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Bayes estimators of a multivariate generalized hyperbolic partial regression model

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Abstract

The matrix-variate generalized hyperbolic distribution belongs to the family of heavy-tailed mixed probability distributions and is considered to be one of the continuous skewed probability distributions. This distribution has wide applications in the field of economics, especially in stock modeling.

This paper includes estimation the parameters of the multivariate semi-parametric regression model represented by the multivariate partial linear regression model when the random error follows the matrix-variate generalized hyperbolic distribution, using the Bayesian method when non-informative prior information is available and under the assumption that the shape parameters and the skewness matrix are known. In addition, the bandwidth parameter is estimated by a suggested way based on the normal distribution rule and the proposed kernel function based on the mixed Gaussian kernel function and studying the findings on the generated data in a way suggested for the model, comparing the estimators depending on the criterion of the mean sum of squares error. The two researchers concluded that the proposed kernel function is better than the Gaussian kernel function in estimate the parameters.

Keywords: matrix-variate generalized hyperbolic distribution, multivariate partial regression model, kernel functions, bandwidth parameter, Bayes method.

1. Introduction

The semi-parametric regression model has gained wide popularity recently due to its advantage in integrating parametric regression models with nonparametric regression models simultaneously.

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Moreover, most economic models are semi-parametric models. AL-Mouel & Mohaisen (2017) studied Bayesian estimation based on the MCMC algorithm for the normal multiple semi-parametric regression model with the conditional ratio between the variance of the parametric part on the variance of the model error, and the variance of the nonparametric part on the variance of the model error. The weights matrix was the penalized spline in addition to the statistical laboratory formation based on the criterion Bayes factor and the application of the findings to experimental data, two nonparametric functions, and for different samples [1]. However, in fact, there are cases in which the random error limit of the studied models has tails, which are heavier than the normal distribution. In such cases, it is important to pay attention for mixed distributions, and from these distributions is the matrix-variate generalized hyperbolic distribution. Thabane & Haq (2004) generalization the multivariate generalized hyperbolic distribution to the matrix-variate generalized hyperbolic distribution as a mixed distribution resulting from the distribution of the matrix normal variance-mean mixture with the matrix generalized inverse Gaussian distribution. They also studied some of its properties and special cases, as well as studying the Bayesian estimation of the multivariate normal linear regression model assuming that the prior distribution of the scale matrix is a matrix generalized inverse Gaussian distribution [16].

The second section deals with the description of the multivariate generalized hyperbolic partial regression model as well as the estimation of the kernel function and the bandwidth parameter by the methods suggested in the third section. In the fourth section, it included finding the posterior probability distributions of the model parameters based on non-informative prior information and Bayes estimators. The fifth section included an experimental study of what was included in the fourth section using the language of Matlab. The sixth section presents the most prominent theoretical and experimental conclusions of the study. The last section presented the most important recommendations.

2. Description of a multivariate generalized hyperbolic partial linear regression model

It is known that the multivariate partial linear regression model is described according to the following equation: [12, 7]

$$Y_{im} = X_i'\beta_m + g_m(T_i) + \varepsilon_{im} i = 1, 2, \dots, n, m = 1, 2, \dots, k. \tag{2.1}$$

Here $X_i'\beta_m$ represents the parametric part of the model, and $g_m(T_i)$ is the nonparametric part of the model, which is an unknown smoothing function .The model can be written in equation (2.1) in the form of matrices as follows: [1]

$$Y = X\beta + W\gamma + \varepsilon \dots \tag{2.2}$$

Whereas, Y is a matrix of response variables of degree $n \times k$, n represents the number of observations, k represents the number of response variables, and X is a non-random matrix that represents observations of parametric explanatory variables of degree $(n \times p + 1)$. The p represents the number of parametric explanatory variables, and the ? represents the parameters matrix of the parametric part of the degree $(p + 1 \times k)$. As for W, it is the design matrix which indicates the kernel weighted of the degree $(n \times s)$. The s represents the number of nonparametric explanatory variables, ? the matrix of parameters of the nonparametric part (Added parameters) of the degree $(s \times k)$, and ε is the matrix of random errors of degree $(n \times k)$. The model can be rewritten in equation (2) as follows: [1]

$$Y_{n\times k} = C_{n\times(p+s+1)}\theta_{(p+s+1)\times k} + \varepsilon_{n\times k} \tag{2.3}$$

As:

$$C = [XW], \qquad \theta = [\beta\gamma]^T$$

It is assumed that the matrix of random errors (ε) follows the matrix-variate generalized hyperbolic distribution. The probability density function of the matrixε can be found using the concept of mixed distributions from the matrix normal variance-mean mixture distribution with the generalized inverse Gaussian distribution as follows: [3, 16]

$$\varepsilon|Z \sim MN_{n,k}(\delta Z, Z\Sigma, I_n), \qquad Z \sim GIG(\lambda, \psi, \nu)$$

As the probability density function of the matrix of random errors conditional by Z takes the following formula:

$$f(\varepsilon Z) = \frac{1}{(2\pi Z)^{\frac{nk}{2}} |\Sigma|^{\frac{n}{2}}} e^{-\frac{1}{2Z} tr(\varepsilon - \delta Z)^T (\varepsilon - \delta Z)\Sigma^{-1}}. \tag{2.4}$$

The probability density function of the random variable Z is as follows: [9, 15]

$$P(Z) = \frac{(\frac{\lambda}{\psi})^{\frac{v}{2}}}{2K_v(\sqrt{\lambda\psi})} Z^{v-1} exp[-\frac{1}{2}((\frac{\psi}{Z}) + \lambda Z)], Z > 0 \tag{2.5}$$

According to the concept of mixed distributions, the probability distribution of the matrix of unconditional random errors by Z is as follows:

$$f(\varepsilon) = \int_0^\infty f(\varepsilon Z)P(Z)dZ$$

$$f(\varepsilon) = \frac{(\frac{\lambda}{\psi})^{\frac{nk}{4}} e^{tr(\varepsilon)^T\delta\Sigma^{-1}} K_{\frac{2v-nk}{2}} \left(\sqrt{\lambda\psi(1 + \frac{tr\varepsilon^T\varepsilon\Sigma^{-1}}{\psi})(1 + \frac{tr\delta^T\delta\Sigma^{-1}}{\lambda})}\right)}{(2\pi)^{\frac{nk}{2}} |\Sigma|^{\frac{n}{2}} K_v(\sqrt{\lambda\psi})} \star \tag{2.6}$$

$$(1 + \frac{tr\varepsilon^T\varepsilon\Sigma^{-1}}{\psi})^{\frac{2v-nk}{4}} (1 + \frac{tr\delta^T\delta\Sigma^{-1}}{\lambda})^{\frac{nk-2v}{4}}$$

As:

λ, ψ, v: represent shape parameters.

