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An Inverse Triple Effect Domination in Graphs

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Abstract

In this paper, an inverse triple effect domination is introduced for any finite graph $G = (V, E)$ simple and undirected without isolated vertices. A subset D^{-1} of $V - D$ is an inverse triple effect dominating set if every $v \in D^{-1}$ dominates exactly three vertices of $V - D^{-1}$. The inverse triple effect domination number $\gamma_{te}^{-1}(G)$ is the minimum cardinality over all inverse triple effect dominating sets in G. Some results and properties on $\gamma_{te}^{-1}(G)$ are given and proved. Under any conditions the graph satisfies $\gamma_{te}(G) + \gamma_{te}^{-1}(G) = n$ is studied. Lower and upper bounds for the size of a graph that has $\gamma_{te}^{-1}(G)$ are putted in two cases when $D^{-1} = V - D$ and when $D^{-1} \neq V - D$. Which properties of a vertex to be belongs to D^{-1} or out of it are discussed. Then, $\gamma_{te}^{-1}(G)$ is evaluated and proved for several graphs.

Keywords: Dominating set; Triple effect domination; Inverse triple effect domination.
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1. Introduction

Domination is one of essential branches in graph theory. The study of domination models has wide area in researches because of its important and applications in large fields of sciences and life. For basic definitions and relations in graph theory see [14, 20, 21]. For a detailed survey of domination, one can see [15, 16, 17]. There are different papers studied the inverse domination such as [1-7]. Where the papers of different types of domination are too many, see [8, 9, 11, 12, 13, 18, 19, 22, 23]. In previous paper [10] we study a new model of domination called triple effect domination . Such that every vertex in the triple effect dominating set D dominates exactly three vertices. We put several theorems and properties of this model. Here, we study the inverse triple effect domination and discuss its bounds and properties, then we applied it on some graphs.

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2. Basic Properties

In this section, the definition of inverse triple effect domination is introduced and its properties are proved.

Definition 2.1. *Let $D \subseteq V(G)$ be a minimum triple effect dominating set in G. If $V - D$ contains triple effect dominating set, then it is called an inverse triple effect dominating set of G with respect to D and denoted by D^{-1}.*

Definition 2.2. *A subset $D^{-1} \subseteq V - D$ is said minimal inverse triple effect dominating set if there is no proper triple effect dominating subset in it. For example see Fig.* 1.

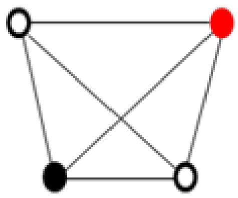

Figure 1: Minimum inverse triple effect dominating set.

Definition 2.3. *An inverse triple effect dominating set is said minimum if its cardinality is smallest over all inverse triple effect dominating sets in G.*

Definition 2.4. *The inverse triple effect domination number denoted by $\gamma_{te}^{-1}(G)$ is the cardinality of the minimum inverse triple effect dominating set. Such set is referred as $\gamma_{te}^{-1}-$set.*

Observation 2.5. *Let G be a graph having an inverse triple effect dominating set. Then:*

1. $\gamma_{te}^{-1}(G) \geq 1$
2. $\gamma_{te}^{-1}(G) \geq \gamma_{te}(G)$.

Remark 2.6. *For any graph G having order n and triple effect dominating set, if $\gamma_{te}(G) > \frac{n}{2}$, then G has no inverse triple effect domination.*

Proposition 2.7. *For any graph G with a minimum triple effect dominating set D, if $v \in V - D$ is dominated by more than three vertices from D, then $v \notin D^{-1}$.*

Proof . Let $v \in V - D$ be dominated by four vertices u_1, u_2, u_3 and u_4 from D. If $v \in D^{-1}$, then it will dominate u_1, u_2, u_3 and u_4, and this contradiction. Thus, $v \notin D^{-1}$. □

Observation 2.8. *For any minimum triple effect dominating set D in G. If every $v \in V - D$ is dominated by four vertices from D, then G has no inverse triple effect domination.*

