Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1099-1107
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Study of the dual ideal of KU-algebra

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this work, we apply the notion of a filter of a KU-Algebra and investigate several properties. The paper defined some filters such as strong filter, n-fold filter and P-filter and discussed a few theorems and examples.

Keywords: KU-algebra, filter KU-algebra, strong filter, P-filter.
2010 MSC: Primary: 20F05; Secondary: 05C05.

1. Introduction

The algebraic structure named BCI /BCK-algebras is come in by Imai and Iseki in 1966[3]. The idea of BCI-algebra which was a popularization of a BCK-algebra presented it by Iseki [4]. Komori [1] introduced the notion of a BCC-algebras ideal and a filter in a BCI-algebra thoughtful by Hoo, in 1991 [2]. A new algebraic named KU-algebra was introduced by Prabpayak and Leerawat which involved a suggestion of the notion of homomorphisms of KU-algebras and a scrutinization of some properties of relevance [6],[7]. This work is introducing the notion of the filter in a KU-Algebra and studying some related properties.

2. Preliminaries

Definition 2.1. *[6] A KU-algebra is an algebra $(X, *, 0)$, where X is a nonempty set, $*$ is a binary operation and 0 is a constant, satisfying the following axioms:for all $x, y, z \in X$:*

i. $((\varpi \diamond \mu) \diamond [(\mu \diamond z)) \diamond (\varpi \diamond z)] = 0,$

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Received: March 2021 *Revised:* May 2021

ii. $\varpi \diamond 0 = 0$,

iii. $0 \diamond \varpi = \varpi$,

iv. $\varpi \diamond \mu$ *and* $\mu \diamond \varpi$ *imply* $\varpi = \mu$.

v. $\varpi \diamond \varpi = 0$,

Definition 2.2. *[5] A nonempty subset* S *of a KU-algebra* Π *is called a KU-subalgebra of* Π *if* $\varpi \diamond \mu \in S, \forall \varpi, \mu \in S$.

Definition 2.3. *[7] Let* $(\Pi, \diamond, 0)$ *and* $(¥, \diamond', 0')$ *be KU-algebras. The mapping* $f : \Pi \to ¥$ *is named a Homomorphism map., if* $f(\varpi \diamond \mu) = f(\varpi) \diamond' f(\mu)$*, for any* $\varpi, \mu \in \Pi$*. A homomorphism map is called a monomorphism (resp., epimorphism) if it is an injective (resp, surjective). An isomorphism map is a bijective homomorphism map.*

3. Main results

Some concepts of filters of a KU-algebra are defined in this part, and a few advantages of these concepts are investigated.

Definition 3.1. *Let* $(\Pi, \diamond, 0)$ *be a KU-algebra and a nonempty subset* I *of* Π*. Then* I *is named a filter of* Π*, if:*

(F_1) *If* $\varpi \in \Im$ *and* $\mu \in \Im$ *then* $y * (y * x) \in F$ *and* $x * (x * y) \in F$.

(F_2) *If* $\varpi \in \Im$ *and* $\varpi \diamond \mu = 0$ *then* $\mu \in \Im\ \forall\, \mu \in \Pi$

Example 3.2. *Let* $\Pi = \{0, 1, 2, 3\}$*. Define* $\diamond$ *as follows:-*

$\diamond$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

The triple $(\Pi, \diamond, 0)$ *is a KU-algebra and the set* $\Im = \{0, 2\}$ *is a filter in* Π.

Proposition 3.3. *Let* $(\Pi, \diamond, 0)$ *be a KU-algebra and* $\{\Im_\alpha, \alpha \in \Gamma\}$ *be a family of filters of* Π*. Then* $\bigcap_{\alpha\in\Gamma} \Im$ *is a filter of* Π.

Proof . Let $\varpi, \mu \in \bigcap_{\alpha\in\Gamma} \Im$, then $\varpi, \mu \in \Im, \forall \alpha \in \Gamma$. Since $\Im_\alpha$ is a filter of $\Pi, \forall \alpha \in \Gamma$.
Hence $(\mu \diamond \varpi) \diamond \varpi \in \Im_\alpha$ *and* $(\varpi \diamond \mu) \diamond \mu \in \Im_\alpha\ \forall\, \Gamma$.
Then $(\mu \diamond \varpi) \diamond \varpi \in \bigcap_{\alpha\in\Gamma} \Im$ *and*$(\varpi \diamond \mu) \diamond \mu \in \bigcap_{\alpha\in\Gamma} \Im$.
Now, let $\mu \in \bigcap_{\alpha\in\Gamma} \Im\ \mu \diamond \varpi = 0$. Then $\mu \in \Im_\alpha$,$\forall \alpha \in \Gamma$ and since $\mu \in \Im_\alpha$ is a filter of Π, $\alpha \in \Gamma$, then $\varpi \in \mu \in \Im_\alpha\ \forall \alpha \in \Gamma$. It follows that $\varpi \in \bigcap_{\alpha\in\Gamma} \Im$. □