$K_v(.)$: represents the modified Bessel function of the third kind of order v which takes the following form: [7, 15]

$$K_v(x) = 0.5 \int_0^\infty t^{v-1} exp(-0.5x(t + t^{-1}))dtx > 0 \tag{2.7}$$

δ : skewness matrix of degree $(n \times k)$.

Equation (2.6) represents the matrix-variate generalized hyperbolic distribution for the random error matrix, which is described as follows:

$$\varepsilon \sim MGH_{n,k}(0, \Sigma, I_n, \lambda, \psi, v, \delta) \leftrightarrow vec(\varepsilon) \sim MGH_{nk}(vec(0), \Sigma \otimes I_n, \lambda, \psi, v, vec(\delta))$$

Since the Y response variables matrix in Equation (2.3) is a linear combination in terms of the random error matrix that follows the matrix-variate generalized hyperbolic distribution, the probability distribution of Y follows the matrix-variate generalized hyperbolic distribution as follows:

The probability density function of the matrix of response variables distribution conditional by Z (YZ)that follows the matrix normal variance-mean mixture distribution is as follows:

$$f(YZ) = \frac{1}{(2\pi Z)^{\frac{nk}{2}} |\Sigma|^{\frac{n}{2}}} e^{-\frac{1}{2Z} tr(Y - C\theta - \delta Z)^T (Y - C\theta - \delta Z)\Sigma^{-1}} \tag{2.8}$$

Depending on the concept of mixed distributions, the probability distribution of Y unconditional by Z is as follows:

$$f(Y) = \frac{(\frac{\lambda}{\psi})^{\frac{nk}{4}} e^{tr(Y-C\theta)^T \delta \Sigma^{-1}}}{(2\pi)^{\frac{nk}{2}} |\Sigma|^{\frac{n}{2}} K_v(\sqrt{\lambda\psi})} K_{\frac{2v-nk}{2}} \left(\sqrt{\lambda\psi(1+\frac{tr(Y-C\theta)^T(Y-C\theta)\Sigma^{-1}}{\psi})(1+\frac{tr\delta^T\delta\Sigma^{-1}}{\lambda})} \right) \star$$
$$\left(1+\frac{tr(Y-C\theta)^T(Y-C\theta)\Sigma^{-1}}{\psi}\right)^{\frac{2v-nk}{4}} \left(1+\frac{tr\delta^T\delta\Sigma^{-1}}{\lambda}\right)^{\frac{nk-2v}{4}} \quad (2.9)$$

This distribution can be expressed descriptively as follows:

$$Y \sim MGH_{(n,k)}(C\theta, \Sigma, I_n, \lambda, \psi, v, \delta) \leftrightarrow vec(Y) \sim MGH_{nk}(vec(C\theta), \Sigma \otimes I_n, \lambda, \psi, \nu, vec(\delta))$$

3. Kernel functions and Bandwidth parameter

The kernel functions are used in estimating the regression functions, the spectral functions, and the probability density functions. The kernel function is a real, symmetric, continuous, and definite function, and its integral is equal to one. The kernel function has other names, including (weight, shape, and window function) and the following table reviews some types of kernel functions: [8, 13]

Table 1: Some of the kernel functions

Ker	**ker(x)**	
Epanchnik	$(\frac{3}{4})(1-x^2)$	$I(\lvert x\rvert \leq 1)$
Quartic	$(\frac{15}{16})(1-x^2)^2$	$I(\lvert x\rvert \leq 1)$
Triweight	$(\frac{35}{32})(1-x^2)^3$	$I(\lvert x\rvert \leq 1)$
Gauss	$(2\pi)^{-0.5}\exp(-\frac{x^2}{2)})$	$I(\lvert x\rvert < \infty)$
Uniform	**0.5**	$I(\lvert x\rvert \leq 1)$

On the other hand, based on the mixed Gaussian kernel function, the researchers proposed a new kernel function as follows:
We assume the mixed Gaussian kernel function is described as follows:

$$(xZ) \sim N(0, Z)$$

That Z is a random variable which follows the generalized inverse Gaussian distribution defined in equation (2.5) and uses the mixed distributions. The proposed kernel function is as follows:

$$ker(x) = \int_0^\infty ker(xZ)P(Z)dZ$$
$$ker(x) = \frac{(\frac{\lambda}{\psi})^{0.25}}{\sqrt{2\pi}K_v(\sqrt{\lambda\psi})} K_{\frac{2v-1}{2}}(\sqrt{\lambda\psi(1+\frac{x^2}{\psi})})(1+\frac{x^2}{\psi})^{\frac{2v-1}{4}} \quad (3.1)$$

Equation (3.1) represents the proposed kernel function and can be called the symmetric generalized hyperbolic kernel function and is described as follows:

$$ker(x) \sim GH(0, 1, \lambda, \psi, v, 0)$$

For example, the kernel function used in the estimation is the Gaussian Kernel function, and depending on the rule of thumb (the rule of normal distribution). The bandwidth parameter at the second kernel degree is as follows: [4, 14]

$$h_{thumb} = 1.06\sigma\hat{n}^{-\frac{1}{5}} \tag{3.2}$$

Where h_{thumb} is a non-random, symmetric and positive parameter, the bandwidth parameter is usually chosen according to the researcher's experience or repetitive methods to obtain the best smoothing parameter (bandwidth parameter). This parameter greatly affects the variance and bias. Moreover, the researchers suggested counting on the rule of thumb and a Gaussian kernel function and assuming the normal probability density function is a mixed function, so the suggested bandwidth parameter is as follows:

$$h_{sug.} = 1.06\sigma\hat{n}^{-\frac{1}{5}} \left(\frac{K_v(\sqrt{\lambda\psi})(\frac{\lambda}{\psi})^{\frac{-5}{4}}}{K_{\frac{2v-5}{2}}(\sqrt{\lambda\psi})} \right)^{1/5} \tag{3.3}$$

Since σrepresents the standard deviation of the sample, it is possible to use any other kernel function according to the following rule: [14]

$$h^* = C_K h_{sug.} \tag{3.4}$$

As:

$$C_K = \left(\frac{2\sqrt{\pi}R(K)}{\mu_2^2(K)} \right)^{\frac{1}{5}} \tag{3.5}$$