Proposition 2.9. *For any minimum triple effect dominating set D in a graph G, if $v \in V -D$ is dominated by four or more vertices in D and it is an isolated vertex in $G[V - D]$, then G has no inverse triple effect domination.*

Proof . Since v is dominated by four vertices u_1, u_2, u_3 and u_4, then $v \notin D^{-1}$ from proposition (2.7). If G has D^{-1}, the four vertices $u_1, u_2, u_3, u_4 \notin D^{-1}$ and since v is not adjacent to any u in $V-D$, then v is not dominated by D^{-1}. Therefore, G has no inverse triple effect domination. □

Proposition 2.10. *For any graph G and any minimum triple effect dominating set D, if $G[V-D]$ is a null graph and there is a vertex $v \in V-D$ that is dominated by four vertices from D, then G without inverse triple effect dominating set.*

Proof . Similar to the proof of Proposition(2.9). □

Proposition 2.11. *For any graph G and any minimum triple effect dominating set D, if $G[V-D]$ is a null graph and there is a vertex $v \in V-D$ that is dominated by one or two vertices in D, then G has no inverse triple effect dominating set.*

Proof . If G has an inverse triple effect dominating set, then $D^{-1} = V-D$. Then, $v \in D^{-1}$ Which will dominates less than three vertices and this contradict our definition. Hence, G has no inverse triple effect dominating set. □

Proposition 2.12. *For any graph G and any minimum inverse triple effect dominating set D^{-1}, if v is an isolated vertex in $G[V-D]$ and it is dominated by three vertices, then $v \in D^{-1}$.*

Proof . If $v \notin D^{-1}$, then $\nexists u \in D^{-1}$ dominates v which is contradict the fact D^{-1} is inverse triple effect dominating set. □

Proposition 2.13. *For any graph G and any minimum triple effect dominating set D, if $\forall v \in V-D, v$ is dominated by three vertices in D, then G has an inverse triple effect dominating set.*

Proof . Since every $v \in V-D$ is dominated by exactly three vertices from D, then $V-D$ is dominates all vertices of D. Hence, $D^{-1} = V-D$. □

Theorem 2.14. *For any graph G with triple effect domination number $\gamma_{te}(G)$ and an inverse triple effect domination number $\gamma_{te}^{-1}(G)$. Then, $\gamma_{te}(G)+\gamma_{te}^{-1}(G) = n$ if one of the following conditions holds:*

1. $\gamma_{te}(G) = \frac{n}{2}$.
2. *$G[V-D]$ is a null graph and every v in $V-D$ is dominated by exactly three vertices from D.*

Proof .
1. Since $\gamma_{te}^{-1}(G) \geq \gamma_{te}(G)$ and $\gamma_{te}(G) = \frac{n}{2}$, then $\gamma_{te}^{-1}(G) = \gamma_{te}(G) = \frac{n}{2}$. Hence, $D^{-1} = V-D$.
2. By Proposition 2.12, $v \in D^{-1}$ for all $v \in V-D$, since G has an inverse triple effect dominating set, then every vertex in $V-D$ is dominates exactly three vertices from D and has no neighborhood in $V-D$. □

Theorem 2.15. *For any graph $G(n,m)$ with an inverse triple effect domination number $\gamma_{te}^{-1}(G)$, if $D^{-1} = V-D$, then*

$$3\gamma_{te}^{-1}(G) \leq m \leq \binom{n}{2} - (n-3)\gamma_{te}^{-1}(G) + \left(\gamma_{te}^{-1}(G)\right)^2$$

