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The generalized (2 + 1) and (3+1)-dimensional with advanced analytical wave solutions via computational applications

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Abstract

The analytical solutions for an important generalized Nonlinear evolution equations NLEEs dynamical partial differential equations (DPDEs) that involve independent variables represented by the (2 + 1)-dimensional breaking soliton equation, the (2 + 1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation, and the (2 +1)-dimensional Bogoyavlenskii's breaking soliton equation (BE), and some new exact propagating solutions to a generalized (3+1)-dimensional KP equation with variable coefficients are constructed by using a new algorithm of the first integral method (NAFIM) and determined some analytical solutions by appointing special values of the parameters. In addition to that, we showed a new variety and unique travelling wave solutions by graphical illustration with symbolic computations.

Keywords: First integral method, Nonlinear evolution equations, Solitary waves solutions, Graphical representation, Symbolic computation

1. Introduction

The present research highlights, the effects of the numerical and analytical solutions to the NLEEs which have become of great importance due to the important role that solutions of these equations play in our sciences life. NLEEs describe complex phenomena which arise in the field of nonlinear sciences such as NLEEs describe complex phenomena which originated in the field of nonlinear sciences,

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the most important of them are mathematical physics, elastic media, nonlinear optics, chaos theory, viscoelasticity, engineering, and so on [4, 6, 8, 16, 26, 28]. The necessary of solitons is because of their presence in a variety of nonlinear differential equations depict many nonlinear phenomena, Industry , nonlinear optics, telecommunication, solid-state physics ,and condensed matter [30, 33]. Nonlinear phenomena are inspected in many applications such as plasma physics, hydrodynamics, biological phenomena, meteorology, quantum mechanics, oceans, etc [36]. Now it is well established that the studies of NLEEs are thriving, by mathematicians and physicists, because these equations depict and discuss the real countenance in a set an of science, engineering and technology fields. Physicists usually discuss the behaviour of the equations for the physical systems, while Mathematicians have been developed new methods for solving these equations as the modified simple equation method [19], the $(G^{'}/G)$-expansion method [11], homotopy perturbation method [31], tanh–coth method[14], first integral method [1, 12, 21], and so on [5, 29].

The formula of the generalized (2 + 1)-dimensional dynamical partial differential equation [13, 23, 24] is

$$u_{\mathrm{xt}} + a u_x u_{\mathrm{xy}} + b u_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = 0 \tag{1.1}$$

Where a and b are permeates. Newly, There are different cases of Eq. (1.1) were investigated in [2, 25].

We'll get the (2 + 1)-dim CBS equation if chosen $a = 4$ and $b = 2$ in Eq. (1.1)

$$u_{\mathrm{xt}} + 4u_x u_{\mathrm{xy}} + 2u_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = 0 \tag{1.2}$$

We'll get the (2 +1)-dimensional Bogoyavlenskii's breaking soliton equation if chosen $a = 4$ and $b = 4$ in Eq. (1.1)

$$u_{\mathrm{xt}} + 4u_x u_{\mathrm{xy}} + 4u_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = \tag{1.3}$$

We'll get the (2 +1)-dimensional breaking soliton equation if chosen $a = -4$ and $b = -2$ in Eq. (1.1)

$$u_{\mathrm{xt}} - 4u_x u_{\mathrm{xy}} - 2u_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = \tag{1.4}$$

We have another application, the (3+1)-dimensional generalized KP equation in the form

$$u_{\mathrm{xt}} + u_{\mathrm{yt}} + u_{\mathrm{xxxy}} + 3(u_x u_y)_x - u_{\mathrm{zz}} = 0 \tag{1.5}$$

This equation was investigated in [17, 18].

This paper consists of first section Preliminaries and Basic Definitions First integral method, a new algorithm of first integral method (AFIM) formula was introduced in second section. Section three includes two parts, first obtained some new and general traveling wave solutions for different types of the generalized (2+1)-dimensional NLEE. In second part some new exact propagating solutions to a generalized (3+1)-dimensional KP equation was constricted by this method. Section four is the graphical representation for all solutions were obtained presented in different figures. Results and discussions for these solutions introduced in five section. Last section represent conclusions.

2. Basic Concepts of FIM

The NPDEs of general formula as

$$w(f, f_x, f_t, f_{\mathrm{xx}}, f_{\mathrm{xt}}, \cdots) \tag{2.1}$$

Conceder $u(x, y, z, t)$ a solution of (2.1), by applying to the wave transforms:

$$u(x, y, z, t) = f(\zeta), \quad \zeta = x + y + z - ct \tag{2.2}$$

The eq. (2.1) transforms to ODE we get:

$$w\left(f, f', f'', \cdots\right), \tag{2.3}$$

A new independent variable:

$$x(\zeta) = f(\zeta), \qquad y(\zeta) = f'(\zeta), \tag{2.4}$$

the system of ODE

$$\begin{aligned} x'(\zeta) &= y(\zeta) \\ y'(\zeta) &= F\left(x(\zeta), y(\zeta)\right) \end{aligned} \tag{2.5}$$

with the qualitative theory of differential equation [37], will obtained an integral of eq. (2.5). Thus we will get a general solution to the eq. (2.5) directly. On the whole, finding the first integration is not easy. So, will applying the Division theorem to choice first integral for eq. (2.5), The exact solution of eq.(2.1) gained by solving eq. (2.5), Now let us rendering the important theorem in this method.

Theorem 2.1 (The Division Theorem). *[10] Assume two polynomials $\Phi(x, y)$ and $\Psi(x, y)$ of independent variables x and y in complex space $\mathbb{C}[x, y]$ and $\Phi(x, y)$ is irreducible in $\mathbb{C}[x, y]$. If $\Psi(x, y)$ vanishes at all zero points of $\Phi(x, y)$, then there exists a polynomial $\beta(X, Y)$ in $\mathbb{C}[x, y]$ such that $\Psi(X, Y) = \Phi(X, Y)\beta(X, Y)$.*

3. New Algorithm of the First Integral Method (NAFIM)

By applying the FIM to the form

$$u''(\zeta) - T\left(u(\zeta, u'(\zeta))\right) u'(\zeta) - R\left(u(\zeta)\right) = 0 \tag{3.1}$$

Then $T\left(u, u'\right) = 0$ and $R(u)$ are polynomials with real coefficients.
Now choose $\mathrm{T}\left(u, u'\right) = 0$, and $\mathrm{R}\left(u\right) = Au^2 + Bu$, so eq. (3.1) change, and becomes

$$u''(\zeta) - Au^2(\zeta) - Bu(\zeta) = 0 \tag{3.2}$$

Using eq. (2.4) and eq. (2.5), then eq. (3.2) becomes a system of ODEs

$$\begin{aligned} X'(\zeta) &= Y(\zeta) \\ Y'(\zeta) &= AX^2(\zeta) + BX(\zeta) \end{aligned} \tag{3.3}$$