And $h_{sug.}$represents the proposed bandwidth parameter based on the Gaussian kernel function, so any derived rule based on the Gaussian kernel function can be inverted by relying on other kernel functions by multiplying it by a multiplier. Therefore, the proposed bandwidth parameter and dependent on the multiplier of the proposed kernel function is as follows:

$$h^* = \left(\frac{K_v(\sqrt{\lambda\psi})K_{\frac{2v-1}{2}}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{-5}{4}}(K_{v+1}(\sqrt{\lambda\psi}))^2} \right)^{\frac{1}{5}} 1.06\sigma n^{-\frac{1}{5}} \left(\frac{K_v(\sqrt{\lambda\psi})(\frac{\lambda}{\psi})^{\frac{-5}{4}}}{K_{\frac{2v-5}{2}}(\sqrt{\lambda\psi})} \right)^{1/5} \tag{3.6}$$

4. Posterior Probability Distributions and Bayes Estimators

In this section, the parameters of the multivariate partial regression model are defined in equation (2.3) are estimated when non-informative prior information is available and under the assumption that the location matrix θ and the scale matrix Σ are unknown matrices and by using a quadratic loss function.

To find the joint prior probability distribution of θ, Σ conditional by a random variable Z we follow the following:

$$P(\theta, \Sigma|Z) \propto P(\theta|\Sigma, Z)P(\Sigma|Z) \tag{4.1}$$

Since $P(\Sigma|Z)$ represents the prior probability distribution of Σ, which is found through Fisher's information of the scale matrix Σ from the conditional likelihood function of Y and after deriving a second partial derivative relative to the scale matrix Σ, the prior probability distribution of Σ is as follows: [11]

$$P(\Sigma|Z) \propto |\Sigma|^{-\frac{k+1}{2}} \tag{4.2}$$

$$P((\theta, \Sigma|Z) \propto |\Sigma|^{-\frac{(p+s+1)}{2}}|\Sigma|^{-\frac{k+1}{2}} \tag{4.3}$$

$$P((\theta, \Sigma|Z) \propto |\Sigma|^{-\frac{p+s+k+2}{2}} \tag{4.4}$$

By combining the joint prior probability distribution of θ, Σ with the probability function of Y conditional by the random variable Z defined in equation (2.8), we obtain the kernel of the joint posterior probability distribution for θ, Σ conditional by Z as follows:

$$\begin{aligned} P(\theta, \Sigma|Y, Z) &\propto P(\theta, \Sigma|Z) f(Y\theta, \Sigma, Z) \\ &\propto |\Sigma|^{-\frac{n+p+s+k+2}{2}} e^{-\frac{1}{2Z} tr(Y - C\theta - \delta Z)^T (Y - C\theta - \delta Z)\Sigma^{-1}} \end{aligned} \tag{4.5}$$

By adding and subtracting the amount $C\theta_{m|Z}$ to the above exponential function and that $\theta_{m|Z}$ represents the conditional maximum likelihood estimator, which was found by deriving normal logarithm of equation (2.8) a partial derivation of θ as follows:

$$\hat{\theta}_{m|Z} = (C^T C)^{-1} C^T Y - (C^T C)^{-1} C^T \delta Z \tag{4.6}$$

And making some mathematical simplifications, we get the following:

$$\begin{aligned} P(\theta, \Sigma|Y, Z) &\propto |\Sigma|^{-\frac{n+k+1}{2}} |\Sigma|^{-\frac{(p+s+1)}{2}} \exp\left(-\frac{1}{2Z} tr B_8 \Sigma^{-1}\right) \\ &\star \exp\left(-\frac{1}{2Z} tr(\theta - \hat{\theta}_{m|Z})^T C^T C (\theta - \hat{\theta}_{m|Z})\Sigma^{-1}\right) \end{aligned} \tag{4.7}$$

As:

$$B_8 = (Y - C\hat{\theta}_{m|Z} - \delta Z)^T (Y - C\hat{\theta}_{m|Z} - \delta Z)$$

We notice from equation (4.7) that $[|\Sigma|^{-\frac{(p+s+1)}{2}} exp(-\frac{1}{2Z} tr(\theta - \hat{\theta}_{m|Z})^T C^T C(\theta - \hat{\theta}_{m|Z})\Sigma^{-1})]$ it represents kernel of the matrix normal variance-mean mixture distribution of θ and the distribution parameters are$(\hat{\theta}_{m|Z}, Z\Sigma, (C^T C)^{-1})$, that $[|\Sigma|^{-\frac{n+k+1}{2}} e^{-\frac{1}{2Z} tr B_8 \Sigma^{-1}}]$ it represents kernel of the inverse wishart distribution of Σ and the distribution parameters are $(\frac{B_8}{Z}, n)$.
Therefore, the joint posterior probability distribution of θ, Σ conditional by the variable Z is as follows:

$$\begin{aligned} P((\theta, \Sigma|Y, Z) = &\frac{|B_8|^{\frac{n}{2}} Z^{-\frac{nk}{2}}}{|\Sigma|^{\frac{n+k+1}{2}} 2^{\frac{nk}{2}} \Gamma_k(\frac{n}{2})} exp(-\frac{1}{2Z} tr B_8 \Sigma^{-1}) \\ &\frac{|C^T C|^{\frac{k}{2}}}{(2\pi Z)^{\frac{(p+s+1)k}{2}} |\Sigma|^{\frac{(p+s+1)}{2}}} e^{-\frac{1}{2Z} tr(\theta - \hat{\theta}_{m|Z})^T C^T C(\theta - \hat{\theta}_{m|Z})\Sigma^{-1}} \end{aligned} \tag{4.8}$$

As: $\Gamma_k(\frac{n}{2})$:is a multivariate Gamma function that is calculated according to the following formula:[10]

$$\Gamma_k(x) = (\pi)^{\frac{(k-1)k}{4}} \prod_{j=1}^{k} \Gamma(x + \frac{(1-j)}{2})$$

In order to obtain the joint posterior probability distribution of θ, Σ unconditioned of the random variable Z, we integrate equation (4.8) relative to the variable Z and based on the concept of the approximate determinant of $|B_8|^{\frac{n}{2}}$— through the following relationship: [6]

$$\det(x) = \exp(tr(\log(x))) \tag{4.9}$$

Returning to equation (4.8), the expression $|B_8|^{\frac{n}{2}}$ can be written as follows:

$$|B_8|^{\frac{n}{2}} = |B_3|^{\frac{n}{2}} |I_k + B_4|^{\frac{n}{2}}$$

As:

$$B_3 = (Y - C\hat{\theta^*})^T (Y - C\hat{\theta^*}), \qquad \hat{\theta^*} = (C^T C)^{-1} C^T Y$$

