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Non-Bayesian estimation of Weibull Lindley burr XII distribution

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Abstract

In this paper, we estimate the four parameters of Weibull Lindley burr distribution by using ordinary least square method and multiple regression least square method. The survival estimate made by using ordinary least square estimator (OLSE) and multiple regression estimator (MRE).

Keywords: Weibull Lindley burr XII distribution (WLBD), The ordinary least squares Method (OLSEM), Multiple Regression Least Squares Method (MRLSM)
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1. Introduction

Offers the [7] comprehensive guidance toward the most commonly used statistical distributions, including normal, lognormal, inverse Gaussian, Pareto, Cauchy, gamma distributions and more. Each distribution includes clear definitions and properties, plus methods of inference, applications. Surles and Padgett in (2001) [13] suggested two parameters Burr Type X which also described as generalized Rayleigh distribution. Kundu and Raqab in (2005) [8] studied different methods to estimate the parameters of this distribution: Maximum likelihood method, Modeled moment method, Weighted least square method, L moment method and compare their performance through Monte Carlo simulation. Al- Naqeep and Hamed in(2009) [3] suggested, as well, a conventional method for estimating the two parameters of generalized Rayleigh distribution for different sample (small, medium and large) and compare the estimators by using mean square error for simulation data. Dhwyia and else in (2012) [6] developed the moment generating function which was derived to help in finding the moment, also cumulative distribution function, then obtaining the least squares

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estimators for the unknown parameters and moment estimator method . Parvin, Ali and Hossein in (2013) [12] considered the estimation of R = P(y ¡ x) where x and y have two parameters of generalized Rayleigh distribution, then obtained the maximum likelihood estimations of parameters with simple iterative procedure for several values of parameters.

2. The ordinary least squares Method (OLSEM)

The ordinary least squares method.is one of most popular procedures in estimating the parameters when the model is linear [9].
Every linear or nonlinear model involve response variable denoted by (y) which is effected by explanatory variable denoted by (x).
If the relationship between variable (Y) and variable (x) is linear, model is represented mathematically by straight line equation as follows [10]:

$$Y = \beta_0 + \beta_1 x + \varepsilon \tag{2.1}$$

Where: β_0 represents the intercept term
β_1 represents the slop term
ε represents the error term
The idea of this method is to minimize the sum of squared differences between observed sample values and the estimate expected values by linear approximation [1]:

$$\varepsilon = Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \tag{2.2}$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right]^2 \tag{2.3}$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [y_i - \hat{y}_i]^2 \tag{2.4}$$

One can obtain The estimation and Simulation of WLBD By Using The ordinary least squares (OLSE) by using the CDF of WLBD which are as follows:

$$F(t_i) = 1 - (\frac{1 + 2\lambda - \lambda(1 + t_i^c)^{-k}}{\lambda + 1}) \exp(-(\lambda + \beta)) \exp((\lambda + \beta)(1 + t_i^c)^{-k}) \tag{2.5}$$

then

$$1 - [F(t_i)] = (\frac{1 + 2\lambda - \lambda(1 + t_i^c)^{-k}}{\lambda + 1}) \exp(-(\lambda + \beta)) \exp((\lambda + \beta)(1 + t_i^c)^{-k}) \tag{2.6}$$

By taking the logarithm of above equation getting:

$$\begin{aligned} \log[1 - \{F(t_i)\}] &= \log[(\frac{1 + 2\lambda - \lambda(1 + t_i^c)^{-k}}{\lambda + 1}) \exp(-(\lambda + \beta)) \exp((\lambda + \beta)(1 + t_i^c)^{-k})] \\ &= -\log(\lambda + 1) - (\lambda + \beta) + \log(1 + 2\lambda - \lambda(1 + t_i^c)^{-k}) + (\lambda + \beta)(1 + t_i^c)^{-k} \end{aligned} \tag{2.7}$$

Comparing the above equation with the simple linear model $Y = \beta_0 + \beta_1 x + \varepsilon$ we get:

$$Y = \log\left[1 - \{F(t_i)\}\right] \tag{2.8}$$

$$x = \log(1 + 2\lambda - \lambda(1 + t_i^c)^{-k}) + (\lambda + \beta)(1 + t_i^c)^{-k} \tag{2.9}$$

$$\beta_0 = -\log(\lambda + 1) - (\lambda + \beta) \tag{2.10}$$

$$\beta_1 = 1 \tag{2.11}$$

$$\begin{aligned}\varepsilon = &\log\left[1 - \{F(t_i)\}\right] + \log(\lambda + 1) + (\lambda + \beta)\\ &- \log(1 + 2\lambda - \lambda(1 + t_i^c)^{-k}) - (\lambda + \beta)(1 + t_i^c)^{-k}\end{aligned} \tag{2.12}$$

By taking the sum square of above equation for the two sides to obtain:

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon^2 = &\sum_{i=1}^{n} [\log\left[1 - \{F(t_i)\}\right] + \log(\lambda + 1) + (\lambda + \beta)\\ &- \log(1 + 2\lambda - \lambda(1 + t_i^c)^{-k}) - (\lambda + \beta)(1 + t_i^c)^{-k}]^2\end{aligned} \tag{2.13}$$