Now, applying the Division theorem to eq. (3.3), there are nontrivial solutions and $q(X,\ Y) = \sum_{i=0}^{M} a_i(X)Y^i$, which is an irreducible polynomial in $\mathbb{C}[X, Y]$, thus

$$q[X(\zeta),\ Y(\zeta)] = \sum_{i=0}^{M} a_i(X(\zeta))Y^i(\zeta) = 0 \tag{3.4}$$

$a_i\left(X\right);\ \{i = 0, 1, 2, \ldots, M\}$are polynomials and $a_M\left(X\right) \neq 0$. By depending on the Division Theorem we will get a polynomial as $g(X) + h\left(X\right)Y$ in a complex domain $C\left[X, Y\right]$ where

$$\frac{dq}{d\zeta}=\frac{dq}{dX}\cdot\frac{dX}{d\zeta}+\frac{dq}{dY}\cdot\frac{dY}{d\zeta}=\big(g(X)X+h(X)Y\big)\sum_{i=0}^{M}a_i(X)Y^i \tag{3.5}$$

We start by sopping $M=1$ in eq. (3.5) gives

$$\sum_{i=0}^{1}a_i^{'}(X)\,Y^{i+1}+\sum_{i=0}^{1}ia_i(X)\,Y^{i-1}\left(AX^2+\mathrm{BX}\right)=(\ g(X)\ +\ h(X)Y)\left(\sum_{i=0}^{1}a_i(X)\,Y^i\right) \tag{3.6}$$

Equating the coefficients Y^i $(i=2,1,0)$ we have;

$$a_1^{'}(X)=a_1(X)\,h(X) \tag{3.7a}$$

$$a_0^{'}(X)=a_1(X)\,g(X)+a_0(X)\,h(X) \tag{3.7b}$$

$$a_1(X)\left(AX^2+\mathrm{BX}\right)=a_0(X)\,g(X) \tag{3.7c}$$

from eq. (3.7a), we conclude that $a_1(X)$ is a constant, we take $\mathrm{a}_1(X)\ =1$, and $\mathrm{h}(X)\ =\ 0$. And balancing the degrees of $g(X)$, $a_1(X)$ and$a_0(X)$, we deduce that degree $g(X)\ =\ 1$ only, then chosen $\mathrm{g}(X)\ =\ A_0X\ +\ B_0$, then we find $a_0(X)$ from eq. (3.7b)

$$a_0(X)=\frac{A_0X^2}{2}+\ B_0X+C_0, \tag{3.8}$$

Where C_0 is an arbitrary integration constant. Then we get a system of nonlinear algebraic equations by substituting $a_1(X)$, $a_0(X)$ and $g(X)$ in eq. (3.7c) and taking all the coefficients of powers X to be zero, when solve this system we have

$$\{A=0,\qquad B=0,\qquad A_0=0,\qquad B_0=0,\qquad C_0=C_0\},$$
$$\{A=0,\qquad B=B_0^2,\qquad A_0=0,\qquad B_0=B_0,\qquad C_0=0\},$$

by the first set we get the travail solution, so neglected and we take only the second set of solutions and substituting in eq. (3.5) we obtain

$$Y(\zeta)=\pm\sqrt{B}X(\zeta), \tag{3.9}$$

respectively. Combining eq. (3.9) with eq. (2.5), we have

$$X_1(\zeta)=C_1e^{\sqrt{B}},\qquad X_2(\zeta)=C_1e^{-\sqrt{B}\zeta}$$

$$Y_1(\zeta)=\frac{C_1\left(\frac{1}{2}C_1Ae^{2\sqrt{B}\zeta}+Be^{\sqrt{B}\zeta}\right)}{\sqrt{B}}+C_2, \tag{3.10a}$$

$$Y_2(\zeta)=-\frac{1}{2}\frac{C_1^2Ae^{-2\sqrt{B}\zeta}}{\sqrt{B}}-C_1\sqrt{B}e^{-\sqrt{B}\zeta}+C_2 \tag{3.10b}$$

Now when $M=2$ in eq. (3.5), and $q(X,Y)=0$ this implies $\frac{dq}{d\zeta}=0$,

$$\sum_{i=0}^{2}a_i^{'}(X)\,Y^{i+1}+\sum_{i=0}^{2}ia_i(X)\,Y^{i-1}\left(AX^2+\mathrm{BX}\right)=\ (\ g(x)\ +\ h(X)Y)\left(\sum_{i=0}^{2}a_i(X)\,Y^i\right) \tag{3.11}$$

We will obtain a system of ODEs when equating the coefficients of Y^i $(i = 3,\ 2, 1, 0)$ for eq. (3.11),

$$a_2'(X) = a_2(X)\,h(X) \tag{3.12a}$$

$$a_1'(X) = a_2(X)\,g(X) + a_1(X)\,h(X) \tag{3.12b}$$

$$a_0'(X) + 2a_2(X)\left(AX^2 + \mathrm{BX}\right) = a_1(X)\,g(X) + a_0(X)\,h(X) \tag{3.12c}$$

$$a_1(X)\left(AX^2 + \mathrm{BX}\right) = a_0(X)\,g(X) \tag{3.12d}$$

from eq. (3.12a), we conclude that $a_2(X)$ is a constant, $h(X) = 0$. we set $a_2(X) = 1$, By taking $a_2(X) = 1$, for simplicity, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we deduce the degree $g(X) =\ 1$ only, a summing that $g(X) = A_0X + B_0$ then we collect the $\mathrm{a}_1(X)$, and $a_0(X)$from eq. (23b) & eq. (3.12c)

$$a_1(X) = \frac{1}{2}A_0X^2 + B_0X + C_0, \quad \text{where } C_0 \text{ is a constant integration.} \tag{3.13a}$$

$$a_0(X) = \frac{1}{8}A_0X^4 + \frac{1}{2}A_0B_0X^3 + \frac{1}{2}B_0^2X^2 + \frac{1}{2}A_0C_0X^2 + B_0C_0X - \frac{2}{3}AX^3 - BX^2 + D_0 \tag{3.13b}$$

by choosing a constant integration D_0 to be zero and combining equation eq. (3.13a) & eq. (3.13b) with eq. (3.12d), when chosen all the coefficients of powers X to be zero, we obtained a system of nonlinear algebraic equations, and solve it we obtian

$$\{A = 0, \quad B = \frac{1}{4}B_0^2, \quad A_0 = 0, \quad B_0 = B_0, \quad C_0 = 0\} \tag{3.14a}$$

$$\{A = A, \quad B = B, \quad A_0 = 0, \quad B_0 = 0, \quad C_0 = 0\}, \tag{3.14b}$$

$$\{A = 0, \quad B = 0, \quad A_0 = 0, \quad B_0 = 0, \quad C_0 = C_0\} \tag{3.14c}$$

from (3.14a) we get solutions same as case M=1. While using eq. (3.14b)(24b) in eq. (3.5), we obtain:

$$Y_1 = \frac{1}{3}\sqrt{6AX + 9B}\ X, \qquad Y_2 = -\frac{1}{3}\sqrt{6AX + 9B}\ X \tag{3.15}$$