Remark 3.4. *Generally, the union of filters of a KU-algebra* Π *is not a filter an isomorphism as shown in example below.*

Example 3.5. *Let* Π*= 0, 1, 2, 3, 4 be a set. Define* $\diamond$ *as follows::*

$\diamond$	*0*	*1*	*2*	*3*	*4*
0	*0*	*1*	*2*	*3*	*4*
1	*0*	*0*	*1*	*0*	*1*
2	*0*	*3*	*0*	*3*	*4*
3	*0*	*1*	*1*	*0*	*1*
4	*0*	*0*	*0*	*0*	*0*

The triple $(\Pi, \diamond, 0)$ *is a KU-algebra and the sets* $L_1 = \{0, 2\}$ *and* $L_2 = \{0, 3\}$ *are two filters of* Π*, the union of the filters but the union of* $2, 3 \in L_1 \bigcup L_2$*, but* $3 \diamond (3 \diamond 2) = 1 \notin L_1 \bigcup L_2$

Proposition 3.6. *Let* $(\Pi, \diamond, 0)$ *be a filter and let* $\{\Im_\alpha, \alpha \in \Gamma\}$ *be a chain of filter of* Π*. Then* $\bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$ *is a filter of* Π*.*

Proof . Let$\varpi, \mu \in \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha \forall\alpha \in \Gamma$ Then there exist $\Im_e, \Im_d \in \{\Im_\alpha\}\alpha \in \Gamma$ such that $\varpi \in \Im_e$ and $\mu \in \Im_d$ So, either $\Im_e \subseteq \Im_d$ or $\Im_d \subseteq \Im_e$ If $\Im_e \subseteq \Im_d$, then $\varpi \in \Im_e$ and $\mu \in \Im_d$ Since $\Im_d$ is a filter of Π, then $(\mu \diamond \varpi) \diamond \varpi \in \Im_d$ and $(\varpi \diamond \mu) \diamond \mu \in \Im_d$.
Similarly, if $\Im_d \subseteq \Im_e$. Then$(\mu \diamond \varpi) \diamond \varpi, (\varpi \diamond \mu) \diamond \mu \in \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$. Now Let $\mu \in \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$ such that $\mu \diamond \varpi = 0$. Then there exists $e \in$ such that $\mu \in \Im_e$. Since$\Im_e$ is a filter of Π, hence $\varpi \in \Im_e$. Thus $\varpi \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$. □

Definition 3.7. *Let* $(\Pi, \diamond, 0)$ *be a KU-algebra, and* $\Im$*be a filter of* Π*. Then I is named a completely closed filter denoted by* ***C.C.F*** *if :* $\varpi \diamond \mu \in \Im$ *, for all* $\varpi, \mu \in \Im$*.*

Example 3.8. *Consider* $\Pi = \{0, 1, 2, 3\}$ *in Example 3.2,* $\Im = \{0, 1, 2\}$ *is a completely closed filter of* Π

Proposition 3.9. *Let* $(\Pi, \diamond, 0)$ *be a KU-algebra, and* $\Im$*is a completely closed filter. Then* $0 \in \Im$*.*

Proof . Let $(\Pi, \diamond, 0)$ be a KU-algebra, Let $\Im$ be a completely closed filter, and $\varpi \in \Im$, it felloes that $\varpi \diamond \varpi \in F$[Since $\Im$ is a completely closed filter. By definition3.7]. Therefor $0 \in \Im$ [Since $\varpi \diamond \varpi = 0$. By definition2.1(v)]. □

Theorem 3.10. *Let* $f : (\Pi, \diamond, 0) \to (¥, \diamond', 0')$ *be a monomorphism from KU-algebras* $(\Pi, \diamond, 0)$ *into* $(¥, \diamond', 0')$ *and* $\Im$ *be a filter of* Π*. Then the image* $f(\Im)$ *is a filter of* ¥*.*

Proof . Let $\Im$ be a filter of Π .