Let $z_1(\beta, \lambda, c, k) = \frac{\partial\left(\sum_{i=1}^{n} \varepsilon_i^2\right)}{\partial \beta}$, $z_2(\beta, \lambda, c, k) = \frac{\partial\left(\sum_{i=1}^{n} \varepsilon_i^2\right)}{\partial \lambda}$,
$z_3(\beta, \lambda, c, k) = \frac{\partial\left(\sum_{i=1}^{n} \varepsilon_i^2\right)}{\partial c}$, $z_1(\beta, \lambda, c, k) = \frac{\partial\left(\sum_{i=1}^{n} \varepsilon_i^2\right)}{\partial k}$
Then

$$\begin{aligned}z_1(\beta, \lambda, c, k) =& 2\sum_{i=1}^{n} \left(\log(1 - F(t_i)) + \beta + \lambda + \log(\lambda + 1) - \log\left(2\lambda + 1 - \lambda\left(1 + t_i^c\right)^{-k}\right)\right.\\ &\left. - (\lambda + \beta)(1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}} \times \left(1 - (1 + t_i^c)^{-k}\right)\right.\end{aligned} \tag{2.14}$$

$$\begin{aligned}z_2(\beta, \lambda, c, k) =& 2\sum_{i=1}^{n} \left(\log(1 - F(t_i)) + \log(1 + \lambda) + \lambda + \beta - \log\left(2\lambda + 1 - \lambda\left(1 + \mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)\right.\\ &\left. - (\lambda + \beta)(1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}} \times \left(\frac{1}{1 + \lambda} + 1 - \frac{2 - (1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}}{2\lambda + 1 - \lambda \cdot (1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}} - (1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}\right)\right.\end{aligned} \tag{2.15}$$

$$\begin{aligned}z_3(\beta, \lambda, c, k) = & 2\sum_{i=1}^{n} \left(\log(1 - F(t_i)) + \beta + \lambda + \log(\lambda + 1) - \log\left(2\lambda + 1 - \lambda\left(1 + \mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)\right.\\ &\left. - (\lambda + \beta)(1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}} \times \left(-\frac{\lambda \cdot (1 + \mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}-1}\, \mathrm{k}\mathrm{t}_i^c \log(\mathrm{t})}{\left(2\lambda + 1 - \lambda\left(1 + \mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)} + (\lambda + \beta)\left(1 + \mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}-1} \mathrm{k}\mathrm{t}_i^c \log(\mathrm{t})\right)\right.\end{aligned} \tag{2.16}$$

$$z_4(\beta,\lambda,c,k) = 2\sum_{i=1}^{n}\left(\log(1-F(t_i))+\beta+\lambda+\log(\lambda+1)-\log\left(2\lambda+1-\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)\right.$$
$$\left.-(\lambda+\beta)(1+\mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}\left(-\frac{\lambda\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)}{2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}}+(\lambda+\beta)\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)\right)\right.$$

We place the partial derivatives $z_1(\beta,\lambda,c,k), z_2(\beta,\lambda,c,k), z_3(\beta,\lambda,c,k)$ and $z_4(\beta,\lambda,c,k)$ to zero as follows:

$$2\sum_{i=1}^{n}\left(\log(1-F(t_i)+\beta+\lambda+\log(\lambda+1)-\log\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)-(\lambda+\beta)(1+\mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}\right).$$
$$\times\left(1-\left(1+t_i^c\right)^{-k}\right)=0 \tag{2.17}$$

$$2\sum_{i=1}^{n}\left(\log(1-F(t_i))+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-\mathrm{k}}\right)-(\lambda+\beta)(1+\mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}\right)$$
$$\times\left(\frac{1}{1+\lambda}+1-\frac{2-\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}}{2\lambda+1-\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}}-\left(1+\mathrm{t}_i^{c}\right)^{-\mathrm{k}}\right)=0 \tag{2.18}$$

$$2\sum_{i=1}^{n}\left(\log(1-F(t_i))+\beta+\lambda+\log(\lambda+1)-\log\left(2\lambda+1-\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)\right.$$
$$\left.-(\lambda+\beta)(1+\mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}\times\left(-\frac{\lambda\cdot\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}-1}\mathrm{kt}_i^{\mathrm{c}}\log(\mathrm{t})}{\left(2\lambda+1-\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)}+(\lambda+\beta)\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}-1}\mathrm{kt}_i^{\mathrm{c}}\log(\mathrm{t})\right)\right)=0 \tag{2.19}$$

$$2\sum_{i=1}^{n}\left(\log(1-F(t_i))+\beta+\lambda+\log(\lambda+1)-\log\left(2\lambda+1-\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\right)-(\lambda+\beta)(1+\mathrm{t}_i^{\mathrm{c}})^{-\mathrm{k}}\right)$$
$$\left(-\frac{\lambda\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)}{2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}}+(\lambda+\beta)\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)\right)=0 \tag{2.20}$$

The equations (2.17), (2.18), (2.19) and (2.20) are system of nonlinear equations and cannot be solved simultaneously, so we can solve them by Newton-Raphson iterative method.
The Jacobian matrix J_{k_2} is the first derivative for each function of $z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k}), z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})$, $z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})$ and $z_4(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})$ with respect to β, λ, c and k .

$$J_{k2}=\begin{bmatrix}\frac{\partial z_1(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\beta} & \frac{\partial z_1(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\lambda} & \frac{\partial z_1(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat c} & \frac{\partial z_1(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat k}\\ \frac{\partial z_2(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\beta} & \frac{\partial z_2(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\lambda} & \frac{\partial z_2(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat c} & \frac{\partial z_2(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat k}\\ \frac{\partial z_3(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\beta} & \frac{\partial z_3(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\lambda} & \frac{\partial z_3(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat c} & \frac{\partial z_3(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat k}\\ \frac{\partial z_4(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\beta} & \frac{\partial z_4(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat\lambda} & \frac{\partial z_4(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat c} & \frac{\partial z_4(\hat\beta,\hat\lambda,\hat c,\hat k)}{\partial\hat k}\end{bmatrix} \tag{2.21}$$

$$\frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{\beta}} = 2\sum_{i=1}^{n}\left(1-(1+t_i^c)^{-k}\right)^2 \tag{2.22}$$