Respectively, combining equation eq. (3.11) with eq. (2.4)

$$X_1(\zeta) = X_2(\zeta) = \frac{3}{2}\frac{B\left(\tanh\left(\frac{1}{2}\zeta\sqrt{B} + \frac{1}{2}C_1\sqrt{B}\right)^2 - 1\right)}{A}, \tag{3.16}$$

then

$$Y(\zeta) = \frac{3}{2}\frac{B^{\frac{3}{2}}\sinh\left(\frac{1}{2}\sqrt{B} + \frac{1}{2}C_1\sqrt{B}\right)}{A\quad\cosh\left(\frac{1}{2}\sqrt{B} + \frac{1}{2}C_1\sqrt{B}\right)^3} + C_2, \tag{3.17}$$

when using eq. (3.14c) in eq. (3.5), and then eq. (2.5) we obtain:

$$X(\zeta) = -\ C_0 \quad + C_1 \tag{3.18}$$

$$Y(\zeta) = \frac{1}{3}A^3C_0^2 - A\ ^2C_1C_0 - \left(\frac{1}{2}\right)B^2C_0 + AC_1^2 + BC_1 + C_2. \tag{3.19}$$

Equations (3.10a), (3.10b), (3.17) and (3.19) represent the general solutions for differential equations of the form eq. (2.2).

4. Application the generalized (2 + 1)-dimensional dynamical partial differential equation

From introduction of our paper the formula of the generalized (2 + 1)-dimensional dynamical partial differential equation is

$$u_{\mathrm{xt}} + au_x u_{\mathrm{xy}} + bu_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = 0$$

By eq. (2.2) we get

$$-cf'' + (a+b)f''f' + f'''' = 0, \tag{4.1}$$

integrate eq. (4.1) we have

$$-cf' + \frac{(a+b)}{2}\left(f'\right)^2 + f''' = 0, \tag{4.2}$$

to reduce the order of the derivation, suppose that $X(\zeta) = f'$ and $Y(\zeta) = f''$, then eq. (4.2) becomes

$$-cX + \frac{(a+b)}{2}X^2 + X'' = 0, \tag{4.3}$$

by comparing with eq. (3.3) and according to concept of the first integral method we get

$$X'(\zeta) = Y(\zeta), \tag{4.4}$$

$$Y'(\zeta) = cX() - \frac{(a+b)}{2}X^2() \tag{4.5}$$

Then we get $A = -\frac{(a+b)}{2}$ and $B = c$.
From eq. (3.10a) and eq. (3.10b) when $M = 1$, we get

$$Y_1(\zeta) = \frac{C_1\left(-\frac{(a+b)}{4}C_1 e^{2\sqrt{c}} + ce^{\sqrt{c}}\right)}{\sqrt{c}} + C_2, \tag{4.6a}$$

$$Y_2(\zeta) = \frac{(a+b)}{4}\frac{C_1^2 e^{-2\sqrt{c}}}{\sqrt{c}} - C_1\sqrt{c}e^{-\sqrt{c}} + C_2, \tag{4.6b}$$

Then to get the solution of eq. (4.2) must be integrating equations (4.6a) and (4.6b), so it's become

$$f_1(\zeta) = \frac{1}{2}\frac{C_1\left(-\frac{(a+b)}{4}\frac{C_1 e^{2\sqrt{c}}}{\sqrt{c}} + 2\sqrt{c}e^{\sqrt{c}}\right)}{\sqrt{c}} + C_2 + C_3, \tag{4.7a}$$

$$f_2(\zeta) = \frac{-(a+b)}{8}\frac{C_1^2 e^{-2\sqrt{c}}}{c} - C_1 e^{-\sqrt{c}} + C_2 + C_3 \tag{4.7b}$$

Also from eq. (3.17) and eq. (3.19) when $M = 2$ we get

$$Y_3(\zeta) = -\frac{3}{(a+b)}\frac{c^{\frac{3}{2}}\sinh\left(\frac{1}{2}\sqrt{c} + \frac{1}{2}C_1\sqrt{c}\right)}{\cosh\left(\frac{1}{2}\sqrt{c} + \frac{1}{2}C_1\sqrt{c}\right)^3} + C_2, \tag{4.8}$$

$$Y_4(\zeta) = -\frac{(a+b)}{6}\zeta^3 C_0^2 + \frac{(a+b)}{2}\zeta^2 C_1 C_0 - \left(\frac{1}{2}\right)c\zeta^2 C_0 - \frac{(a+b)}{2}C_1^2\zeta + cC_1\zeta + C_2, \tag{4.9}$$

integrating equations (4.8) and (4.9), we have

$$f_3(\zeta) = \frac{3}{(a+b)}\frac{\mathrm{c}}{\cosh\left(\frac{1}{2}\sqrt{c} + \frac{1}{2}C_1\sqrt{c}\right)^3} + C_2 + C_3, \tag{4.10}$$

$$f_4(\zeta) = -\frac{(a+b)}{24}^4 C_0^2 + \frac{(a+b)}{6}\,^3 C_1 C_0 - \left(\frac{1}{6}\right)c^3 C_0 - \frac{(a+b)}{4}C_1^2\zeta^2 + \frac{c}{2}C_1^2 + C_2 + C_3, \tag{4.11}$$

5. The (2 + 1)-dimensional (CBS) equation

The CBS equation was introduced by two scientists Bogoyavlenskii and Schiff. Where, Bogoyavlenskii utilized the modified Lax formalism, while, the scientist Schiff used a different method to obtain CBS equation, by curtailments the self-dual Yang–Mills equation [10, 27, 15] The formula of (CBS) equation can be written be as

$$u_{\mathrm{xt}} + 4u_x u_{\mathrm{xy}} + 2u_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = 0$$