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$C(C^T C)^{-1} C^T$:Is idempotent matrix of rank$(p+s+1)$, through equation (4.9):

$$|B_8|^{\frac{n}{2}} = |B_3|^{\frac{n}{2}} \exp\left(\frac{n}{2} tr(\log(I_k + B_4))\right) \tag{4.10}$$

Depending on the Maclaurin series, the: [6]

$$tr\left(\log(I_k + B_4)\right) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} tr(B_4^p)$$

And by using the Maclaurin series of the exponential function in equation (4.10):

$$|B_8|^{\frac{n}{2}} = |B_3|^{\frac{n}{2}} \sum_{r=0}^{\infty} \frac{n^r}{r! 2^r} \left[\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} tr(B_4^p)\right]^r$$

And taking the terms of the series up to the second order, we get:

$$|B_8|^{\frac{n}{2}} \cong |B_3|^{\frac{n}{2}} \left[1 + \frac{n}{2} tr B_4 + (\frac{n}{2})^2 \frac{(tr B_4)^2}{2}\right]$$

Accordingly, the joint posterior probability distribution of θ, Σ unconditional of Z and according to mixed distributions is approximate as follows:

$$P(\theta, \Sigma | Y) = \int_0^{\infty} P(\theta, \Sigma | Y, Z) P(Z) dZ$$

$$P(\theta, \Sigma | Y) \cong \frac{|B_3|^{\frac{n}{2}} |C^T C|^{\frac{k}{2}} (\frac{\lambda}{\psi})^{\frac{\upsilon}{2}} e^{tr(Y - C\hat{\theta^*})^T \delta_1^* \Sigma^{-1}} e^{tr(\theta - \hat{\theta^*})^T C^T (-\delta) \Sigma^{-1}}}{(2\pi)^{\frac{(p+s+1)k}{2}} |\Sigma|^{\frac{n+(p+s+1)+k+1}{2}} 2^{\frac{nk}{2}} \Gamma_k(\frac{n}{2}) 2K_{\upsilon}(\sqrt{\lambda\psi})} \star$$

$$\left[\frac{2K_{\frac{2\upsilon - k(n+(p+s+1))}{2}}(\sqrt{\lambda\psi(1 + \frac{tr\Theta^*}{\psi})(1 + \frac{tr\delta_2^*}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{2\upsilon - k(n+(p+s+1))}{4}} (\frac{1 + tr\delta_2^*/\lambda}{1 + tr\Theta^*/\psi})^{\frac{2\upsilon - k(n+(p+s+1))}{4}}} + \frac{n}{2} tr\delta_1^{*T} \delta_1^* B_3^{-1}\right.$$

$$\frac{2K_{\frac{2\upsilon - k(n+(p+s+1))+4}{2}}\left(\sqrt{\lambda\psi(1 + \frac{tr\Theta^*}{\psi})(1 + \frac{tr\delta_2^*}{\lambda})}\right)}{(\frac{\lambda}{\psi})^{\frac{2\upsilon - k(n+(p+s+1))+4}{4}} (\frac{1 + \delta_2^*/\lambda}{1 + \frac{tr\Theta^*}{\psi}})^{\frac{2\upsilon - k(n+(p+s+1))+4}{4}}} - ntr(Y - C\theta^*)^T \delta_1^* B_3^{-1}$$

$$\frac{2K_{\frac{2\upsilon - k(n+(p+s+1))+2}{2}}\left(\sqrt{\lambda\psi(1 + tr\Theta^*/\psi)(1 + \frac{tr\delta_2^*}{\lambda})}\right)}{(\frac{\lambda}{\psi})^{\frac{2\upsilon - k(n+(p+s+1))+2}{4}} (\frac{1 + \delta_2^*/\lambda}{1 + \frac{tr\Theta^*}{\psi}})^{\frac{2\upsilon - k(n+(p+s+1))+2}{4}}} + \frac{n^2}{8} (tr\delta_1^{*T} \delta_1^* B_3^{-1})^2$$

$$\frac{2K_{\frac{2\upsilon - k(n+(p+s+1))+8}{2}}\left(\sqrt{\lambda\psi(1 + tr\Theta^*/\psi)(1 + \frac{tr\delta_2^*}{\lambda})}\right)}{(\frac{\lambda}{\psi})^{\frac{2\upsilon - k(n+(p+s+1))+8}{4}} (\frac{1 + tr\delta_2^*/\lambda}{1 + \ tr\Theta^*/\psi})^{\frac{2\upsilon - k(n+(p+s+1))+8}{4}}} + \frac{n^2}{2} (tr(Y - C\hat{\theta^*})^T \delta_1^* B_3^{-1})^2$$

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$$\frac{2K_{\frac{2v-k(n+(p+s+1))+4}{2}}\left(\sqrt{\lambda\psi(1+\frac{tr\Theta^*}{\psi})(1+\frac{tr\delta_2^*}{\lambda})}\right)}{(\frac{\lambda}{\psi})^{\frac{2v-k(n+(p+s+1))+4}{4}}(\frac{1+\frac{tr\delta_2^*}{\lambda}}{1+\frac{tr\Theta^*}{\psi}})^{\frac{2v-k(n+(p+s+1))+4}{4}}} - \frac{n^2}{2}(tr\delta_1^{*T}\delta_1^* B_3^{-1})$$
$$\star\left(tr(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-1}\right)\frac{2K_{\frac{2v-k(n+(p+s+1))+6}{2}}\left(\sqrt{\lambda\psi(1+\frac{tr\Theta^*}{\psi})(1+\frac{tr\delta_2^*}{\lambda})}\right)}{(\frac{\lambda}{\psi})^{\frac{2v-k(n+(p+s+1))+6}{4}}(\frac{1+\frac{tr\delta_2^*}{\lambda}}{1+\frac{tr\Theta^*}{\psi}})^{\frac{2v-k(n+(p+s+1))+6}{4}}}\Bigg] \tag{4.11}$$

As:

$$\Theta^* = \left(B_3 + (\theta-\hat{\theta}^*)^T C^T C(\theta-\hat{\theta}^*)\right)\Sigma^{-1}, \delta_2^* = \left(\delta_1^{*T}\delta_1^* + \delta^T C(C^TC)^{-1}C^T\delta\right)\Sigma^{-1}$$