$$\frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{\lambda}} = 2\sum_{i=1}^{n}\left(\frac{1}{1+\lambda}+1-\frac{2-(1-t_i^c)^{-k}}{2\lambda+1-\lambda\,(1-t_i^c)^{-k}}-(1+t_i^c)^{-k}\right)\left(1-(1+t_i^c)^{-k}\right) \tag{2.23}$$

$$\begin{aligned}\frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{c}} =& 2\sum_{i=1}^{n}\left(\frac{\lambda\cdot(1-t_i^c)^{-k-1}\,kt_i^c\log(t_i)}{\left(2\lambda+1-\lambda\,(1-t_i^c)^{-k}\right)}+(\lambda+\beta)\,(1+t_i^c)^{-k-1}\,kt_i^c\log(t_i)\right)\\ &\times(1-(1+t_i^c)^{-k})\\ &+\frac{2}{1+t_i^c}\left(\log(1-F(t_i)+\log(1+\lambda)+\lambda+\beta-\log(2\lambda+1-\lambda(1-t_i^c)^{-k})\right.\\ &\left.-(\lambda+\beta)(1+t_i^c)^{-k}\right)(1+t_i^c)^{-k}\,kt_i^c\log(t_i)\end{aligned} \tag{2.24}$$

$$\begin{aligned}\frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{k}} =& 2\sum_{i=1}^{n}\left(-\frac{\lambda\,(1-t_i^c)^{-k}\log\,(1-t_i^c)}{2\lambda+1-\lambda\,(1-t_i^c)^{-k}}+(\lambda+\beta)\,(1+t_i^c)^{-k}\log\,(1+t_i^c)\right)\left(1-(1+t_i^c)^{-k}\right)\\ &+2\left(\log(1-F(t_i)+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\,(1-t_i^c)^{-k}\right)\right.\\ &\left.-(\lambda+\beta)(1+t_i^c)^{-k}\right)(1+t_i^c)^{-k}\log\,(1+t_i^c)\end{aligned} \tag{2.25}$$

$$\frac{\partial z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{\beta}} = \frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{\lambda}} \tag{2.26}$$

$$\begin{aligned}\frac{\partial z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{\lambda}} =& 2\sum_{i=1}^{n}\left(1+\frac{1}{\lambda+1}-\frac{2-(1+t_i^c)^{-k}}{2\lambda+1-\lambda\,(1+t_i^c)^{-k}}+(1+t_i^c)^{-k}\right)^2\\ +2(\log(1-F(t_i)&+\beta+\lambda+\log(\lambda+1)-\log\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)-(\lambda+\beta)\,(1+t_i^c)^{-k}\\ &\times\left(-\frac{1}{(\lambda+1)^2}+\frac{\left(2-(1+t_i^c)^{-k}\right)^2}{\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)^2}\right)\end{aligned} \tag{2.27}$$

$$\begin{aligned}\frac{\partial z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial\hat{c}} =& 2\sum_{i=1}^{n}\left(-\frac{\lambda\,(1+t_i^c)^{-k-1}\,kt_i^c\log(t_i)}{\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)}+(\lambda+\beta)\,(1+t_i^c)^{-k-1}\,kt_i^c\log(t_i)\right)\\ &\left(\frac{1}{1+\lambda}+1-\frac{2-(1+t_i^c)^{-k}}{2\lambda+1-\lambda\,(1+t_i^c)^{-k}}-(1+t_i^c)^{-k}\right)\\ +2(\log(1-F(t_i)&+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)-(\lambda+\beta)\,(1+t_i^c)^{-k}\\ &\times(-\frac{(1+t_i^c)^{-k-1}\,kt_i^c\log(t_i)}{\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)}+\frac{\left(2-(1+t_i^c)^{-k}\right)\lambda\,(1+t_i^c)^{-k-1}\,kt_i^c\log(t_i)}{\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)^2}\\ &\left.+(1+t_i^c)^{-k-1}\,kt_i^c\log(t_i)\right)\end{aligned} \tag{2.28}$$

$$\frac{\partial z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{k}} = 2\sum_{i=1}^{n}\left(-\frac{\lambda\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)}{2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}} + (\lambda+\beta)\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)\right) \quad (2.29)$$

$$\times\left(\frac{1}{1+\lambda}+1-\frac{2-\left(1+t_i^c\right)^{-k}}{2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}}-\left(1+t_i^c\right)^{-k}\right) \quad (2.30)$$

$$+2(\log(1-F(t_i)+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)-(\lambda+\beta)\left(1+t_i^c\right)^{-k} \quad (2.31)$$

$$\left(-\frac{\left(1+t_i^c\right)^{-k}\log\left(1-t_i^c\right)}{2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}}+\frac{\left(2-\left(1+t_i^c\right)^{-k}\right)\lambda\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)}{\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)^2}+\left(1+t_i^c\right)^{-k}\log\left(1+t_i^c\right)\right) \quad (2.32)$$

$$\frac{\partial z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{\beta}} = \frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{c}} \quad (2.33)$$

$$\frac{\partial z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{\lambda}} = \frac{\partial z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{c}} \quad (2.34)$$