Then $a = 4$ and $b = 2$, leads to $A = -3$ and $B = c$, and by equations (4.7a), (4.7b), (4.10) and (4.11), the analytical solutions of (CBS) equation become

$$u_1(x,y,t) = \frac{1}{2}\frac{C_1\left(-\frac{3}{2}\frac{C_1 e^{2\sqrt{c}(x+y-ct)}}{\sqrt{c}} + 2\sqrt{c}e^{\sqrt{c}(x+y-ct)}\right)}{\sqrt{c}} + C_2(x+y-ct) + C_3, \tag{5.1}$$

$$u_2(x,y,t) = \frac{-3}{4}\frac{C_1^2 e^{-2\sqrt{c}(x+y-ct)}}{c} - C_1 e^{-\sqrt{c}(x+y-ct)} + C_2(x+y-ct) + C_3, \tag{5.2}$$

$$u_3(x,y,t) = \frac{1}{2}\frac{\mathrm{c}}{\cosh\left(\frac{\sqrt{c}}{2}(x+y-ct) \;+\; \frac{1}{2}C_1\sqrt{c}\right)^3} + C_2(x+y-ct) + C_3, \tag{5.3}$$

$$u_4(x,y,t) = -\frac{1}{4}(x+y-ct)^4 C_0^2 + \left(C_1C_0 \;-\left(\frac{cC_0}{6}\right)\right)(x+y-ct)^3 + \left(\frac{c}{2}C_1 - \frac{3}{2}C_1^2\right)\star$$
$$(x+y-ct)^2 + C_2(x+y-ct) + C_3, \tag{5.4}$$

6. The (2 +1)-dimensional Bogoyavlenskii's breaking Soliton equation

This equation was investigated by Bogoyavenskii, which specializes in studying the wave Riemann scattered along the x-axis with a wave on the y-axis [15]. With formula of equation as

$$u_{\mathrm{xt}} + 4u_x u_{\mathrm{xy}} + 4u_{\mathrm{xx}} u_y + u_{\mathrm{xxxy}} = 0$$

Then $a = 4$ and $b = 4$. Then parameters A, B of this equation become$A = -\frac{(a+b)}{2} = -\frac{8}{2} = -4$ and $B = c$, and according to equations (4.7a), (4.7b), (4.10) and (4.11) we have the exact solutions as

$$u_1(x,y,t) = \frac{1}{2}\frac{C_1\left(-\frac{C_1 e^{2\sqrt{c}(x+y-ct)}}{\sqrt{c}} + 2\sqrt{c}e^{\sqrt{c}(x+y-ct)}\right)}{\sqrt{c}} + C_2(x+y-ct) + C_3, \tag{6.1}$$

$$u_2(x,y,t) = -\frac{C_1^2 e^{-2\sqrt{c}(x+y-ct)}}{c} - C_1 e^{-\sqrt{c}(x+y-ct)} + C_2(x+y-ct) + C_3, \tag{6.2}$$

$$u_3(x,y,t) = \frac{3}{8}\frac{\mathrm{c}}{\cosh\left(\frac{\sqrt{c}}{2}(x+y-ct) \;+\; \frac{1}{2}C_1\sqrt{c}\right)^3} + C_2(x+y-ct) + \;\; C_3, \tag{6.3}$$

$$u_4(x,y,t) = -\frac{1}{3}(x+y-ct)^4 C_0^2 + \left(\frac{4C_1C_0}{3} - \left(\frac{cC_0}{6}\right)\right)(x+y-ct)^3 \; + \left(\frac{c}{2}C_1 - 2C_1^2\right)\star$$
$$(x+y-ct)^2 + C_2(x+y-ct) + C_3, \tag{6.4}$$

7. The (2 + 1)-dimensional breaking Soliton equation

We'll obtain this equation if chosen $a=-4$ and $b=-2$ in Eq. (1.1):

$$u_{\mathrm{xt}}-4u_xu_{\mathrm{xy}}-2u_{\mathrm{xx}}u_y+u_{\mathrm{xxxy}}=0$$

Then we get $A=3$ and $B=c$, and by equations (4.7a), (4.7b), (4.11) and (5.1), the analytical solutions given as

$$u_1(x,y,t)=\frac{1}{2}\frac{C_1\left(\frac{3}{2}\frac{C_1e^{2\sqrt{c}(x+y-ct)}}{\sqrt{c}}+2\sqrt{c}e^{\sqrt{c}(x+y-ct)}\right)}{\sqrt{c}}+C_2(x+y-ct)+C_3, \tag{7.1}$$

$$u_2(x,y,t)=\frac{3}{4}\frac{C_1^2e^{-2\sqrt{c}(x+y-ct)}}{c}-C_1e^{-\sqrt{c}(x+y-ct)}+C_2(x+y-ct)+C_3, \tag{7.2}$$

$$u_3(x,y,t)=\frac{1}{2}\frac{\mathrm{c}}{\cosh\left(\frac{\sqrt{c}}{2}(x+y-ct)\ +\ \frac{1}{2}C_1\sqrt{c}\right)^3}+C_2(x+y-ct)+C_3, \tag{7.3}$$

$$u_4(x,y,t)=\frac{1}{4}(x+y-ct)^4C_0^2+\left(C_1C_0\ -\left(\frac{cC_0}{6}\right)\right)(x+y-ct)^3\ +\left(\frac{c}{2}C_1-\frac{3}{2}C_1^2\right)\star$$
$$(x+y-ct)^2+C_2(x+y-ct)+C_3, \tag{7.4}$$

8. The generalized (3+1)-dimensional KP equation

The (3+1)-dimensional KP equation is integrable and discusses the evolution of shallow-water waves when viscosity and the surface tension are less [32].

$$u_{\mathrm{xt}}+u_{\mathrm{yt}}+u_{\mathrm{xxxy}}+3(u_xu_y)_x-u_{\mathrm{zz}}=0$$

