinds of results can be very useful in resolving fractional differential equations with the numerical approach and that is another kind of research to be pursued.

References

[1] B. Ahmed and S. K. Ntouyas, *Initial-value problems for hybrid Hadamard fractional differential equations*, Elect. J. Diff. Eq. 2014(161) (2014) 1–8.

[2] B. Ahmed, S. K. Ntouyas and A. Alsaedi, *Existence results for a system of coupled hybrid fractional differential equations*, Sci. World J. Article ID 426438 (2014) 6 pages.

[3] S. Ali Khan, K. Shah and R. Ali Kha, *On coupled system of nonlinear hybrid differential equation with arbitrary order*, Matrix Sci. Math. 1(2) (2017) 11–16.

[4] R. Almeida, *A Gronwall inequality for a general Caputo fractional operator*, arxiv: 1/ 05.100 / 9v1 [math. CA],(2017) 1–14.

[5] K. Boukerrioua, *Note on some nonlinear integral inequalities and applications to differential equations*, Int. J. Diff. Eq. Article ID 456216 (2011) 15 pages.

[6] C. Corduneanu, *Principles of Differential and Integral Equations*, Allyn & Bacon, Boston, 1971.

[7] B.C. Dhage, *Quadratic perturbations Of periodic boundary value problems of second order ordinary differential equations*, Diff. Equ. Appl. 2(4) (2010) 465–486.

[8] M. A. E. Herzallah and D. Baleanu , *On Fractional Order Hybrid Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V., 2006.

[10] S. Lin, *Generalized Gronwall inequalities and their applications to fractional differential equations*, J. Ineq. Appl. 549 (2013), 9 pages .

[11] H. Lu, S. Sun, D. Yang and H. Tengi, *Theory of fractional hybrid differential equations with linear perturbations of second type*, Boundary Value Prob. (2013) 16 pages.

[12] K. Nouri, M. Nazari and B. Keramati, *Existence results for a coupled system of fractional integro-differential equations with time-dependent delay*, J. Fixed Point Theory Appl. (2017) 17 pages.

[13] I. Petràš, *Fractional Derivatives, Fractional Integrals, and Fractional Differential Equations in Matlab*, Engineering Education and Research Using MATLAB, Technical University of Košice, Slovak Republic.

[14] D. Qian, Z. Gong and C. Li, *A generalized Gronwall inequality and its application to fractional differential equations with Hadamard derivatives*, http://nsc10.cankaya.edu.tr/proceedings/PAPERS/Symp2-Fractional%20 Calculus%20Applications/Paper26.pdf.

[15] J. Shao and F. Meng, *Gronwall-Bellman type inequalities and their applications to fractional differential equations*, Abst. Appl. Anal. J., Article ID 217641 (2013) 7 pages.

[16] P. Thiramanus, J. Tariboon and S. K Ntouyas, *Integral inequalities with 'maxima' and their applications to Hadamard type fractional differential equations*, J. Ineq. Appl. 398 (2014) 15 pages.

[17] H. Ye, J. Gao and Y. Ding, *A generalized Gronwall inequality and its applications to a fractional differential equation*, J. Math. Anal. Appl. 328 (2007) 1075–1081.

[18] Y. Zhaoa, S. Suna, Z. Hana and Q. Li, *Theory of fractional hybrid differential equations*, Comput. Math. Appl. 62 (2011) 1312–1324.