(i) Let $\varpi, \mu \in f(\Im)$. Then there exist c, d$\in \Im$ such that $\varpi = f(c), \mu = f(d)$: Then $(\varpi \diamond' \mu) \diamond' \mu = (f(c) \diamond' f(d)) \diamond' f(d) = (f(c \diamond d)) \diamond' f(d) = f((c \diamond d) \diamond d)$ and since Then $(c \diamond d) \diamond d \in I$, then $(\varpi \diamond' \mu) \diamond' \mu \in f(\Im)$. Similarly, $(\mu \diamond' \varpi) \diamond' \varpi \in f(\Im)$.

(ii) Let $\mu \in f(\Im)$ such that $\mu \diamond' \varpi = 0'$. Then there exist $c \in \Im andd \in \Im$ such that $\varpi = f(c)$ and $\mu = f(d)$. Now, $\mu \diamond' \varpi = f(d) \diamond' f(c) = f(d \diamond c) = 0' = f(0)$ and since f is an injective, then $d \diamond c = 0$. Thus, $c \in \Im$, [by definition3.1]. So, $\varpi = f(c) \in f(F)$. Therefore $f(\Im)$ is a filter of ¥.

□

Theorem 3.11. *Let* $f : (\Pi, \diamond, 0) \to (¥, \diamond', 0')$ *be an epimorphism from KU-algebras* $(\Pi, \diamond, 0)$ *into* $(¥, \diamond', 0')$ *If* $\Im$ *is a filter of* ¥ *. Then* $f^{-1}(\Im)$ *is a filter of* Π

Proof . Suppose that $\Im$ is a filter of Π.
(i) Let $\varpi, \mu \in f^{-1}(\Im)$, it follows that $f(\varpi), f(\mu) \in \Im$ and since $\Im$ is a filter of ¥, then $(f(\varpi) \diamond' (f(\mu)) \diamond' f(\mu)) \in \Im$. Thus, $f(\varpi \diamond \mu) \diamond' f(\mu)) = f(\varpi \diamond \mu) \diamond \mu) \in \Im$, then $(\varpi \diamond \mu) \diamond \mu \in f^{-1}(\Im)$. Similarly$(\mu \diamond \varpi) \diamond \varpi \in f^{-1}(\Im)$
(ii) Let $\mu \in f^{-1}(\Im)$ such that$\mu \diamond \varpi = 0$, then $f(\mu) \in \Im$. we have $f(\mu) \diamond' f(\varpi) = f(\mu \diamond \varpi) = f(0) = 0'$, and since I is a filter of Π . Then $f(\varpi) \in \Im$. Hence $\varpi \in f^{-1}(\Im)$. Hence $f^{-1}(\Im)$ is a filter of Π. □

Theorem 3.12. *Let $\{(\Pi, \diamond_\alpha, 0_\alpha) : \alpha \in \Gamma\}$ be a family of KU-algebras such that $\varpi_\alpha \in \Pi$. Then $(\prod_{\alpha\in\Gamma} \Pi_\alpha, \bigotimes, (0'_\alpha))$ is $\prod_{\alpha\in\Gamma}$ KU-algebras.*

Proof .

(i) Let $(\varpi_\alpha), (\mu_\alpha), (z_\alpha) \in \prod_{\alpha\in\Gamma} \forall\alpha \in \Gamma$

Then $(\varpi_\alpha) \bigotimes (\mu_\alpha)[(\mu_\alpha) \bigotimes z_\alpha) \bigotimes (\varpi_\alpha \bigotimes z_\alpha)]$ imply $(\varpi_\alpha \diamond_\alpha \mu_\alpha) \diamond_\alpha [(\mu_\alpha \diamond_\alpha z_\alpha) \diamond_\alpha (\varpi_\alpha \diamond_\alpha z_i)] = 0'_\alpha$, since is Π_i a KU-algebra

,

(ii) Let $(\varpi_\alpha) \in \prod_{\alpha\in\Gamma} \Pi_\alpha, \forall\alpha \in \Gamma$ .Then $(\varpi_\alpha) \bigotimes (0'_\alpha) = (\varpi_\alpha \diamond_\alpha 0_\alpha) = (0'_\alpha)$[By definition2.1(ii)]

(iii) Let $(\varpi_\alpha) \in \prod_{\alpha\in\Gamma} \Pi_\alpha, \forall\alpha \in \Gamma$ .Then $(0'_\alpha) \bigotimes (\varpi_\alpha) = (0_\alpha \diamond_\alpha \varpi_\alpha) = (\varpi_\alpha)$[By definition2.1(iii)]