By eq. (2.2), we get

$$-cf''-cf''+f''''+3\left(\left(f'\right)^2\right)'-f''=0, \tag{8.1}$$

$$-(2c+1)f''+3\left(\left(f'\right)^2\right)'+f''''=0, \tag{8.2}$$

Integrate eq. (8.2) we have

$$-(2c+1)f'+3\left(f'\right)^2+f'''=0, \tag{8.3}$$

To reduce the order of the derivation, suppose that $X(\zeta)=f'$ and $Y()=f''$then eq. (8.3) becomes

$$-(2c+1)X+3X^2+X''=0, \tag{8.4}$$

by comparing with eq. (3.3) and according to concept of the first integral method we get

$$\begin{aligned}X'(\zeta)&=Y(\zeta),\\ Y'(\zeta)&=\ (2c+1)X(\zeta)-3X^2(\zeta),\end{aligned} \tag{8.5}$$

then we get $A = -3$ and $B = (2c + 1)$ with integrating equations (3.10a), (3.10b), (3.17) and (3.19), we have the exact solutions of eq. (1.5) as form

$$u_1(x, y, z, t) = \frac{1}{2}\frac{C_1\left(-\frac{3}{2}\frac{C_1 e^{2\sqrt{2c+1}(x+y+z-ct)}}{\sqrt{2c+1}} + 2\sqrt{2c+1}e^{\sqrt{2c+1}(x+y+z-ct)}\right)}{\sqrt{2c+1}} + C_2(x + y + z - ct) + C_3, \tag{8.6}$$

$$u_2(x, y, z, t) = \frac{-3}{4}\frac{C_1^2 e^{-2\sqrt{2c+1}(x+y+z-ct)}}{2c+1} - C_1 e^{-\sqrt{2c+1}(x+y+z-ct)} + C_2(x + y + z - ct) + C_3, \tag{8.7}$$

$$u_3(x, y, z, t) = \frac{1}{2}\frac{2c+1}{\cosh\left(\frac{1}{2}(x + y + z - ct)\sqrt{2c+1} + \frac{1}{2}C_1\sqrt{2c+1}\right)^3} + C_2(x + y + z - ct) + C_3, \tag{8.8}$$

$$u_4(x, y, z, t) = -\frac{1}{4}(x + y + z - ct)^4 C_0^2 + \left(C_1C_0 - \left(\frac{C_0}{6}\right)(2c + 1)\right)(x + y + z - ct)^3 + \left(\frac{(2c+1)}{2}C_1 - \frac{3}{2}C_1^2\right)(x + y + z - ct)^2 + C_2(x + y + z - ct) + C_3, \tag{8.9}$$

9. Graphical representation

The aims of this part are to give the illustrations of our new solutions in different dimensions, the physical descriptive we given more completely about the illustrations of eq. (1.2), eq. (1.3), eq. (1.4) and eq. (1.5) by using Maple.

Figure 1 Includes two figures 1.1 and 1.2 of the Plots for the solitary wave solutions $u_1(x, y, t)$, $u_2(x, y, t), u_3(x, y, t)$, *and* $u_4(x, y, t)$ of eq. (1.2) with $m = 1$ and $m = 2$ respectively when $A = -4, B = c = 0.05, C_0 = 1$, $C_1 = 1$, $C_2 = 1, C_3 = 1$, $t = 0.01$, where $(F_{12} - a_1)$, $(F_{12} - b_1)$, $(F_{22} - a_2)$ and $(F_{22} - b_2)$ are the plotting of three dimensions with different sides, $(F_{12} - c_1)$ and $(F_{22} - c_2)$ are the contour plotted with same values. In figure 1.1, $(F_{11} - d_1)$ and $(F_{11} - f_1)$ is the complex plotting with the values $B = c = -2$ *and* $y = 0.01$.