We notice from equation (4.11) that it is difficult to find the posterior marginal probability distributions of θ,Σ unconditional of the random variable Z, so the posterior marginal distribution of the location matrix Σ conditional of Z will be found by integrating equation (4.8) relative to the scale matrix Σ as follows:

$$P((\theta|Y,Z) = \int_\Sigma P((\theta,\Sigma|Y,Z)d\Sigma$$
$$P((\theta|Y,Z) = \frac{|B_8|^{\frac{n}{2}}|C^TC|^{\frac{k}{2}}\Gamma_k(\frac{n+(p+s+1)}{2})}{(\pi)^{\frac{(p+s+1)k}{2}}\Gamma_k(\frac{n}{2})}|B_8+(\theta-\theta_{(m|Z)})^T C^T C(\theta-\theta_{(m|Z)})|^{-\frac{n+(p+s+1)}{2}} \tag{4.12}$$

It is possible to rewrite equation (4.12) as follow:

$$P((\theta|Y,Z) = R(k,(p+s+1),r)\frac{|B_8|^{-\frac{(p+s+1)}{2}}|C^TC|^{\frac{k}{2}}}{|I_k+(\theta-\theta_{(m|Z)})^T C^T C(\theta-\theta_{(m|Z)})B_8^{-1}|^{\frac{n+(p+s+1)}{2}}} \tag{4.13}$$

As:

$$R(k,(p+s+1),r) = \frac{\Gamma_k(\frac{n+(p+s+1)}{2})}{(\pi)^{\frac{(p+s+1)k}{2}}\Gamma_k(\frac{n}{2})}, \quad \text{r is degree of freedom}$$

And using property of Box and Tiao. [2]

$$|I_k + P_{k\times(p+s+1)}Q_{(p+s+1)\times k}| = |I_{(p+s+1)} + Q_{(p+s+1)\times k}P_{k\times(p+s+1)}|$$

Thus:

$$P((\theta|Y,Z) = \frac{R(k,(p+s+1),r)|B_8|^{-\frac{(p+s+1)}{2}}|C^TC|^{\frac{k}{2}}}{|I_{(p+s+1)}+(\theta-\theta_{(m|Z)})B_8^{-1}(\theta-\theta_{(m|Z)})^T C^T C|^{\frac{n+(p+s+1)}{2}}}$$

And surely:

$$R(k,(p+s+1),r) = R(p+s+1),k,r)$$
$$\frac{\Gamma_k(\frac{n+(p+s+1)}{2})}{(\pi)^{\frac{(p+s+1)k}{2}}\Gamma_k(\frac{n}{2})} = \frac{\Gamma_{(p+s+1)}(\frac{n+(p+s+1)}{2})}{(\pi)^{\frac{(p+s+1)k}{2}}\Gamma_{(p+s+1)}(\frac{n-k+(p+s+1)}{2})}$$

Hence, the posterior marginal probability distribution of the matrix θ conditioned by the random variable Z is as follows:

$$P((\theta|Y,Z) = \frac{|B_8|^{-\frac{(p+s+1)}{2}}|C^TC|^{\frac{k}{2}}\Gamma_{(p+s+1)}(\frac{n+(p+s+1)}{2})}{(\pi)^{\frac{(p+s+1)k}{2}}\Gamma_{(p+s+1)}(\frac{n-k+(p+s+1)}{2})}$$
$$\star\,|I_{(p+s+1)}+(\theta-\hat{\theta}_{(m|Z)})B_8^{-1}(\theta-\hat{\theta}_{(m|Z)})^T C^T C|^{-\frac{n+(p+s+1)}{2}} \tag{4.14}$$

We conclude from equation (4.14) that the posterior marginal probability distribution of θ conditioned by the random variable Z is skewed matrix-t distribution of the parameters $(\theta_{(m|Z}, B_8, (C^TC)^{-1}, r = n-k+1)$ and is described as follows:

$$\theta|Y,Z \sim Mt_{(p+s+1),k}(\hat{\theta}_{(m|Z}, B_8, (C^TC)^{-1}, r) \leftrightarrow vec(\hat{\theta}|Y,Z) \sim Mt_{(p+s+1)k}(vec(\hat{\theta}_{(m|Z}), B_8 \otimes (C^TC)^{-1}, r)$$

Accordingly, the posterior marginal probability distribution for θ unconditional of the variable Z and based on the concept of the approximate determinant defined in equation (4.9) of $|I_{(p+s+1)} + (\theta - \hat{\theta}_{m|Z})B_8^{-1}(\theta - \hat{\theta}_{m|Z})^T C^TC|^{-\frac{n+(p+s+1)}{2}}$ and $|B_8|^{-\frac{(p+s+1)}{2}}$ and conducting some mathematical operations and taking the limits of the series for the first order will be as follows:

$$
\begin{aligned}
P(\theta|Y) &= \int_0^\infty P(\theta|Y,Z)P(Z)dZ \\
P(\theta|Y) &\cong \frac{|B_3|^{-\frac{(p+s+1)}{2}}|(C^TC)|^{\frac{k}{2}}(\frac{\lambda}{\psi})^{\frac{\upsilon}{2}}\Gamma_{(p+s+1)}\left(\frac{n+(p+s+1)}{2}\right)}{\pi^{\frac{(p+s+1)k}{2}}\Gamma_{(p+s+1)}\frac{n-k+(p+s+1)}{2}2K_\upsilon\sqrt{\lambda\psi}}\Bigg[\frac{2K_\upsilon\sqrt{\lambda\psi}}{(\frac{\lambda}{\psi})^{\frac{\upsilon}{2}}} \\
&- \xi_1 tr(\theta-\hat{\theta}^*)B_3^{-1}(\theta-\hat{\theta}^*)^T C^TC + \xi_2 tr(\theta-\theta^*)\delta_1^{*T}\delta_1^* B_3^{-2}(\theta-\hat{\theta}^*)^T C^TC \\
&- \xi_3 tr(\theta-\hat{\theta}^*)(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-2}(\theta-\hat{\theta}^*)^T C^TC - \xi_3 tr(\theta-\hat{\theta}^*)B_3^{-1}\delta^T C \\
&+ \xi_4 tr(\theta-\hat{\theta}^*)\delta_1^{*T}\delta_1^* B_3^{-2}\delta^T C - \xi_5 tr(\theta-\hat{\theta}^*)(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-2}\delta^T C \\
&- \xi_2 tr(C^TC)^{-1}C^T\delta B_3^{-1}\delta^T C + \xi_6 tr(C^TC)^{-1}C^T\delta\delta_1^{*T}\delta_1^* B_3^{-2}\delta^T C \\
&- \xi_4 tr(C^TC)^{-1}C^T\delta(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-2}\delta^T C - \xi_7 tr\delta_1^{*T}\delta_1^* B_3^{-1} \\
&+ \xi_8 tr(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-1} + \xi_9 tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(\theta-\hat{\theta}^*)B_3^{-1}(\theta-\hat{\theta}^*)^T C^TC \\
&- \xi_{10} tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(\theta-\hat{\theta}^*)\delta_1^{*T}\delta_1^* B_3^{-2}(\theta-\hat{\theta}_1^*)^T C^TC \\
&+ \xi_{11} tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(\theta-\hat{\theta}^*)(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-2}(\theta-\hat{\theta}^*)^T C^TC \\
&+ 2\xi_9 tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(\theta-\hat{\theta}^*)B_3^{-1}\delta^T C \\
&+ 4\xi_{10} tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(\theta-\theta^*)(Y-C\theta^*)^T\delta_1^* B_3^{-2}\delta^T C \\
&+ \xi_{10} tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(C^TC)^{-1}C^T\delta B_3^{-1}\delta^T C - \xi_{12} tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(C^TC)^{-1}C^T\delta\delta_1^{*T}\delta_1^* B_3^{-2}\delta^T C \\
&+ \xi_{13} tr\delta_1^{*T}\delta_1^* B_3^{-1} tr(C^TC)^{-1}C^T\delta(Y-C\theta^*)^T\delta_1^* B_3^{-2}\delta^T C \\
&- \xi_{14} tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(\theta-\theta^*)B_3^{-1}(\theta-\theta^*)^T C^TC \\
&+ \xi_{11} tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(\theta-\theta^*)\delta_1^{*T}\delta_1^* B_3^{-2}(\theta-\theta^*)^T C^TC \\
&- 4\xi_9 tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(\theta-\theta^*)(Y-C\theta^*)^T\delta_1^* B_3^{-2}(\theta-\theta^*)^T C^TC \\
&- 4\xi_9 tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(\theta-\hat{\theta}^*)B_3^{-1}\delta^T C + 4\xi_{10} tr(Y-C\hat{\theta}^*)^T\delta_1^* B_3^{-1} tr(\theta-\theta^*)\delta_1^{*T}\delta_1^* B_3^{-2}\delta^T C \\
&- \xi_{11} tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(\theta-\theta^*)(Y-C\theta^*)^T\delta_1^* B_3^{-2}\delta^T C \\
&- \xi_{11} tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(C^TC)^{-1}C^T\delta B_3^{-1}\delta^T C \\
&+ \xi_{13} tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(C^TC)^{-1}C^T\delta\delta_1^{*T}\delta_1^* B_3^{-2}\delta^T C \\
&- 4\xi_{10} tr(Y-C\theta^*)^T\delta_1^* B_3^{-1} tr(C^TC)^{-1}C^T\delta(Y-C\theta^*)^T\delta_1^* B_3^{-2}\delta^T C \Bigg]
\end{aligned}
\tag{4.15}
$$

As:

$$\xi_1 = \frac{n+(p+s+1)}{2}, \qquad \xi_2 = \frac{n+(p+s+1)}{2}\frac{2K_{v+2}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+2}{2}}}$$

$$\xi_3 = (n+(p+s+1))\frac{2K_{v+1}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+1}{2}}}, \qquad \xi_4 = (n+(p+s+1))\frac{2K_{v+3}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+3}{2}}},$$

$$\xi_5 = 2(n+(p+s+1))\frac{2K_{v+2}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+2}{2}}}, \qquad \xi_6 = \frac{n+(p+s+1)}{2}\frac{2K_{v+4}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+4}{2}}},$$

$$\xi_7 = \frac{(p+s+1)}{2}\frac{2K_{v+2}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+2}{2}}}, \qquad \xi_8 = (p+s+1)\frac{2K_{v+1}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+1}{2}}}$$

$$\xi_9 = \frac{(p+s+1)(n+(p+s+1))}{4}\frac{2K_{v+2}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+2}{2}}}, \quad \xi_{10} = \frac{(p+s+1)(n+(p+s+1))}{4}\frac{2K_{v+4}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+4}{2}}}$$

$$\xi_{11} = \frac{(p+s+1)(n+(p+s+1))}{2}\frac{2K_{v+3}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+3}{2}}}, \quad \xi_{12} = \frac{(p+s+1)(n+(p+s+1))}{4}\frac{2K_{v+6}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+6}{2}}}$$

$$\xi_{13} = \frac{(p+s+1)(n+(p+s+1))}{2}\frac{2K_{v+5}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+5}{2}}}, \quad \xi_{14} = \frac{(p+s+1)(n+(p+s+1))}{2}\frac{2K_{v+1}(\sqrt{\lambda\psi})}{(\frac{\lambda}{\psi})^{\frac{v+1}{2}}}$$

We notice from equation (30) that it is difficult to find the Bayesian estimator, so it will be found through the concept of mixed distributions and according to the properties of the mathematical expectation as follows: [5, 16].

$$\hat{\theta} = E_Z E_{(\theta|Z}((\theta|Y,Z)$$
$$\hat{\theta}_B = (C^TC)^{-1}C^TY - (C^TC)^{-1}C^T\delta\frac{K_{v+1}(\sqrt{\lambda\psi})}{K_v(\sqrt{\lambda\psi})}(\frac{\lambda}{\psi})^{-0.5} \tag{4.16}$$

As for the posterior marginal probability distribution of the scale matrix Σ conditioned by the random variable Z, it can be found by integrating equation (4.8) relative to the location matrix θ as follows:

$$P(\Sigma|Y,Z) = \int_\theta P(\theta,\Sigma|Y,Z)d\theta$$
$$= \frac{|B_8|^{\frac{n}{2}}Z^{-\frac{nk}{2}}}{|\Sigma|^{\frac{n+k+1}{2}}2^{\frac{nk}{2}}\Gamma_k(\frac{n}{2})}exp(-\frac{1}{2Z}trB_8\Sigma^{-1}) \tag{4.17}$$