(iv) Let $(\varpi_\alpha), (\mu_\alpha) \in \prod_{\alpha\in\Gamma} \Pi_\alpha$, such that $(\varpi_\alpha) \bigotimes (\mu_\alpha) = (0'_\alpha)$ and $(\mu_\alpha) \bigotimes (\varpi_\alpha) = (0'_\alpha)$
imply $(\varpi_\alpha \diamond_\alpha \mu_\alpha) = (0'_\alpha), and(\mu_\alpha) \diamond_\alpha \varpi_\alpha = (0'_\alpha)$. Then $(\varpi_\alpha \diamond_\alpha \mu_\alpha) = (0_\alpha), and(\mu_\alpha) \diamond_\alpha \varpi_\alpha = (0_\alpha)$
So, $\varpi_\alpha = (\mu_\alpha)$[By definition2.1(iiii)], it follows that $\varpi_\alpha = (\mu_\alpha)$.

(v) Let $(\varpi_\alpha) \in \prod_{\alpha\in\Gamma} \Pi_\alpha, \forall\alpha \in \Gamma$. Then $(\varpi\alpha) \bigotimes (\varpi\alpha) = ((\varpi\alpha) \diamond (\varpi\alpha)) \forall\alpha \in \Gamma = (0'_\alpha)[(\varpi\alpha) \diamond_\alpha (\varpi\alpha)) = 0_\alpha$ since Π_i is a KU-algebra]. Therefore,$(\prod_{\alpha\in\Gamma} \Pi_\alpha, \bigotimes, (0'_\alpha))$ is $\prod_{\alpha\in\Gamma}$ KU-algebras.

□

Theorem 3.13. $\left(\prod_{\alpha\in\Gamma} \Pi_\alpha, \bigotimes, (0'_\alpha)\right)$ *be a $\prod_{\alpha\in\Gamma} \Pi_\alpha$ KU-algebras and $\{\Im_\alpha \alpha \in \Gamma\}$ be a family of filters of Π_α. Then $\prod_{\alpha\in\Gamma} \Im_\alpha$ is a $\prod_{\alpha\in\Gamma}$ filter of the product $\prod_{\alpha\in\Gamma} \Pi_\alpha$.*

Proof .

(i) Let $\varpi = (\varpi_\alpha), \mu = (\mu_\alpha), \in \prod_{\alpha\in\Gamma} \Im_\alpha \forall(\varpi_\alpha), (\mu_\alpha) \in \Gamma_\alpha$ and $\alpha \in \Gamma$.
Now, $(\varpi \bigotimes \mu) \bigotimes \mu = ((\varpi_\alpha) \bigotimes (\mu_\alpha)) \bigotimes (\mu_\alpha) = (\varpi_\alpha \diamond_\alpha \mu_\alpha) \diamond_\alpha (\mu_\alpha) \in \prod_{\alpha\in\Gamma} \Im_\alpha$ [Since $(\varpi_\alpha \diamond_\alpha \mu_\alpha) \diamond_\alpha (\mu_\alpha) \in \Im_\alpha$, by definition3.1 (F1)],

(ii) Let $(\varpi_\alpha), \in \prod_{\alpha\in\Gamma} \Im_\alpha$such that $(\mu_\alpha) \bigotimes (\varpi_\alpha) = (0'_\alpha) \forall\alpha \in \Gamma$ $\mu_\alpha, \varpi_\alpha \in \Pi_\alpha$ Then$(\mu_\alpha \diamond \varpi_\alpha) = (0_\alpha)$ So $\mu_\alpha \in \Im_\alpha, \mu_\alpha \diamond \varpi_\alpha = 0_\alpha, \forall\alpha \in \Gamma$, Hence $\varpi_\alpha \in \Im_\alpha$, [Since $\Im_\alpha$are filters ofΠ_α], then $\varpi_\alpha \prod_{\alpha\in\Gamma} \Im_\alpha$.
Hence Π_α. Then $\prod_{\alpha\in\Gamma} \Im_\alpha$ is a $\prod_{\alpha\in\Gamma}$ filter of $\prod_{\alpha\in\Gamma} \Pi_\alpha$.

□

Definition 3.14. *The filter $\Im$ of a KU-algebra Π is named a strong filter and denoted by a **S.F** if: $z \diamond \varpi \in \Im, \mu \in \Im$ imply $z \diamond (\mu \diamond \varpi) \in \Im, \forall\varpi, z \in \Pi$.*

Example 3.15. *Let* Π= *0, 1, 2, 3, 4 be a set. Define* $\diamond$ *as follows::*

$\diamond$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

The triple $(\Pi, \diamond, 0)$ *is a KU-algebra and the set*$L = \{0, 1, 3\}$ *is a S.F of* Π.