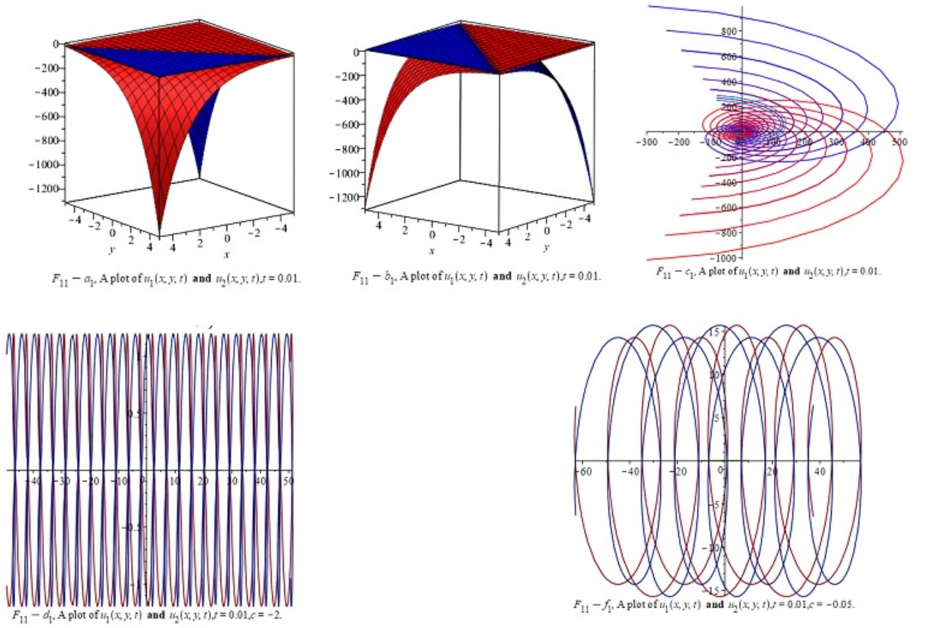


Figure 1.1

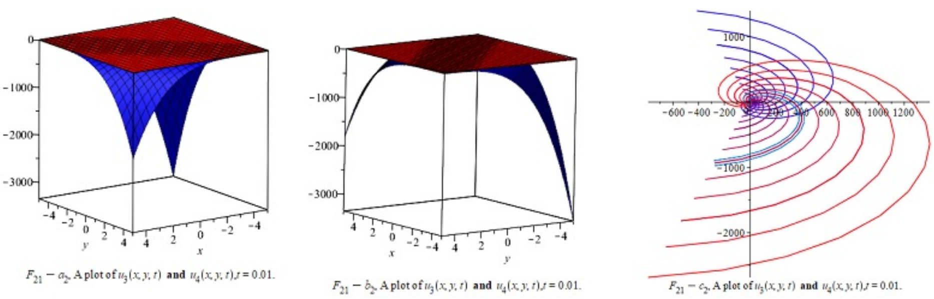
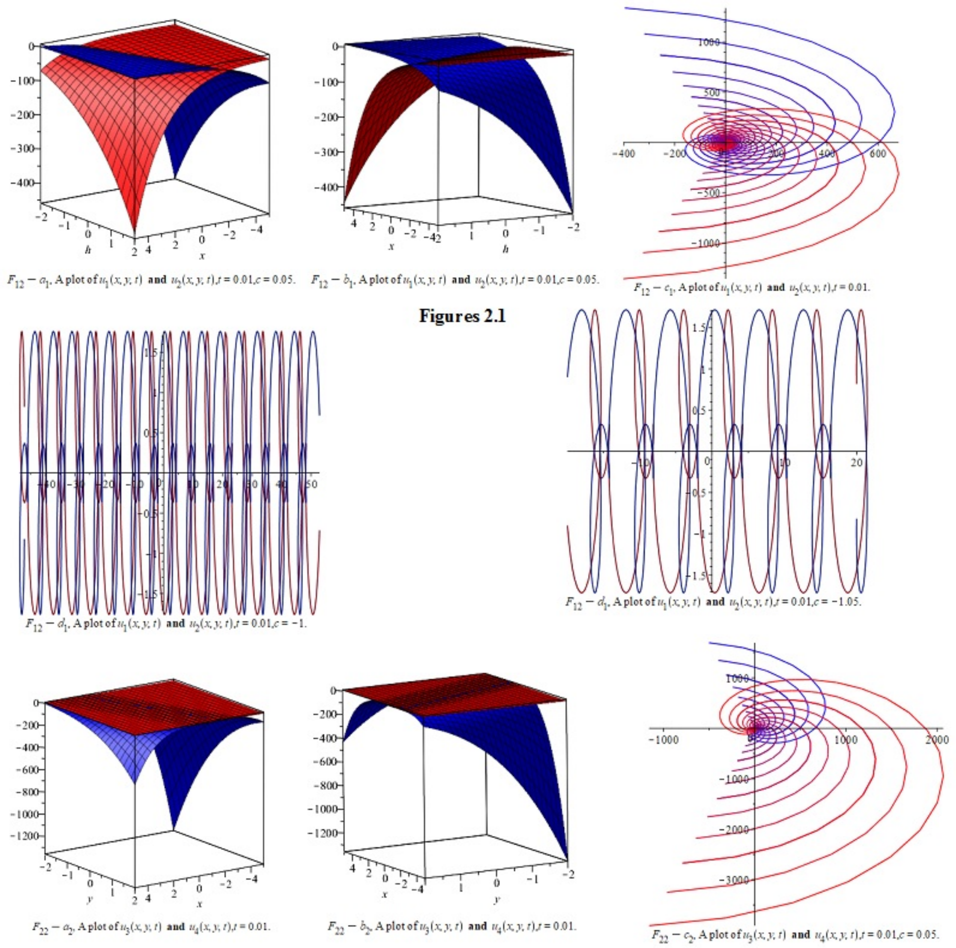


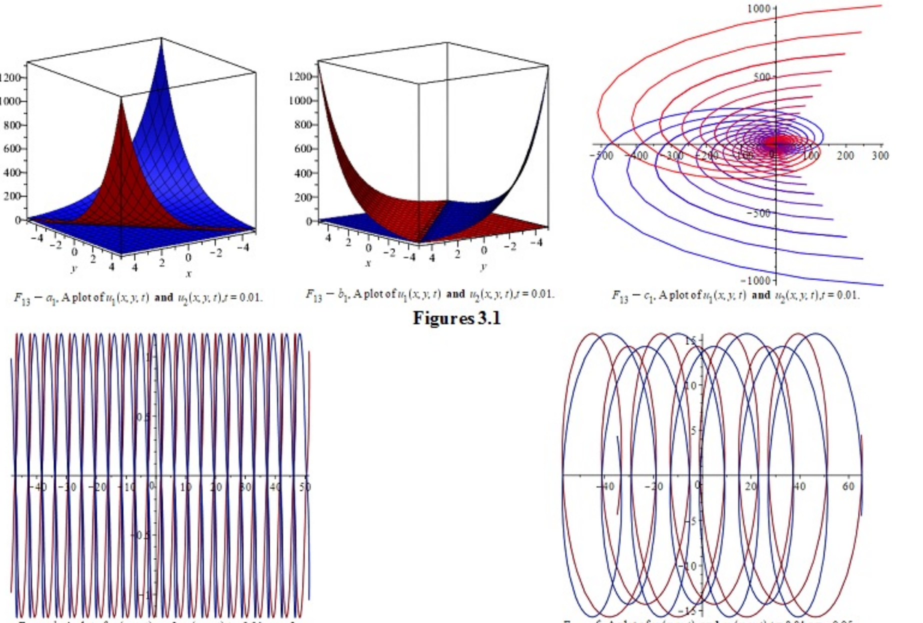
Figure 1.2

Figure 2 represents with two figures 2.1 and 2.2 of the Plots for $u_1(x, y, t)$, $u_2(x, y, t)$, $u_3(x, y, t)$, and $u_4(x, y, t)$ of eq. (1.3) with $m = 1$ and $m = 2$ respectively when $A = -4, B = c = 0.05, C_0 = 1, C_1 = 1, C_2 = 1, C_3 = 1, t = 0.01$, where $(F_{12} - a_1)$, $(F_{12} - b_1)$, $(F_{22} - a_2)$ and $(F_{22} - b_2)$ are the plotting of three dimensions with different sides, $(F_{12} - c_1)$ and $(F_{22} - c_2)$ are the contour plotted with same values, but in figure 1.1, $(F_{12} - d_1)$ and $(F_{12} - f_1)$ are the complex plotting with the values $y = 0.01$ *and* $B = c = -1, -1.05$ respectively.



Figures 2.2

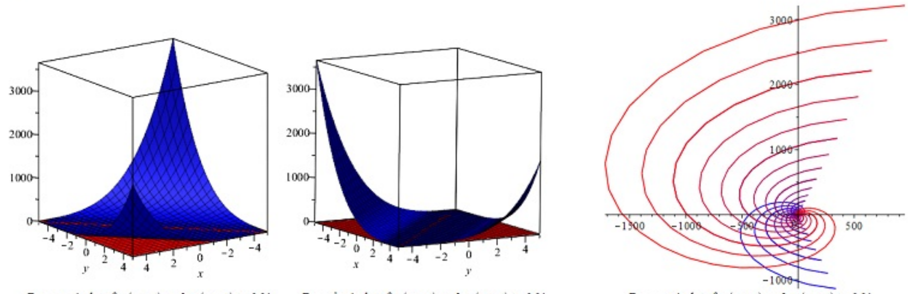
Figure 3 includes figures 3.1 and 3.2 of the Plots for $u_1(x,y,t)$, $u_2(x,y,t)$, $u_3(x,y,t)$, and $u_4(x,y,t)$ of eq. (4) with m=1 and m=2 respectively when $A = 3, B = c = 0.05, C_0 = 1$, $C_1 = 1$, $C_2 = 1, C_3 = 1$, $t = 0.01$, where $(F_{13} - a_1)$, $(F_{13} - b_1)$, $(F_{23} - a_2)$ and $(F_{23} - b_2)$ are the plotting of three dimensions with different sides, $(F_{12} - c_1)$ and $(F_{22} - c_2)$ are the contour plotted with same values, but in figure 1.1, $(F_{13} - d_1)$ and $(F_{13} - f_1)$ are the complex plotting with the values $y = 0.01$ and $B = c = -2$ and $y = 0.01$. respectively.