$$\frac{\partial z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{c}} = 2\sum_{i=1}^{n}\left(-\frac{\lambda\left(1+t_i^c\right)^{-k-1}kt_i^c\log(t_i)}{\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)}+(\lambda+\beta)\left(1+t_i^c\right)^{-k-1}kt_i^c\log(t_i)\right)^2$$
$$+2(\log(1-F(t_i)+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)-(\lambda+\beta)\left(1+t_i^c\right)^{-k}$$
$$\times\left(\frac{\lambda\left(1+t_i^c\right)^{-k-2}k^2\left(t_i^c\right)^2\log(t_i)^2}{\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)}-\frac{\lambda\left(1+t_i^c\right)^{-k-1}kt_i^c\log(t_i)^2}{\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)}\right.$$
$$+\frac{\lambda\left(1+t_i^c\right)^{-k-2}k\left(t_i^c\right)^2\log(t_i)^2}{\left(2\lambda+1-\lambda\left(1+t_i^c\right)^{-k}\right)}+\frac{\lambda^2\left(\left(1+t_i^c\right)^{-k-1}\right)^2k^2\left(t_i^c\right)^2\log(t_i)^2}{\left(2\lambda+1-\lambda\cdot\left(1+t_i^c\right)^{-k}\right)^2}$$
$$-(\lambda+\beta)\left(1+t_i^c\right)^{-k-2}k^2\left(t_i^c\right)^2\log(t_i)^2+(\lambda+\beta)\left(1+t_i^c\right)^{-k-1}kt_i^c\log(t_i)^2$$
$$\left.-(\lambda+\beta)\left(1+t_i^c\right)^{-k-2}k\left(t_i^c\right)^2\log(t_i)^2\right) \quad (2.35)$$

$$\frac{\partial z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{k}} = 2\sum_{i=1}^{n}\left(-\frac{\lambda\,(1+t_i^c)^{-k}\,kt_i^c\log(t_i)}{(1+t_i^c)\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)}+\frac{(\lambda+\beta)\,(1+t_i^c)^{-k}\,kt_i^c\log(t_i)}{1+t_i^c}\right)$$
$$\times\,(-\frac{\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\log(1+\mathrm{t_i}^{\mathrm{c}})}{2\lambda+1-\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}}+(\lambda+\beta)\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\log(1+\mathrm{t_i}^{\mathrm{c}}))$$
$$+2(\log(1-F(t_i)+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)-(\lambda+\beta)\,(1+t_i^c)^{-k}$$
$$\left(\times\frac{\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\log(1+\mathrm{t_i}^{\mathrm{c}})\,\mathrm{kt_i}^{\mathrm{c}}\log(\mathrm{t_i})}{(1+\mathrm{t_i}^{\mathrm{c}})\left(2\lambda+1-\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\right)}-\frac{\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\,\mathrm{t_i}^{\mathrm{c}}\log(\mathrm{t_i})}{(1+\mathrm{t_i}^{\mathrm{c}})\left(2\lambda+1-\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\right)}\right.$$
$$+\frac{\lambda^2\left((1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\right)^2\mathrm{kt_i}^{\mathrm{c}}\log(\mathrm{t_i})\log(1+\mathrm{t_i}^{\mathrm{c}})}{(1+\mathrm{t_i}^{\mathrm{c}})\left(2\lambda+1-\lambda\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\right)^2}$$
$$\left.-\frac{(\lambda+\beta)\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\log(1+\mathrm{t_i}^{\mathrm{c}})\,\mathrm{kt_i}^{\mathrm{c}}\log(\mathrm{t_i})}{1+\mathrm{t_i}^{\mathrm{c}}}+\frac{(\lambda+\beta)\,(1+\mathrm{t_i}^{\mathrm{c}})^{-\mathrm{k}}\,\mathrm{t_i}^{\mathrm{c}}\log(\mathrm{t_i})}{1+\mathrm{t_i}^{\mathrm{c}}}\right) \tag{2.36}$$

$$\frac{\partial z_4(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{\beta}} = \frac{\partial z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{k}} \tag{2.37}$$

$$\frac{\partial z_4(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{\lambda}} = \frac{\partial z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{k}} \tag{2.38}$$

$$\frac{\partial z_4(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{c}} = \frac{\partial z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{k}} \tag{2.39}$$

$$\frac{\partial z_4(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})}{\partial \hat{k}} = 2\sum_{i=1}^{n}\left(-\frac{\lambda\,(1+t_i^c)^{-k}\log(1+t_i^c)}{2\lambda+1-\lambda\,(1+t_i^c)^{-k}}+(\lambda+\beta)\,(1+t_i^c)^{-k}\log(1+t_i^c)\right)^2$$
$$+2(\log(1-F(t_i)+\log(1+\lambda)+\lambda+\beta-\log\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)-(\lambda+\beta)\,(1+t_i^c)^{-k}$$
$$\left(\frac{\lambda\,(1+t_i^c)^{-k}\log(1+t_i^c)^2}{2\lambda+1-\lambda\,(1+t_i^c)^{-k}}+\frac{\lambda^2\left((1+t_i^c)^{-k}\right)^2\log(1+t_i^c)^2}{\left(2\lambda+1-\lambda\,(1+t_i^c)^{-k}\right)^2}\right.$$
$$\left.-(\lambda+\beta)(1+t_i^c)^{-k}\log(1+t_i^c)^2\right) \tag{2.40}$$

Also, the Jacobian matrix here must be non - singular and symmetric matrix.
Now,

$$\begin{bmatrix}\hat{\beta}_{k+1}\\ \hat{\lambda}_{k+1}\\ \hat{c}_{k+1}\\ \hat{k}_{k+1}\end{bmatrix}=\begin{bmatrix}\hat{\beta}_{k}\\ \hat{\lambda}_{k}\\ \hat{c}_{k}\\ \hat{k}_{k}\end{bmatrix}-J_{k_2}^{-1}\begin{bmatrix}z_1(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})\\ z_2(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})\\ z_3(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})\\ z_4(\hat{\beta},\hat{\lambda},\hat{c},\hat{k})\end{bmatrix} \tag{2.41}$$