Proposition 3.16. *Let* $(\Pi, \diamond, 0))$ *be a KU-algebra. Then any* ***S.F*** *of* π *is a filter.*

Proof . Directly by definition3.14 □

Remark 3.17. *Generally, the proposition3.16 is not correct if it is conversed as shown below.*

Example 3.18. *Let* Π= *0, 1, 2, 3, 4 be a set. Define* $\diamond$ *as follows::*

$\diamond$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	1	0	3	3
3	0	0	1	0	2
4	0	0	0	0	0

The triple $(\Pi, \diamond, 0)$ *is a KU-algebra and the set L=* $\{0,\ 2\}$ *is a filter of* π, *but not a* ***S.F*** *since z=1,* ϖ*=4,* $\mu = 2, 1 \diamond 4 = 2,\ \ 1 \diamond (2 \diamond 4) = 3 \notin L$

Proposition 3.19. *The intersection of a family of* ***S.F*** *of a KU-algebra* Π *is a S.F of* Π.

Proof . Let $\{\Im_\alpha, \alpha \in \Gamma\}$ be a family of **S.F** of . It follows that by definition 3.14
$\{\Im_\alpha, \alpha \in \Gamma\}$ is a family of filters of Π and by proposition 3.3 imply $\bigcap_{\alpha\in\Gamma} \Im$is a filter of Π.
Now, let $z \diamond \varpi \in \bigcap_{\alpha\in\Gamma} \Im\ \ \mu \in \bigcap_{\alpha\in\Gamma} \Im$ such that $\varpi, z \in \Pi$,
$\Rightarrow z \diamond \varpi \in \Im,\ \ \mu \in \Im$
$\Rightarrow z \diamond (\mu \diamond \varpi) \in \Im_\alpha, (\forall \alpha \in \Gamma)$ [Since $\Im_\alpha$ is a **S.F** of Π],
$\Rightarrow z \diamond (\mu \diamond \varpi) \in \bigcap \Im_\alpha$. Hence $\bigcap \Im_\alpha$ is a **S.F** of Π. □

Remark 3.20. *The union of a family of a* ***S.F*** *of a KU-algebra* Π *may be not a* ***S.F*** *as shown below*

Example 3.21. *Consider* $X = \{0, 1, 2, 3, 4, 5\}$ *with binary operation* $''\star''$ *defined by the following table:*

$\star$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	1	3
2	0	1	0	2	1	4
3	0	1	0	0	1	1
4	0	1	0	2	0	2
5	0	0	0	0	0	0

The subset $L_1 = \{0,2,3\}$ *and* $L_2 = \{0,2,4\}$ *are two* ***S.F*** *of* Π, *but* $L_1 \bigcup L_2 = \{0,2,3,4\}$ *is not a* ***S.F****of* Π, *since* $\varpi = 3, \mu = 4, z = 0$ *but* $(3 \diamond 4) \diamond 4 = 1 \notin L_1 \bigcup L_2$

Proposition 3.22. *Let* $(\Pi, \diamond, 0)$ *be a filter and let* $\{\Im_\alpha, \alpha \in \Gamma\}$ *be a chain of* ***S.F*** *of* X*.Then* $\bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$ *is a* ***S.F*** *of* Π.

Proof . Let $\{\Im_\alpha, \alpha \in \Gamma\}$ be a chain of **S.F** of Π. It follows that by using definition 3.14, then $\{\Im_\alpha, \alpha \in \Gamma\}$ is a chain of filters of Π and by proposition3.6, we have

$\bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$is a filter of Π . Now, let $z \diamond \varpi \in \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$, $\mu \in \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$ such that $\varpi, z \in \Pi$. Then there exist$\Im_e, \Im_d \in \{\Im_\alpha, \alpha \in \Gamma\}$ such that $z \diamond \varpi \in \Im_e$and $\mu \in \Im_d$

$\Rightarrow \Im_e \subseteq \Im_d$ or $\subseteq \Im_e$ if $\Im_e \subseteq \Im_d \Rightarrow z \diamond \varpi \in \Im_d, \mu \in \Im_d$

$\Rightarrow \exists d \in \Gamma$ such that $z \diamond (\mu \diamond \varpi) \in \Im_d$ [Since$\Im_d$ is a S.F of Π ,],

$\Rightarrow z \diamond (\mu \diamond \varpi) \in \bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$ Similarly, $\Im_d \subseteq \Im_e$, therefore $\bigcup\limits_{\alpha\in\Gamma} \Im_\alpha$ is a **S.F** of Π □