Equation (4.17) represents the inverse wishart distribution of parameters $(\frac{B_8}{Z}, n)$ for the posterior marginal probability distribution of Σ conditional by Z. Based on mixed distributions and the approximate determinant previously defined in Equation (4.9) for $|B_8|^{\frac{n}{2}}$, following the same steps in finding the joint posterior probability distribution ofθ, Σ unconditioned by Z, we conclude that the posterior marginal probability distribution of the scale matrix Σ unconditioned by Z is approximate as follows:

$$P(\Sigma Y) = \int_0^\infty P(\Sigma Y, Z)P(Z)dZ$$

$$\begin{aligned}
P(\Sigma Y) \cong{}& \frac{|B_3|^{\frac{n}{2}} e^{tr(Y-C\theta^*)^T\delta_1^*\Sigma^{-1}}}{|\Sigma|^{\frac{n+k+1}{2}} 2^{\frac{nk}{2}} \Gamma_k(\frac{n}{2}) K_v(\sqrt{\lambda\psi})} \Bigg[\frac{K_{\frac{2v-nk}{2}}(\sqrt{\lambda\psi(1+\frac{tr\tau_1}{\psi})(1+\frac{tr\tau_2}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{-nk}{4}} (\frac{1+\frac{tr\tau_2}{\lambda}}{1+\frac{tr\tau_1}{\psi}})^{\frac{2v-nk}{4}}} + \frac{n}{2} tr\delta_1^{*T}\delta_1^* B_3^{-1} \\
& \frac{K_{\frac{2v-nk+4}{2}}(\sqrt{\lambda\psi(1+\frac{tr\tau_1}{\psi})(1+\frac{tr\tau_2}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{-nk+4}{4}} (\frac{1+\frac{tr\tau_2}{\lambda}}{1+\frac{tr\tau_1}{\psi}})^{\frac{2v-nk+4}{4}}} - ntr(Y-C\theta^*)^T\delta_1^* B_3^{-1} \\
& \frac{K_{\frac{2v-nk+2}{2}}(\sqrt{\lambda\psi(1+\frac{tr\tau_1}{\psi})(1+\frac{tr\tau_2}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{-nk+2}{4}} (\frac{1+\frac{tr\tau_2}{\lambda}}{1+\frac{tr\tau_1}{\psi}})^{\frac{2v-nk+2}{4}}} + \frac{n^2}{8}(tr\delta_1^{*T}\delta_1^* B_3^{-1})^2 \\
& \frac{K_{\frac{2v-nk+8}{2}}(\sqrt{\lambda\psi(1+\frac{tr\tau_1}{\psi})(1+\frac{tr\tau_2}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{-nk+8}{4}} (\frac{1+\frac{tr\tau_2}{\lambda}}{1+\frac{tr\tau_1}{\psi}})^{\frac{2v-nk+8}{4}}} + \frac{n^2}{2}(tr(Y-C\theta^*)^T\delta_1^* B_3^{-1})^2 \\
& \frac{K_{\frac{2v-nk+4}{2}}(\sqrt{\lambda\psi(1+\frac{tr\tau_1}{\psi})(1+\frac{tr\tau_2}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{-nk+4}{4}} (\frac{1+\frac{tr\tau_2}{\lambda}}{1+\frac{tr\tau_1}{\psi}})^{\frac{2v-nk+4}{4}}} - \frac{n^2}{2}(tr\delta_1^{*T}\delta_1^* B_3^{-1}) \\
& (tr(Y-C\theta^*)^T\delta_1^* B_3^{-1}) \frac{K_{\frac{2v-nk+6}{2}}(\sqrt{\lambda\psi(1+\frac{tr\tau_1}{\psi})(1+\frac{tr\tau_2}{\lambda})})}{(\frac{\lambda}{\psi})^{\frac{-nk+6}{4}} (\frac{1+\frac{tr\tau_2}{\lambda}}{1+\frac{tr\tau_1}{\psi}})^{\frac{2v-nk+6}{4}}} \Bigg]
\end{aligned} \tag{4.18}$$

As:

$$\tau_1 = B_3\Sigma^{-1}, \qquad \tau_2 = \delta_1^{*T}\delta_1^*\Sigma^{-1}$$
$$B_3 = (Y-C\hat{\theta}^*)^T(Y-C\hat{\theta}^*), \hat{\theta}^* = (C^TC)^{-1}C^TY, \qquad \delta_1^* = (\delta - C(C^TC)^{-1}C^T\delta)$$

It is difficult to find the Bayesian estimator for Σ from equation (4.18). Therefore, it will be found through the concept of mixed distributions and according to the properties of the mathematical expectation as follows:

$$\begin{aligned}
\hat{\Sigma} &= E_Z E_{(\Sigma|Z}((\Sigma|Y,Z) \\
\hat{\Sigma}_B &= \frac{(Y-C\hat{\theta}^*)^T(Y-C\hat{\theta}^*)}{n-k-1} \frac{K_{v-1}(\sqrt{\lambda\psi})}{K_v(\sqrt{\lambda\psi})} (\frac{\lambda}{\psi})^{0.5} - \frac{(Y-C\hat{\theta}^*)^T\delta_1^* + \delta_1^{*T}(Y-C\hat{\theta}^*)}{n-k-1} \\
&+ \frac{\delta_1^{*T}\delta_1^*}{n-k-1} \frac{K_{v+1}(\sqrt{\lambda\psi})}{K_v(\sqrt{\lambda\psi})} (\frac{\lambda}{\psi})^{-0.5}
\end{aligned} \tag{4.19}$$

5. Simulation

This section discusses the application of the mechanism reached in the sections (2, 3, 4) to data generated in a suggested method for a multivariate partial regression model with random error following the matrix-variate generalized hyperbolic distribution.

5.1. The Proposed Method for Generating Random Data

It is difficult to generate random data from the multivariate partial regression model when the random error limit follows the matrix-variate generalized hyperbolic distribution. Therefore, it is resorted to generate these data through mixed distributions, as the matrix normal variance-mean mixture distribution and the generalized inverse Gaussian distribution previously mentioned were used.

Whereas, random observations were generated from the distribution of the multivariate standard normal$Z$, and since $Z = (\varepsilon|Z - \delta Z)(Z\Sigma)^{-0.5}$ and ε represents the matrix of random errors of the model from which the observations are to be generated and through the concept of mixed distributions as follow:

$$\varepsilon|Z = Z\ (Z\Sigma)^{0.5} + \delta Z \ldots \tag{5.1}$$

$$\varepsilon = \int_Z (\varepsilon|Z \ \ P(Z)dZ$$

$$\varepsilon = Z\Sigma^{0.5} \frac{K_{\frac{2v+1}{2}}(\sqrt{\lambda\psi})}{K_v(\sqrt{\lambda\psi})(\frac{\lambda}{\psi})^{\frac{1}{4}}} + \delta \frac{K_{v+1}(\sqrt{\lambda\psi})}{K_v(\sqrt{\lambda\psi})(\frac{\lambda}{\psi})^{\frac{1}{2}}} \tag{5.2}$$

Equation (5.2) represents the matrix of random errors, which follows the matrix-variate generalized hyperbolic distribution.