By applying this equation (2.41), we get estimates for parameters of the WLBD by ordinary least squares method.
Then, error term is formulated as:

$$\begin{bmatrix} \varepsilon_{k+1}(\hat{\beta}) \\ \varepsilon_{k+1}(\hat{\lambda}) \\ \varepsilon_{k+1}(\hat{c}) \\ \varepsilon_{k+1}(\hat{k}) \end{bmatrix} = \left[\begin{bmatrix} \hat{\beta}_{k+1} \\ \hat{\lambda}_{k+1} \\ \hat{c}_{k+1} \\ \hat{k}_{k+1} \end{bmatrix} - \begin{bmatrix} \hat{\beta}_{k} \\ \hat{\lambda}_{k} \\ \hat{c}_{k} \\ \hat{k}_{k} \end{bmatrix} \right] \tag{2.42}$$

Where $\hat{\beta}_0, \hat{\lambda}_0, \hat{c}_0$ and $\hat{k}_0$ are the initial values which are assumed, also $\varepsilon_{k+1}(\hat{\beta})$, $\varepsilon_{k+1}(\hat{\lambda})$, $\varepsilon_{k+1}(\hat{c})$ and $\varepsilon_{k+1}(\hat{k})$ are assumed too.

3. Multiple Regression Least Squares Method (MRLSM)

Multiple Regression Least Squares Method(MIRISM):
The idea of this method is to minimize the sum of squared diffrences between observed simble values and the estimated expected values by linear approximation. We use technique which is proposcued from (Kindermann, Lariccia)(1983)[2] to conversion of non-linear model to linear model and using the estimation of the Multiple Linear Regression model parameters.

By using the following equation:

$$E\left(\hat{F}\left(t_{(i)}\right)\right) = F\left(t_{(i)}\right) \tag{3.1}$$

$\hat{F}\left(t_{(i)}\right)$ is unbiased estimate for distribution function $F\left(t_{(i)}\right)$ and replacement
$F\left(t_{(i)}\right)$ by the plotting position formula:

$$P_i = \frac{i}{n+1}, i = 1, 2, \ldots, n \tag{3.2}$$

By using the distribution function of Weibull Lindley Buree VII distribution , which are as follows [5]:

$$F\left(t_i\right) = 1 - \left(\frac{1 + 2\lambda - \lambda(1 + t_i^c)^{-k}}{\lambda + 1}\right) \exp(-(\lambda + \beta)) \exp((\lambda + \beta)(1 + t_i^c)^{-k}) \tag{3.3}$$

$$1 - \left[F\left(t_i\right)\right] = \left(\frac{1 + 2\lambda - \lambda(1 + t_i^c)^{-k}}{\lambda + 1}\right) \exp(-(\lambda + \beta)) \exp((\lambda + \beta)(1 + t_i^c)^{-k}) \tag{3.4}$$

By taking the logarithm of above equation getting:

$$\begin{aligned} \log\left[1 - \left\{F\left(t_i\right)\right\}\right] &= \log\left[\left(\frac{1 + 2\lambda - \lambda(1 + t_i^c)^{-k}}{\lambda + 1}\right) \exp(-(\lambda + \beta)) \exp((\lambda + \beta)(1 + t_i^c)^{-k})\right] \\ &= -\log(\lambda + 1) - (\lambda + \beta) + \log(1 + 2\lambda - \lambda(1 + t_i^c)^{-k}) + (\lambda + \beta)(1 + t_i^c)^{-k} \end{aligned} \tag{3.5}$$

Now, taking partial derivatives of equation (3.5) w.r.t β, λ, c and k respectively:

$$\frac{\partial}{\partial \beta} \log\left[1 - F\left(t_{(i)}\right)\right] = -1 + (1 + t_i^c)^{-k} \tag{3.6}$$

$$\frac{\partial}{\partial\lambda}\log\left[1-F\left(t_{i(i)}\right)\right]=-\frac{1}{\lambda+1}-1+\frac{2-\left(1+t_i^c\right)^{-k}}{1+2\lambda-\lambda\left(1+t^c\right)^{-k}}+\left(1+t_i^c\right)^{-k} \tag{3.7}$$

$$\frac{\partial}{\partial c}\log\left[1-F\left(t_{(i)}\right)\right]=\frac{\lambda\left(1+\mathrm{t_i}^{\mathrm{c}}\right)^{-\mathrm{k}}\mathrm{kt_i}^{\mathrm{c}}\log(\mathrm{t}_i)}{\left(1+\mathrm{t_i}^{\mathrm{c}}\right)\left(1+2\lambda-\lambda\left(1+\mathrm{t_i}^{\mathrm{c}}\right)^{-\mathrm{k}}\right)}-\frac{(\lambda+\beta)\left(1+\mathrm{t_i}^{\mathrm{c}}\right)^{-\mathrm{k}}\mathrm{kt_i}^{\mathrm{c}}\log(\mathrm{t_i})}{1+\mathrm{t_i}^{\mathrm{c}}} \tag{3.8}$$

$$\frac{\partial}{\partial k}\log\left[1-F\left(t_{(i)}\right)\right]=\frac{\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\log\left(1+\mathrm{t}_i^{\mathrm{c}}\right)}{1+2\lambda-\lambda\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}}-(\lambda+\beta)\left(1+\mathrm{t}_i^{\mathrm{c}}\right)^{-\mathrm{k}}\log\left(1+\mathrm{t_i}^{\mathrm{c}}\right) \tag{3.9}$$

By estimating the first order Taylor series [11]
about $\theta=(\beta_0,\lambda_0,c_0,k_0)$

$$f(\theta+h)=f(\theta)+h_1f_1'(\theta)+h_2f_2'(\theta)+h_3f_3'(\theta)+f_4'(\theta) \tag{3.10}$$