Theorem 3.23. *(*$\prod\limits_{\alpha\in\Gamma} \Pi_\alpha, \bigotimes, (0'_\alpha)$*) be a* $\prod\limits_{\alpha\in\Gamma} \Pi_\alpha$ *KU-algebras and* $\{\Im_\alpha \alpha \in \Gamma\}$ *be a family of* ***S.F*** *of* Π_α*. Then* $\prod\limits_{\alpha\in\Gamma} \Im_\alpha$ *is a*$\prod\limits_{\alpha\in\Gamma}$ ***S.F*** *of the product* $\prod\limits_{\alpha\in\Gamma} \Pi_\alpha$ *.*

Proof . Let $\{\Im_\alpha \alpha \in \Gamma\}$ be a family of **S.F** of Π_α.

Implies $\Im_\alpha \alpha \in \Gamma$ is a family of filter of [By definition3.14].

Let $\varpi = (\varpi_\alpha), \mu = (\mu_\alpha), z = (z_\alpha)$ such that $z \bigotimes \varpi \in \prod\limits_{\alpha\in\Gamma} \Im_\alpha$, $\mu \in \prod\limits_{\alpha\in\Gamma} \Im_\alpha$

Now,$(z \bigotimes \varpi) = ((z_\alpha) \bigotimes (\varpi_\alpha)), = (z_\alpha \diamond_\alpha \varpi_\alpha) \in \prod\limits_{\alpha\in\Gamma} \Im_\alpha$, $\mu_\alpha \prod\limits_{\alpha\in\Gamma} \Im_\alpha$

then $(z_\alpha \diamond_\alpha \varpi_\alpha) \in \Im_\alpha$, $\mu_\alpha \in \Im_\alpha$

$\Rightarrow z_\alpha \diamond_\alpha (\mu_\alpha \diamond \varpi_\alpha) \in \Im_\alpha$[Since $\Im_\alpha$ be a family of **S.F** of Π_α],

So,$z \bigotimes (\mu \bigotimes \varpi) = (z_\alpha) \diamond_\alpha (\mu_\alpha \diamond \varpi_\alpha)$,

$= z_\alpha \diamond_\alpha (\mu_\alpha \diamond \varpi_\alpha) \in \prod\limits_{\alpha\in\Gamma} \Im_\alpha$.

Therefore $\prod\limits_{\alpha\in\Gamma} \Im_\alpha$ is a $\prod\limits_{\alpha\in\Gamma}$ **S.F** of the product $\prod\limits_{\alpha\in\Gamma} \Pi_\alpha$ □

Theorem 3.24. *Let* $f : (\Pi, \diamond, 0) \to (¥, \diamond', 0')$ *be a monomorphism from KU-algebras* $(\Pi, \diamond, 0)$ *into* $(¥, \diamond', 0')$*and* $\Im$ *be a* ***S.F*** *of of* Π*. Then the image* $f(\Im)$ *is a* ***S.F*** *of of* ¥*.*

Proof . Let $\Im$be a **S.F** of Π and by proposition 3.16, we have I is a filter of Π, and by Theorem 3.10, then f $(\Im)$ is a filter of ¥ .

Let $z \diamond' \varpi \, \inf(\Im), \mu \in f(\Im)$,

$\Rightarrow \exists \mu, \kappa \in \Pi, \upsilon \in \Im$ such that$\varpi = f(\mu), \mu = f(\upsilon), z = f(\kappa)$,

$\Rightarrow z \diamond' \varpi = f(\kappa) \diamond' f(\mu) \in f(\Im)$,[Since $\Im$ is a **S.F** of Π],

$= f(\kappa \diamond \mu) \in f(\Im), f(\nu) \in f(\Im)$

$\Rightarrow \kappa \diamond \mu \in \Im, \nu \in \Im$

$\Rightarrow \kappa \diamond (\nu \diamond \mu) \in \Im$ [Since $\Im$ is a **S.F** of Π],

$\Rightarrow f(\kappa \diamond (\nu \diamond \mu)) \in f(\Im)$

$\Rightarrow f(\kappa) \diamond' (\nu) \diamond' f(\mu)) \in f(\Im)$

$\Rightarrow z \diamond' (\mu \diamond' \varpi) \in f(\Im)$. Therefore $f(\Im)$ is a **S.F** of of ¥ □