Proof . Since $D^{-1} = V - D$, then $V = D \cup D^{-1}$, then there are two cases :
Case1: Let $G[D]$ and $G\left[D^{-1}\right]$ are two null subgraphs, let G has a minimum number of edges, there exists three edges between every vertex in D^{-1} to D.Then, the number of edges between D and D^{-1} equals $3\left|D^{-1}\right| = 3\gamma_{ar}^{-1}(G)$. Thus, $m \geq 3\gamma_{ar}^{-1}(G)$.
Case 2: Suppose that $G[D]$ and $G\left[D^{-1}\right]$ are two complete subgraphs.
Let $m_1 = \frac{\left|D^{-1}\right|\left|D^{-1}-1\right|}{2}$ and $m_2 = \frac{\left|V-D^{-1}\right|\left|V-D^{-1}-1\right|}{2}$ represent the number of edges inside $G[D]$ and $G\left[D^{-1}\right]$ respectively. Where the number of edges between D and D^{-1} equals $m_3 = 3\gamma_{te}^{-1}(G)$. Thus, the number of edges in graph G equals:
$m = m_1 + m_2 + m_3 \leq \binom{n}{2} - (n-3)\gamma_{te}^{-1}(G) + \left(\gamma_{te}^{-1}(G)\right)^2$. □

Theorem 2.16. *For any graph $G(n,m)$ with an inverse triple effect domination number $\gamma_{te}^{-1}(G)$ and D^{-1} is a γ_{te}^{-1} -set. If $D^{-1} \neq V - D$, then :*

$$n - \gamma_{te} + 2\gamma_{te}^{-1} \leq m \leq \binom{\gamma_{te}}{2} + \binom{\gamma_{te}^{-1}}{2} + \binom{n - \gamma_{te} - \gamma_{te}^{-1}}{2} + 3\gamma_{te}^{-1} + 2\gamma_{te}$$

Proof . Since $D^{-1} \neq V - D$, then let $V - D = D^{-1} \cup W$ where $D^{-1} \cap W = \emptyset$, we get two cases:
Case 1: Suppose that $G[D], G\left[D^{-1}\right]$ and $G[W]$ are null graphs. Since D^{-1} is triple effect dominating set, then there are three edges from every vertex in D^{-1} to $D \cup W$ say (m_1), then $m_1 = 3\gamma_{te}^{-1}(G)$. Therefore, there are one edge at least from every vertex in W to D (say m_2).
Then, $m_2 = |W| = n - |D| - \left|D^{-1}\right| = n - \gamma_{te} - \gamma_{te}^{-1}$, where $G = D \cup D^{-1} \cup W$. Hence, $m \geq 3\gamma_{te}^{-1} + n - \gamma_{te} - \gamma_{te}^{-1} = n - \gamma_{te} + 2\gamma_{te}^{-1}$. Then, $m \geq n - \gamma_{te} + 2\gamma_{te}^{-1}$.
Case 2: Suppose that $G[D], G\left[D^{-1}\right]$ and $G[W]$ are complete subgraphs. Let m_1, m_2, m_3 be the number of edges of $G[D], G\left[D^{-1}\right]$ and $G[W]$ respectively which are equal to $\binom{\gamma_{te}}{2}$, $\binom{\gamma_{te}^{-1}}{2}$ and $\binom{n - \gamma_{te} - \gamma_{te}^{-1}}{2}$. As case one, the number of edges between D^{-1} and $D \cup W$ equals $m_4 = 3\gamma_{te}^{-1}(G)$.

So that, there is at most $2|D|$ edges from D to W where, there are two edges from every vertex in D to W (if there exists $v \in D, v$ dominates three vertices in W, then v is not dominated by D^{-1}, since it has no neighborhood in D^{-1}). Then, the number of edges between set D and set W say m_5 equals to $2|D| = 2\gamma_{te}$. Hence, $m = m_1 + m_2 + m_3 + m_4 + m_5$ which get the required identity. □

Theorem 2.17. *For any graph $G(n,m)$ and any minimum inverse triple effect dominating set D^{-1}. If $D^{-1} \neq V - D$ such that $H = V - D - D^{-1}$, then:*

1. *There is no vertex v in D or D^{-1} such that $|N(v) \cap H| = 3$.*
2. *For any $v \in D$, if v has one neighborhood in H, then it has another two neighborhoods in D^{-1}.*
3. *For any $u \in D^{-1}$, if u has one neighborhood in H, then it has another two neighborhoods in D.*
4. *$|H| \leq \lceil \frac{n}{2} \rceil$.*