Comparing the above equation with (3.5) we get:

$$f(\theta+h)=E\left(\log\left[1-F\left(t_{(i)}\right)\right]\right) \tag{3.11}$$

$$f(\theta)=-\log(\lambda_0+1)-(\lambda_0+\beta_0)+\log(1+2\lambda_0-\lambda_0(1+t_i^{c_0})^{-k_0})+(\lambda_0+\beta_0)(1+t_i^{c_0})^{-k_0} \tag{3.12}$$

$$f_1'(\theta)=-1+(1+t_i^{c_0})^{-k_0} \tag{3.13}$$

$$f_2'(\theta)=-\frac{1}{\lambda_0+1}-1+\frac{2-\left(1+t_i^{c_0}\right)^{-k_0}}{1+2\lambda_0-\lambda_0\left(1+t^{c_0}\right)^{-k_0}}+\left(1+t_i^{c_0}\right)^{-k_0} \tag{3.14}$$

$$f_3'(\theta)=\frac{\lambda_0\left(1+\mathrm{t_i}^{c_0}\right)^{-\mathrm{k_0}}\mathrm{k_0t_i}^{c_0}\log(\mathrm{t}_i)}{\left(1+\mathrm{t_i}^{c_0}\right)\left(1+2\lambda_0-\lambda_0\left(1+\mathrm{t_i}^{c_0}\right)^{-\mathrm{k_0}}\right)}-\frac{(\lambda_0+\beta_0)\left(1+\mathrm{t_i}^{c_0}\right)^{-\mathrm{k_0}}\mathrm{k_0t_i}^{c_0}\log(\mathrm{t_i})}{1+\mathrm{t_i}^{c_0}} \tag{3.15}$$

$$f_4'(\theta)=\frac{\lambda_0\left(1+\mathrm{t}_i^{c_0}\right)^{-\mathrm{k_0}}\log\left(1+\mathrm{t}_i^{c_0}\right)}{1+2\lambda_0-\lambda_0\left(1+\mathrm{t}_i^{c_0}\right)^{-\mathrm{k_0}}}-(\lambda_0+\beta_0)\left(1+\mathrm{t}_i^{c_0}\right)^{-\mathrm{k_0}}\log\left(1+\mathrm{t_i}^{c_0}\right) \tag{3.16}$$

$$h_1=\beta-\beta_0,h_2=\lambda-\lambda_0,h_3=c-c_0,h_4=k-k_0 \tag{3.17}$$

Substitute equations from (3.11) to (3.17) in equation (3.10), we get:

$$\begin{aligned}E\left(\log\left[1-F\left(t_{(i)}\right)\right]\right)=&-\log(\lambda_0+1)-(\lambda_0+\beta_0)+\log(1+2\lambda_0-\lambda_0(1+t_i^{c_0})^{-k_0})\\&+(\lambda_0+\beta_0)(1+t_i^{c_0})^{-k_0}+(\beta-\beta_0)(-1+(1+t_i^{c_0})^{-k_0})\\&+(\lambda-\lambda_0)(-\frac{1}{\lambda_0+1}-1+\frac{2-\left(1+t_i^{c_0}\right)^{-k_0}}{1+2\lambda_0-\lambda_0\left(1+t^{c_0}\right)^{-k_0}}+\left(1+t_i^{c_0}\right)^{-k_0})\\&+(c-c_0)\left[\frac{\lambda_0\left(1+\mathrm{t_i}^{c_0}\right)^{-\mathrm{k_0}}\mathrm{k_0t_i}^{c_0}\log(\mathrm{t}_i)}{\left(1+\mathrm{t_i}^{c_0}\right)\left(1+2\lambda_0-\lambda_0\left(1+\mathrm{t_i}^{c_0}\right)^{-\mathrm{k_0}}\right)}\right.\\&\left.-\frac{(\lambda_0+\beta_0)\left(1+\mathrm{t_i}^{c_0}\right)^{-\mathrm{k_0}}\mathrm{k_0t_i}^{c_0}\log(\mathrm{t_i})}{1+\mathrm{t_i}^{c_0}}\right]\\&+(k-k_0)\left[\frac{\lambda_0\left(1+\mathrm{t}_i^{c_0}\right)^{-\mathrm{k_0}}\log\left(1+\mathrm{t}_i^{c_0}\right)}{1+2\lambda_0-\lambda_0\left(1+\mathrm{t}_i^{c_0}\right)^{-\mathrm{k_0}}}-(\lambda_0+\beta_0)\left(1+\mathrm{t}_i^{c_0}\right)^{-\mathrm{k_0}}\log\left(1+\mathrm{t_i}^{c_0}\right)\right]\end{aligned} \tag{3.18}$$

When it is selected $(\beta_0, \lambda_0, c_0, k_0) = (0, \lambda, 0, k)$ in the equation (3.18), we get:

$$\begin{aligned}\log\left[1 - F\left(t_{(i)}\right)\right] &= -\log(\lambda+1) - \lambda + \log(1 + 2\lambda - 2^{-k}\lambda) + 2^{-k}\lambda + \beta(-1 + 2^{-k}) \\ &\quad + c(\frac{2^{-k}\lambda k\log(t_i)}{2(1+2\lambda-2^{-k}\lambda)} - \frac{2^{-k}\lambda k\log(t_i)}{2}) \\ &\quad + k(\frac{2^{-k}\lambda k\log(2)}{1+2\lambda-2^{-k}\lambda} - 2^{-k}\lambda\log(2)) \\ &= -\log(\lambda+1) - \lambda + \log(1 + 2\lambda - 2^{-k}\lambda) + 2^{-k}\lambda + \beta(-1 + 2^{-k}) \\ &\quad + k(\frac{2^{-k}\lambda k\log(2)}{1+2\lambda-2^{-k}\lambda} - 2^{-k}\lambda\log(2)) \\ &\quad + c(\frac{2^{-k}\lambda k}{2(1+2\lambda-2^{-k}\lambda)} - \frac{2^{-k}\lambda k}{2})\log(t_i)\end{aligned} \tag{3.19}$$