Theorem 3.25. *Let* $f : (\Pi, \diamond, 0) \to (¥, \diamond', 0')$ *be an epimorphism from KU-algebras* $(\Pi, \diamond, 0)$ *into* $(¥, \diamond', 0')$ *If* $\Im$ *is a* ***S.F*** *of* ¥*. Then* $f^{-1}(\Im)$ *is a* ***S.F*** *of* Π

Proof . Let $\Im$ be a **S.F** of ¥ imply it is a filter of ¥, and by Theorem3.11

$\Rightarrow f^{-1}(\Im)$ is a filter of Π

Let $z \diamond' \varpi \in f^{-1}(\Im), \mu \in f^{-1}(\Im)$,

$\Rightarrow f(z) \diamond' f(\varpi) \in \Im, f(\mu) \in \Im$

$\Rightarrow (f(z) \diamond' f(\mu)) \diamond' f(\varpi) \in \Im$[Since $\Im$ is a **S.F** of ¥],

$\Rightarrow f(z \diamond (\mu \diamond \varpi)) \in \Im$,

$\Rightarrow z \diamond (\mu \diamond \varpi) \in f^{-1}(\Im)$,

$\Rightarrow f^{-1}(\Im)$ is a **S.F** of Π. □

For any elements ϖ and μ in a KU-algebra Π, we have $\varpi^n \diamond \mu$ is denotes $\varpi \diamond (\varpi \diamond \varpi \diamond \mu)$, where ϖ occurs n times.

Definition 3.26. *Let Π be a KU-algebra and I be a filter of Π. Then $\Im$is named n-fold strong filter of Π and denoted by **n-fold S.F** if there exists a fixed natural number n such that for any $\varpi, \mu, z \in \Pi$ if $z^n \diamond \varpi \in \Im, \mu \in \Im$ impl$\mu z^n \diamond (\mu \diamond \varpi) \in \Im, \forall \varpi, z \in \Pi$.*

Remark 3.27. *The **1-fold S.F** of Π is precisely a **S.F** of Π*

Example 3.28. *Let Π = {0, 1, 2, 3, 4} be a set in example 3.5. then the filter $\Im$ = {0, 3} is a 2-fold S.F of Π.*

Proposition 3.29. *Let Π be a KU-algebra, and $\{\Im_\alpha, \alpha \in \Gamma\}$ be a family of n-fold S.F of Π. Then $\bigcap_{\alpha\in\Gamma} \Im_\alpha$ is an n-fold S. F of Π.*

Proof . It is easy. □

Proposition 3.30. *Let Π be a KU-algebra, and $\{\Im_\alpha, \alpha \in \Gamma\}$ be a chain of **of n-fold S.F of** of Π. Then $\bigcup_{\alpha\in\Gamma} \Im_\alpha$ is a **n-fold S.F of** of Π.*

Proof . It is easy. □

Theorem 3.31. *Let $f : (\Pi, \diamond, 0) \to (¥, \diamond', 0')$ be a monomorphism from KU-algebras $(\Pi, \diamond, 0)$ into $(¥, \diamond', 0')$and $\Im$ be a **n-fold S.F** of of Π. Then the image $f(\Im)$ is a **n-fold S.F** of ¥*

Definition 3.32. *Let Π be a KU-algebra and $\Im$ be a filter of Π. Then $\Im$ is named a p-filter and denoted by P.F if: for any $\varpi, \mu \in \Im$ imply $(z \diamond \varpi) \diamond (z \diamond \mu) \in \Im, \forall z \in \Pi$.*

Example 3.33. *Assume that Π= {0, 1, 2, 3, 4} be a set. Define $\diamond$ as follows::*

$\diamond$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

*The subset $\Im = \{0, 2\}$ is a **P.F** of a KU-algebra.*

Proposition 3.34. *Assume Π be a KU-algebra. Then any P.F of Π is a filter*

Proof . It is clear by definition 3.32 □

Remark 3.35. *Generally, the proposition 3.34 is not correct if it is conversed as shown below.*

Example 3.36. *Assume* Π *that KU-algebra in example(3.21).*$\Im = \{0, 2\}$ *is a filter but not a P.F of* Π *, since* $(3 \diamond 0) \diamond (3 \diamond 2) = 1 \notin \Im$.