Proof .
1. Let $v \in D$, then v dominates exactly three vertices from H. Therefore, v has no neighborhood in D^{-1} and v is not dominated by D^{-1}. Which is a contradiction since D^{-1} is a γ_{te}^{-1}−set. Hence, $N(v) \cap H \neq 3$. Similarly we can prove this case if $v \in D^{-1}$.
2. Since every vertex in D dominates three vertices in G, and it dominates one vertex in H, then it dominates two vertices in D^{-1}.
3. Similar to proof 2 above.
4. Since $|D| \geq \lceil \frac{n}{4} \rceil$ and $\left|D^{-1}\right| \geq \lceil \frac{n}{4} \rceil$, then $|H| \leq \lceil \frac{n}{2} \rceil$. □

3. Applications in Some Graphs

In this section, we study the triple effect domination on some graphs and evaluate their $\gamma_{te}^{-1}-$ set and $\gamma_{te}^{-1}(G)$. We show that there are some graphs haven't this type of domination. While, some of them will have.

Observation 3.1. *Any graph has no triple effect domination, then it has no inverse triple effect domination such that path graph P_n, cycle graph C_n, helm graph H_n and big helm graph $\mathcal{H}_n$.*

Proposition 3.2. *The complete graph $K_n\,(n \geq 4)$ has an inverse triple effect domination if and only if $n = 4, 5, 6$. Furthermore, $\gamma_{te}^{-1}\,(K_n) = \gamma_{te}\,(K_n) = n - 3$.*

Proof . It is clear when $n = 4, 5, 6$, then K_n has an inverse triple effect domination number equals to $n - 3$, by similar technique of [10, Proposition 3.2]. For example see Fig 2 .
But if $n \geq 7$, then K_n has no inverse triple effect domination according to Remark 2.6, since $\gamma_{te}\,(K_n) > \frac{n}{2}$. □

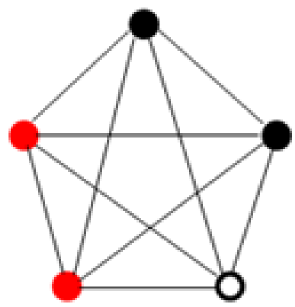

Figure 2: Inverse triple effect dominating set in k_5.

Theorem 3.3. *For any integer $n \geq 3$, the wheel graph W_n has an inverse triple effect domination, such that $\gamma_{te}^{-1}\,(W_n) = \gamma_{te}\,(W_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof . let us table the vertices of W_n as: $v_1, v_2, \ldots, v_{n+1}$. To choose a set D^{-1}, with respect to D that chosen in [10, Theorem 3.4]. The following two cases are obtained according to n :
Case 1. If $n \equiv 0, 2(\text{mod}3)$, let D^{-1} contains one vertex from every three consecutive vertices of C_n, where $D^{-1} = \left\{v_{3i-1}, i = 1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil\right\}$ is dominating set. Every vertex in D dominates three vertices, v_{n+1} and another two vertices adjacent with it, except when $n \equiv 2(\text{mod}3)$, there are two vertices v_2 and v_n of D^{-1} dominate v_1, v_{n+1} and another vertex. Therefore, D^{-1} is $\gamma_{te}^{-1}-$ set and $\gamma_{te}^{-1} = |D^{-1}| = \left\lceil \frac{n}{3} \right\rceil$.
Case 2 . If $n \equiv 1(\text{mod}3)$, then we can take $D^{-1} = \left\{v_{3i-1}, i = 1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil - 1\right\} \cup \{v_n\}$.
Hence, D^{-1} is a $\gamma_{te}^{-1}-$ set and $\gamma_{te}^{-1} = |D^{-1}| = \left\lceil \frac{n}{3} \right\rceil$. For example see Fig 3.
Now, to prove that D^{-1} is minimum, let M is an inverse triple effect dominating set in G,such that $|M| < |D^{-1}|$, then there exist at least one vertex in $V - D$ don't dominated by any vertex of M. Hence, M is not inverse triple effect dominating set and D^{-1} is minimum inverse triple effect dominating set. □