The following algorithm shows the proposed method for generating random data from a matrix-variate generalized hyperbolic distribution:

1. Assume we have the number of observations ($n = 100$) and the number of response variables ($k = 2$).
2. Generate random numbers from the standard normal distribution with n observations and k variables of response and let the standard normal random number matrix be$Z$.
3. We put $(\varepsilon|Z = Z \star (Z\Sigma)^{0.5} + \delta Z$, knowing that Z represents step (2).
4. We find the generated observations ε which represent the random error observations generated from the matrix-variate generalized hyperbolic distribution taking into account the assumed values of the shape parameters (λ, ψ, ν) and skewness matrix (δ) defined in Table 2.
5. For the purpose of generating observations from the partial multivariate regression model, we generate the observations of the two explanatory variables $(p, s = 2)$ for the parametric and nonparametric part (X_1, X_2) and (t_1, t_2) through the following equation:

   $$X_j = 2\bar{X}_j u_j$$

   Since $u_j$represents the standard uniform distribution, $\bar{X}_j$represents the arithmetic mean and they are usually assumed values ,and the nonparametric part W represents the kernel weights one time we take it to the Gaussian kernel function and another we take it to the proposed kernel function which was previously defined in equation (3.1) and based on the rule of thumb to choose the bandwidth parameter for both functions and the nonparametric variables (t_1, t_2) is a standard normal variables.
6. Randomly assumed values are given for θ and Σ and for shape parameters they are given random values depending on the state of the studied distribution $\lambda, \psi, v > 0$ and as in Table 2 below.

Table 2: Approved default values for all parameters

	λ	φ	υ	$\theta_{(p+s+1)\times k}$	$\Sigma_{k\times k}$	δ
1	***5***	***0.5***	***2.5***	$\begin{bmatrix} 2 & 1.4 & 5 & 3 & 4.1 \\ 1.5 & 3 & 2.5 & 1 & 3.4 \end{bmatrix}^T$	$\begin{bmatrix} 1.5 & 2 \\ 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 0.6498 \\ 0.9347 \end{bmatrix}^T$
2	***7***		***3.5***			

7. After substituting step (4) and observations of the two explanatory variables for the parametric part, the kernel weights matrix for the nonparametric part defined in step (5) and the assumed values of the parameters defined in step (6), we obtain 4 models based on the combination between the assumed values of the response matrix Y.

5.2. Estimation of model parameters

The location matrix θ and the scale matrix Σ were estimated in a Bayes method when non-informative prior information was available and under the quadratic loss function. The comparison between the estimates was also made using the mean sum of squares errors and depending on all the combinations between the default values shown in Table 2 by using a program Matlab-R2016a.

Table 3: Mean sum of squares error for the estimator of location matrix θ and scale matrix Σ

Models $(\lambda, \varphi, \upsilon)$	***Gaussian kernel function***		***proposed kernel function***	
	θ	Σ	θ	Σ
first(5,0.5,2.5)	***0.0472***	***0.0014***	***0.0173***	***0.0026***
second(5,0.5,2.5)	***0.0633***	***0.0097***	***0.0405***	***0.0081***

We notice from Table 3 that the best estimator for θ and Σ it was at the proposed kernel function and third model, this estimate is as follows:

$$\hat{\theta}_B = \begin{bmatrix} 1.97431.3448 & 5.12903.0614 & 4.0104 \\ 1.56492.8636 & 2.64491.1322 & 3.2859 \end{bmatrix}^T, \qquad \hat{\Sigma}_B = \begin{bmatrix} 1.5412 & 2.0051 \\ 2.0051 & 2.9627 \end{bmatrix}$$

5.3. Matching Generated Data

The following figures show the matrix of generated and estimated response variables that follow the matrix-variate generalized hyperbolic distribution for two models (2,3) as they are chosen according to the following (the highest , lowest mean sum of squares errors) respectively of θ and for the proposed kernel function:

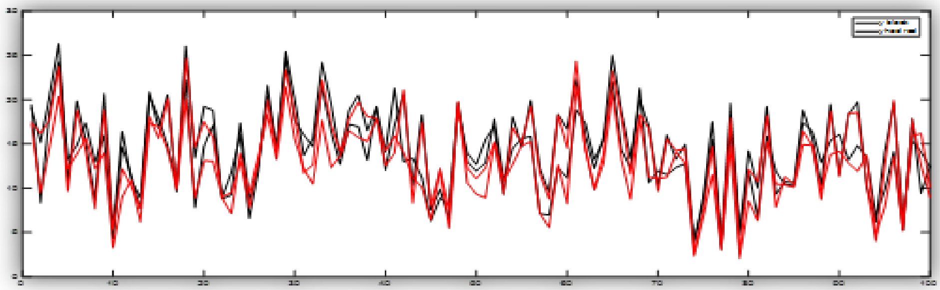

Figure 1: generated and estimated observations of the second model

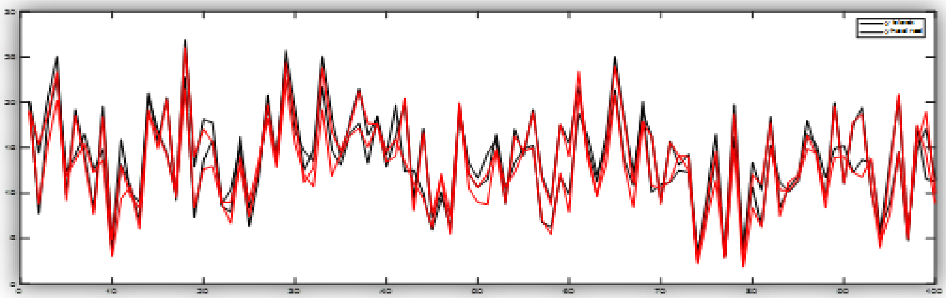

Figure 2: generated and estimated observations of the third model

6. Conclusions

The most important theoretical and experimental conclusions of this research were as follows:

1. The posterior marginal probability distribution of θ when the error is twisted is an uncommon approximate distribution.
2. The posterior marginal probability distribution for Σ when the error is twisted is an uncommon approximate distribution.
3. Find accurate Bayesian estimates for the approximate posterior marginal probability distributions.
4. The parameters estimators under the proposed kernel function of twisted error override the estimates for the Gaussian kernel function.

7. Recommendations

We recommend conducting an applied side of what was reached in the research on data related to economics and comparing the proposed kernel function with other kernel functions defined in Table 1 to know their efficiency, In addition, balanced loss functions are used to estimate the parameters of the model.

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