Proposition 3.37. *Let* Π *be a KU-algebra and I be a* ***P.F*** *of* Π*. Then* $\Im$ *is a* ***C.C.F*** *of* Π

Proof . Let $\Im$ be a P.F of Π, and by definition3.34 it follows that$\Im$ is a filter of Π. Now, let $\varpi, \lambda \in \Im, z \in \Pi$, and since $\Im$ is a P.F of Π, then $(z \diamond \varpi) \diamond (z \diamond \mu) \in \Im$. Now, choose $z = 0$ it follows that $(0 \diamond \varpi) \diamond (0 \diamond \mu) = \varpi \diamond \mu \in \Im$, [By definition2.1(iii)], imply $\Im$ is a C.C.F of Π. Therefore $\Im$ is a C.C.F of Π. □

Proposition 3.38. *Let* Π *be a KU-algebra and* $\{\Im_\alpha, \alpha \in \Gamma\}$ *be a family of* ***P.F*** *of* Π*. Then* $\bigcap_{\alpha\in\Gamma} \Im$ *is a* ***P.F*** *of* Π*.*

Proof . Let $\{\Im_\alpha, \alpha \in \Gamma\}$ be a family of P.F of Π.

$\Rightarrow \{\Im_\alpha, \alpha \in \Gamma\}$ be a family filters of Π

$\Rightarrow \bigcap_{\alpha\in\Gamma} \Im$ is a filter of Π [By proposition3.3)],

Now, let $\varpi, \mu, z \in \bigcap_{\alpha\in\Gamma} \Im$. Then $\varpi, \mu, z, \in \Im_\alpha \forall\alpha \in \Gamma$ and since $\Im_\alpha$ is a P.F of Π, $\forall\alpha \in \Gamma$, then $(z \diamond \varpi) \diamond (z \diamond \mu) \in \Im_\alpha \forall\alpha \in \Gamma, \Rightarrow (z \diamond \varpi) \diamond (z \diamond \mu) \bigcap_{\alpha\in\Gamma} \Im$

$\Rightarrow \bigcap_{\alpha\in\Gamma} \Im$ is a P.F of Π. □

Theorem 3.39. $(\prod_{\alpha\in\Gamma} \Pi_\alpha, \bigotimes, (0'_\alpha))$ *be a* $\prod_{\alpha\in\Gamma} \Pi_\alpha$ *KU-algebras and* $\{\Im_\alpha \alpha \in \Gamma\}$ *be a family of* ***P.F*** *of* Π_α*. Then* $\prod_{\alpha\in\Gamma} \Im_\alpha$ *is a* $\prod_{\alpha\in\Gamma}$ ***P.F*** *of the product* $\prod_{\alpha\in\Gamma} \Pi_\alpha$ *.*

Proof . Let $\{\Im_\alpha, \alpha \in \Gamma\}$ be a family of **P.F** of Π.

$\Rightarrow \{\Im_\alpha, \alpha \in \Gamma\}$ be a family filters of Π

Let $\varpi, \mu, z, \in \prod_{\alpha\in\Gamma} \Im_\alpha$ Such that $\varpi = (\varpi_\alpha), \mu = (\mu_\alpha), z = (z_\alpha)$

By theorem 3.13,we get $\prod_{\alpha\in\Gamma} \Im_\alpha$ is a filter of Π.

Now, $(z \bigotimes \varpi) \bigotimes (z \bigotimes \mu) = ((z_\alpha) \bigotimes (\varpi_\alpha)) \bigotimes ((z_\alpha) \bigotimes (\mu_\alpha))$,

$= ((z_\alpha \diamond_\alpha \varpi_\alpha) \diamond_\alpha (z_\alpha \diamond_\alpha \mu_\alpha)) \in \prod_{\alpha\in\Gamma} \Im_\alpha$ imply $(z \bigotimes \varpi) \bigotimes (z \bigotimes \varpi) \in \prod_{\alpha\in\Gamma} \Im_\alpha$

Therefore $\prod_{\alpha\in\Gamma} \Im_\alpha$ is a $\prod_{\alpha\in\Gamma}$ **P.F** of the product $\prod_{\alpha\in\Gamma} \Pi_\alpha$ □

4. Open problems

(1) Is the positive implicative filter an implicative filter?

(2) Is the positive implicative filter a commutative filter?

(3) Is the strong positive implicative filter a strong implicative filter?

(4) Is the strong positive implicative filter a strong commutative filter?

5. Conclusions

We have studied the dual ideal of a KU-Algebra which is named the filter. And then, we discussed few results of the filter of a KU-Algebra. After that, we introduced the strong filter and n-fold strong filter. Also, we study the positive implicative filter Moreover, the product of some filters and the product of some strong filters are established. The main purpose of our future work is to investigate the fadedness of fuzzy filters with special properties such as a bipolar intuitionist (interval value) fuzzy n-fold filter in some algebras. Moreover, the filter may have a lot of applications in different branches of theoretical physics and computer science, for example, artificial intelligence, graph theory and code theory.

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