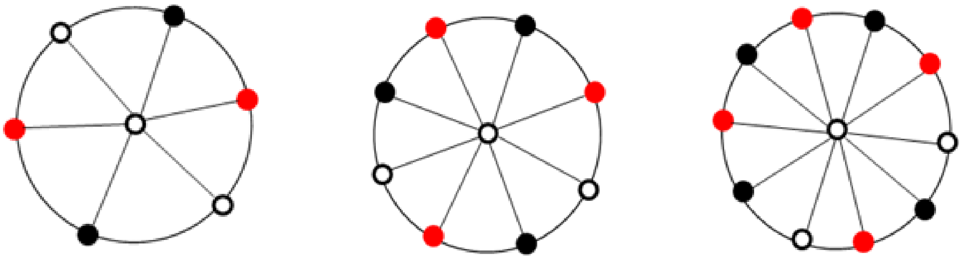

Figure 3: Inverse triple effect dominating set in W_n

Theorem 3.4. *For a complete bipartite graph $K_{n,m}$, we have* $\gamma_{te}^{-1}(K_{n,m}) = \begin{cases} 3 & \text{if } m = n = 3 \\ n+m-6 & \text{if } n \wedge m > 3 \end{cases}$

Proof . Let V_1 and V_2 be the two sets of $K_{n,m}$ vertices, where $|V_1| = n$ and $|V_2| = m$.
Case 1. It is clear $D^{-1} = V_1$ or $D^{-1} = V_2$.
Case 2. If $n, m > 3$, then D^{-1} must be contains $n-3$ vertices of V_1 and $m-3$ vertices of V_2 where all the $n-3$ vertices will dominate the three vertices of V_2. Also, all $m-3$ vertices of V_2 will dominate the three vertices of V_1. Hence, $\gamma_{te}^{-1}(K_{n,m}) = n+ m-6$. For example see Fig 4. □

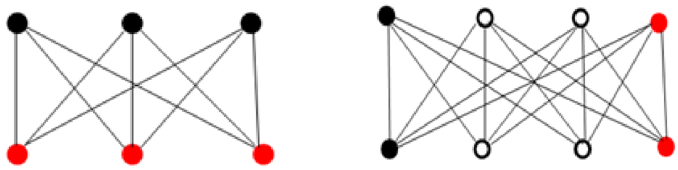

Figure 4: Inverse triple effect dominating set in $K_{n,m}$

Proposition 3.5. *The tadpole graph $T_{m,n}$ has no inverse triple effect domination.*

Proof . Since every vertex $v \in V - D$ can't dominates three vertices in G. □

Proposition 3.6. *Let G be the barbell graph $B_{n,n} (n \geq 4)$, then :* $\gamma_{te}^{-1}(B_{n,n}) = 2n-6$ *if and only if* $n = 4, 5, 6$.

Proof . Since $B_{n,n}$ have two copies of K_n graph joined by a bridge, and since $\gamma_{te}^{-1}(K_n) = n-3$ if and only if $n = 4, 5, 6$ according to Proposition 3.2. Then, D^{-1} of $B_{n,n}$ contains all vertices of D^{-1} in the two copies of K_n, where the bridge must be lies in two vertices belong to D^{-1} or to $V - D^{-1}$ together. For example see Fig 5 . □

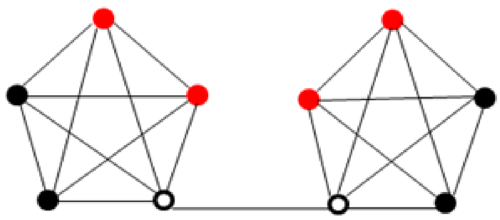

Figure 5: Inverse triple effect dominating set in $B_{5,5}$

4. Conclusion

In this paper, we introduce the inverse type of the triple effect domination model. Several bounds and properties are proved. So we apply this type of domination on more graphs to evaluate their inverse triple effect domination numbers.

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