the equation (3.19) represents multiple linear regression with one variables as following:

$$Y = b_0 + b_1 X + \varepsilon \tag{3.20}$$

where Y is the dependent variable.
X is the independent or predictor variables.
b_0 and b_1 are unknown parameters.
ε is the error term.
To estimate the parameters of this linear models which is similar method in the previous section, the computations become very complicated.Thos, it will employ the methods of linear algebra to make the computations be more efficient.
Suppose, the size of the random sample is equal n, then:

$$Y_i = b_0 + b_1 X_{i1} + \varepsilon_i, i = 1, 2, \dots, n \tag{3.21}$$

The goal in least squares regression is that minimizes the sum squer of errors:

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i^2 = \sum_{i=1}^{n}&\left[\log\left[1 - F\left(t_{(i)}\right)\right] - \log(\lambda+1) - \lambda + \log(1+2\lambda-2^{-k}\lambda)\right. \\ &+ 2^{-k}\lambda + \beta(-1+2^{-k}) + k(\frac{2^{-k}\lambda k\log(2)}{1+2\lambda-2^{-k}\lambda} - 2^{-k}\lambda\log(2)) \\ &+ c(\frac{2^{-k}\lambda k}{2(1+2\lambda-2^{-k}\lambda)} - \frac{2^{-k}\lambda k}{2})\log(t_i)]^2\end{aligned} \tag{3.22}$$

Then with $\vec{Y} = [Y_1, Y_2, \dots, Y_n]^T, \vec{\varepsilon} = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]^T, \vec{B} = [\mathrm{b}_0,\ \mathrm{b}_1,]^T$, and

$X = \begin{pmatrix} 1 & \log\left(t_{(1)}\right) \\ 1 & \log\left(t_{(2)}\right) \\ \vdots & \vdots \\ \vdots & \\ 1 & \log\left(t_{(n)}\right) \end{pmatrix}$ Then, in the case that $X^r X$ is invertible, the least squares estimator for the coefficient vector $\vec{B}$, is given by:

$$\hat{B} = \left(X^T X\right)^{-1} X^T Y \tag{3.23}$$

Comparing the equation (3.21) with (3.19) we get:

$$Y_i = E\left(\log\left[1 - F\left(t_{(i)}\right)\right]\right) = \log\left[1 - P_i\right] \tag{3.24}$$

$$\begin{aligned} b_0 =& -\log(\lambda+1) - \lambda + \log(1+2\lambda-2^{-k}\lambda) \\ & + 2^{-k}\lambda + \beta(-1+2^{-k}) + k\left(\frac{2^{-k}\lambda k\log(2)}{1+2\lambda-2^{-k}\lambda} - 2^{-k}\lambda\log(2)\right) \end{aligned} \tag{3.25}$$

$$b_1 = c\left(\frac{2^{-k}\lambda k}{2(1+2\lambda-2^{-k}\lambda)} - \frac{2^{-k}\lambda k}{2}\right)\log(t_i)]^2 \tag{3.26}$$

$$X_i = \log\left(t_{(i)}\right) \tag{3.27}$$

$$\begin{aligned} \varepsilon_i =& \log\left[1 - F\left(t_{(i)}\right)\right] - \log(\lambda+1) - \lambda + \log(1+2\lambda-2^{-k}\lambda) \\ & + 2^{-k}\lambda + \beta(-1+2^{-k}) + k\left(\frac{2^{-k}\lambda k\log(2)}{1+2\lambda-2^{-k}\lambda} - 2^{-k}\lambda\log(2)\right) \\ & + c\left(\frac{2^{-k}\lambda k}{2(1+2\lambda-2^{-k}\lambda)} - \frac{2^{-k}\lambda k}{2}\right)\log(t_i) \end{aligned} \tag{3.28}$$

and

$$\hat{B} = \begin{bmatrix} -\log(\hat{\lambda}_{\text{MRLS}}+1) - \hat{\lambda}_{\text{MRLS}} + \log(1+2\hat{\lambda}_{\text{MRLS}} - 2^{-k}\hat{\lambda}_{\text{MRLS}} + 2^{-k}\lambda + \beta(-1+2^{-k}) + \\ \\ \hat{k}_{MRLS}\left(\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}\log(2)}{1+2\hat{\lambda}_{MRLS} - 2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}} - 2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\log(2)\right) \\ \hat{c}_{MRLS}\left(\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}}{2(1+2\hat{\lambda}_{MRLS} - 2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS})} - \frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}}{2}\right) \end{bmatrix}, \tag{3.29}$$

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}, \vec{Y} = \begin{bmatrix} -\log\left[1-P_1\right] \\ -\log\left[1-P_2\right] \\ \vdots \\ -\log\left[1-P_n\right] \end{bmatrix} \tag{3.30}$$