$F_{13}-a_1$. A plot of $u_1(x,y,t)$ and $u_2(x,y,t)$, $t=0.01$. $F_{13}-b_1$. A plot of $u_1(x,y,t)$ and $u_2(x,y,t)$, $t=0.01$. $F_{13}-c_1$. A plot of $u_1(x,y,t)$ and $u_2(x,y,t)$, $t=0.01$.

$F_{13}-d_1$. A plot of $u_1(x,y,t)$ and $u_2(x,y,t)$, $t=0.01, c=-2$. $F_{13}-f_1$. A plot of $u_1(x,y,t)$ and $u_2(x,y,t)$, $t=0.01, c=-0.05$.

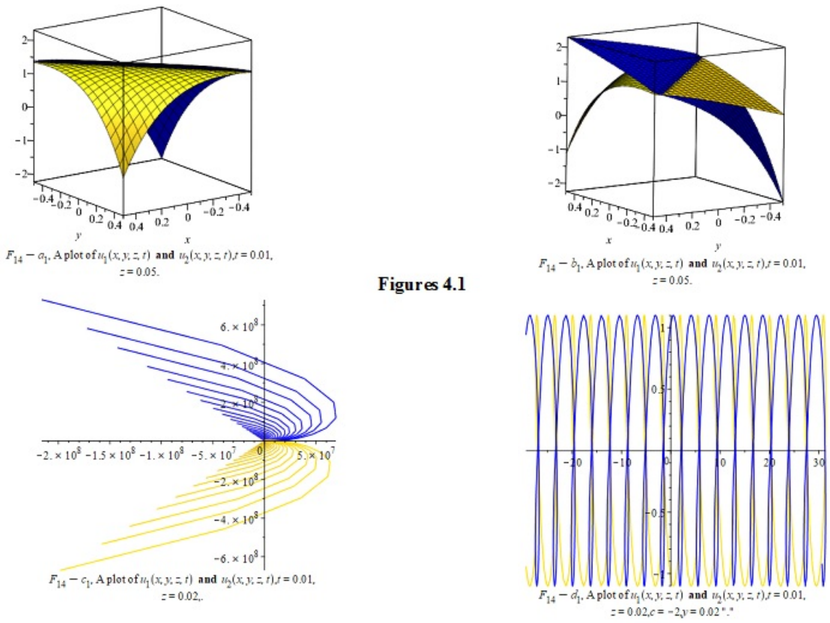
Figures 3.1



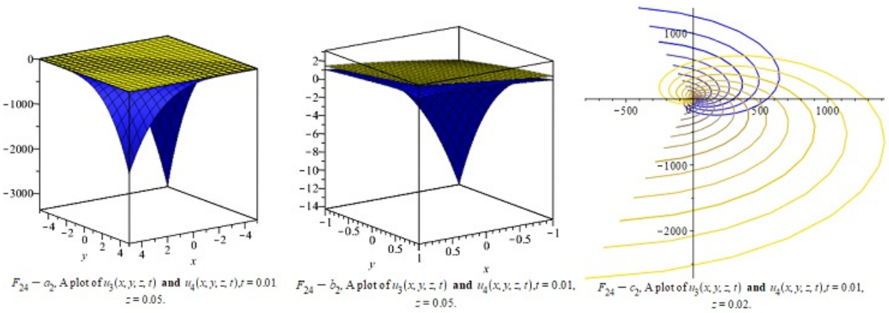
$F_{23}-a_2$. A plot of $u_3(x,y,t)$ and $u_4(x,y,t)$, $t=0.01$. $F_{23}-b_2$. A plot of $u_3(x,y,t)$ and $u_4(x,y,t)$, $t=0.01$. $F_{23}-c_2$. A plot of $u_3(x,y,t)$ and $u_4(x,y,t)$, $t=0.01$.

Figures 3.2

Figure 4 represent with two figures 4.1 and 4.2 of the Plots for $u_1(x,y,z,t)$, $u_2(x,y,z,t)$, $u_3(x,y,z,t)$, and $u_4(x,y,z,t)$ of eq. (1.5) with $m=1$ and $m=2$ respectively when $A=-4, B=2c+1$, $c=0.05, C_0=1$, $C_1=1$, $C_2=1, C_3=1$, $t=0.01$, where $(F_{14}-a_1)$, $(F_{14}-b_1)$, $(F_{24}-a_2)$ and $(F_{24}-b_2)$ are the plotting of three dimensions with different sides, $(F_{14}-c_1)$ and $(F_{24}-c_2)$ are the contour plotted with same values, but in figure 1.1, $(F_{14}-d_1)$ and $(F_{14}-f_1)$ are the complex plotting with the values $y=0.01$ and $B=c=-1,-1.05$ respectively.



Figures 4.1



Figures 4.2

10. Result and discussion

In this section, we compare the similarity and differences new families of solutions which are was gained using our recently suggest technique with the help of contemporary software (Maple) and comparing with different Mathematical methods:

- **First:** In [3, 34] there are some solutions were obtained for the equation (1.1) by using different methods, but in this paper, a different and new solutions in hyperbolic functions form were obtained.
- **Secondly:** In [35, 9] there are some solutions were obtained for the equation (1.4) by using different methods, but in this paper the results are new and variant. However, the results are new and variant when the frames are same. These solutions and application of waves are modern and different from the existing solutions. This method is a powerful tool for obtaining a new and modern exact solutions of NEEs.

- **Thirdly:** in [22, 20] the exact solutions are discussed of equation (1.5) by using different methods, but in this paper the results are new and variant.

However, the results of these solutions and application of waves are modern and different from the existing solutions. This method is a direct tool for get new and modern exact solutions of nonlinear equations.

11. Conclusion

In this text, we obtained new and different solution of a (2 + 1)-dimensional generalized (NLEEs) including independent variables, and some new propagating solutions to a generalized (3+1)-dimensional KP equation with variable coefficients using solutions by (NAFIM). The gained solution it is important for explain the nonlinear phenomena in physics. The summary, that newly method is successful and direct can be used in many nonlinear phenomena in physics.

References

[1] M. Abdoon, *Programming first integral method general formula for the solving linear and nonlinear equations*, Appl. Math. 6 (2015) 568.