Substitute in equation (3.23), then the general formula for estimating the parameters is:

$$\begin{bmatrix} -\log(\hat{\lambda}_{\text{MRLS}}+1)-\hat{\lambda}_{\text{MRLS}}+\log(1+2\hat{\lambda}_{\text{MRLS}}-2^{-k}\hat{\lambda}_{\text{MRLS}}+2^{-k}\lambda+\beta(-1+2^{-k})+ \\ \hat{k}_{MRLS}(\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}\log(2)}{1+2\hat{\lambda}_{MRLS}-2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}}-2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\log(2)) \\ \hat{c}_{MRLS}(\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}}{2(1+2\hat{\lambda}_{MRLS}-2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS})}-\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}}{2}) \end{bmatrix} =$$

$$\left(\begin{bmatrix} 1 & 1 & \dots & 1 \\ \log\left(t_{(1)}\right) & \log\left(t_{(2)}\right) & \dots & \log\left(t_{(n)}\right) \end{bmatrix}\begin{bmatrix} 1 & \log\left(t_{(1)}\right) \\ 1 & \log\left(t_{(2)}\right) \\ \vdots & \vdots \\ \vdots & \\ 1 & \log\left(t_{(n)}\right) \end{bmatrix}\right)^{-1}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \log\left(t_{(1)}\right) & \log\left(t_{(2)}\right) & \cdots & \log\left(t_{(n)}\right) \end{bmatrix}\begin{bmatrix} -\log\left[1-P_1\right] \\ -\log\left[1-P_2\right] \\ \vdots \\ -\log\left[1-P_n\right] \end{bmatrix} \tag{3.31}$$

and

$$\begin{bmatrix} -\log(\hat{\lambda}_{\text{MRLS}}+1)-\hat{\lambda}_{\text{MRLS}}+\log(1+2\hat{\lambda}_{\text{MRLS}}-2^{-k}\hat{\lambda}_{\text{MRLS}}+2^{-k}\lambda+\beta(-1+2^{-k})+ \\ \hat{k}_{MRLS}(\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}\log(2)}{1+2\hat{\lambda}_{MRLS}-2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}}-2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\log(2)) \\ \hat{c}_{MRLS}(\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}}{2(1+2\hat{\lambda}_{MRLS}-2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS})}-\frac{2^{-\hat{k}_{MRLS}}\hat{\lambda}_{MRLS}\hat{k}_{MRLS}}{2}) \end{bmatrix} = \tag{3.32}$$

$$\begin{bmatrix} n & \sum_{i=1}^{n}\log\left(t_{ii}\right)) & \sum_{i=1}^{n} t_{(i)} \\ \sum_{i=1}^{n}\log\left(t_{(i)}\right) & \sum_{i=1}^{n}\left[\log\left(t_{(i)}\right)\right]^2\left(t_{ii}\right)\log\left(t_{(i)}\right) & \sum_{i=1}^{n}\left(t_{(i)}\right)^2 \end{bmatrix}^{-1}$$

$$\times\begin{bmatrix} -\sum_{i=1}^{n}\log\left[1-P_i\right] \\ -\sum_{i=1}^{n}\log\left(t_{(i)}\right)\log\left[1-P_i\right] \end{bmatrix} \tag{3.33}$$

4. Conclusion

In this chapter ; we estimate of the parameters of WLBD by using the ordinary least squares Methods (OLSM) and Multiple regression least squares Methods (MRLSM).In this approach, we approximate survival estimate with respect both ordinary least square estimator and multiple regression estimator.

References

[1] M. A. Al-Fawzan, *Methods for estimating the parameters of the Weibull distribution.* King Abdulaziz City for Science and Technology, Saudi Arabia (2000).

[2] H. ALkanani and M. A. Abbas , *the non Bayesian estimation s method for parameters of exponentiated weibull (Ew) distribution InternationalJournal of Mathematics and statistic*, 2(5)(2014)81-94.

[3] H. Al - Nachawati and S. E. Abu - Youssef , *A Bayesian Analysis of Order Statistics from the Generalized Rayleigh Distribution*, Applied Mathematics Sciences, 3(27)(2009):1315-1325.

[4] A. A. Al- Naqeeb , *Suggested Method of Location & Scale Parameters Estimates for Rayleigh distribution According to the Expected Value of the Standardized Order Statistics by Simulation*, Al-Tiqani Journal, 23(6)(2010):1-14.

[5] S. F. Ateya, *Estimation under modified Weibull distribution based on right censored generalized order statistics*. Journal of Applied Statistics, 40(12),(2013) 2720-2734.

[6] S. H. Dhwyia , F. A. Faten , A. I. Nathier and A. A. Hani ,*Proposed Methods for Estimating Parameters of the Generalized Rayleigh Distribution in the Presence of One Outlier*, American Journal of Mathematics and Statistics, 2(6)(2012):178-183.

[7] N.L. Johnson , S. Kotz, & N. Balakrishnan, *Continuous Univariate Distributions*, 2nd ed, John Wiley and Sons, Inc., New York,USA, Volume 1, (1994), 1-77, .

[8] D. Kundu and M. Z. Raqab , *Generalized Rayleigh Distribution: Different Methods of Estimation*,Int. J. Modern Math. Sci. 2014, 10(2):103-115

[9] J. Ling, J. Pan, . *A new method for selection of population distribu- tion and parameter estimation*. Reliability Engineering System Safety, 60(3),(1998) 247-255.

[10] D. C. Montgomery, G. C. Runger, *Applied statistics and probability for engineers*. John Wiley Sons (2010).

[11] E. Parzen, *Nonparametric statistical data modeling*. Journal of the American statistical association, 74(365),(1979) 105-121.

[12] F. Parvin , A. Ali and J. K. Hossein , *Estimating* $R = P(Y < X)$ *in the Generalized Rayleigh Distribution with Different Scale Parameters*, Applied Mathematics Sciences, 7(2)(2013):87-92.

[13] J. G. Surles, and W. J. Padgett, *Inference for Reliability and Stress - Strength for a Scaled Burr Type X Distribution*, Lifetime Data Analysis, 7(2001):187-200.