[2] Al-Amr and O. Mohammed, *Exact solutions of the generalized (2+1)-dimensional nonlinear evolution equations via the modified simple equation method*, Comput. Math. Appl. 69(5) (2015) 390–397.

[3] M. Al-Amr, *Exact solutions of the generalized (2+ 1)-dimensional nonlinear evolution equations via the modified simple equation method*, Comput. Math. Appl. 5 (2015) 390–397.

[4] A. Aleixo and A. Balantekin, *Algebraic construction of coherent states for nonlinear quantum deformed systems*, J. Phys. A: Math. Theor. 16 (2012) 165302.

[5] T. Bo. and G. YI-Tian, *On the generalized Tanh method the (2+1)-dimensional breaking soliton equations*, Mod. Phys. Lett. A. 38 (1995) 2937–2941.

[6] B. Buti, V. Galinski, V. Shevchenko, G. Lakhina, B. Tsurutani, B. Goldstein, B. Diamond and M. Medvedev, *Evolution of Nonlinear Alfvén Waves in Streaming Inhomogeneous Plasmas*, Astrophysical J. 523 (1999) 849.

[7] O. Bogoyavlenskii, *Breaking solitons in 2+ 1-dimensional integrable equations*, Russian Math. Surv. 45(4) (1990).

[8] R. Cherniha and S. Kovalenko, *Lie symmetries and reductions of multi-dimensional boundary value problems of the Stefan type*, J. Phys. A: Math. Theo. 44 (2011) 485202.

[9] M. Darvishi and M. Najafi, *Some exact solutions of the (2+ 1)-dimensional breaking soliton equation using the three-wave method*, Int. J. Comput. Math. Sci. 6 (2012) 13–16.

[10] T. Ding and C. Li, *Ordinary differential equations*, Theor. Math. Phys. 137 (1996) 1367–1377.

[11] M. Ekici, D. Duran and A. Sonmezoglu, *Constructing of exact solutions to the (2+ 1)-dimensional breaking soliton equations by the multiple (G'/G)-expansion method*, J. Adv. Math. Stud. 1 (2014) 27–30.

[12] L. Faeza, *First integral method for constructing new exact solutions of the important nonlinear evolution equations in physics*, Phys. Conf. Ser. 2020.

[13] Y. Guldem and D. Durmus, *Solution of the (2+1) donation breaking soliton equation by using two different methods*, J. Eng. Tech. App. Sci. 1 (2016) 13–18.

[14] G. Hami and F. Ömer, *Benjamin-Bona-Mahony equation by using the sn-ns method and the tanh-coth method*, Math. Moravica, 1 (2017) 95–103.

[15] T. Kobayashi and K. Toda, *The Painlevé test and reducibility to the canonical forms for higher-dimensional soliton equations with variable-coefficients*, Symmet. Integ. Geom. Meth. Appl. 2 (2006) 063.

[16] A. Lisok, A. Trifonov and A. Shapovalov, *The evolution operator of the Hartree-type equation with a quadratic potential*, J. Phys. A: Math. Gen. 37 (2004) 4535.

[17] W. Ma, A. Abdeljabbar and M. GamilAsaad, *Wronskian and Grammian solutions to a (3+ 1)-dimensional generalized KP equation*, Appl. Math. Comput. 24 (2011) 10016–10023.

[18] W. Ma and A. Abdeljabbar, *A bilinear Bäcklund transformation of a (3 + 1)-dimensional generalized KP equation*, Appl. Math. Lett. 25(10) (2012) 1500–1504.

[19] K. Melike, A. Arzu and B. Ahmet, (2016) *Solving space-time fractional differential equations by using modified simple equation method*, Commun. Theor. Phys. 5 (2016) 563–568.

[20] M. Mirzazadeh, *A couple of solutions to a (3+ 1)-dimensional generalized KP equation with variable coefficients by extended transformed rational function method*, Elect. J. Math. Anal. Appl. 1 (2015) 188–194.

[21] A. Mohamed, *First integral method: A general formula for nonlinear fractional Klein-Gordon equation using advanced computing language*, Amer. J. Comput. Math. 5 (2015) 127–134.

[22] S. Mohyud-Din, A. Irshad, N. Ahmed and U. Khan, *Exact solutions of (3+ 1)-dimensional generalized KP equation arising in physics*, Results Phys. 7 (2017) 3901–3909.

[23] M. Najafi, S. Arbabi and M. Najafi, *New application of sine-cosine method for the generalized (2 + 1)-dimensional nonlinear evolution equations*, Int. J. Adv. Math. Sci. 1 (2013) 45–49.

[24] M. Najafi, M. Najafi and S. Arbabi, *New exact solutions for the generalized (2+ 1)-dimensional nonlinear evolution equations by Tanh-Coth method*, Int. J. Modern Theo. Phys. 2 (2013) 79–85.

[25] M. Najafi, M. Najafi and S. Arbabi, *New application of* $(G^{'}/G)$*-expansion method for generalized (2 + 1)-dimensional nonlinear evolution equations*, Int. J. Eng. Math. 5 (2013) Article ID 746910.

[26] V. Orlov and O. Kovalchuk, *Research of one class of nonlinear differential equations of third order for mathematical modelling the complex structures*, IOP Conf. Ser.: Mater. Sci. Eng. 365 (2018) 042045.

[27] Y. Peng, *New types of localized coherent structures in the Bogoyavlenskii-Schiff equation*, Int. J. Theor. Phys. 45 (2006) 1779–1783.

[28] A. Pullen, A. Benson and L. Moustakas, *Nonlinear evolution of dark matter subhalos and applications to warm dark matter*, Astrophysical J. 792(1) (2014) 24.

[29] A. Seadawy, N. Cheemaa and A. Biswas, *Optical dromions and domain walls in (2+ 1)-dimensional coupled system*, Optik, 227 (2020) 165669.

[30] Y. Tang, S. Tao, M. Zhou and Q. Guan, *Interaction solutions between lump and other solitons of two classes of nonlinear evolution equations*, Nonlinear Dyn. 89(1) (2017) 429–442 .

[31] M. Tarig and B. Jafar, (2011) *Homotopy perturbation method and Elzaki transform for solving system of nonlinear partial differential equations*, Worl. Appl. Sci. J. 7 (2011) 944–948.

[32] A. Wazwaz, *Integrable (2+ 1)-dimensional and (3+ 1)-dimensional breaking soliton equations*, Physica Scripta, 81 (2010) 035005.

[33] W. X. Ma, *Abundant lumps and their interaction solutions of (3+1)-dimensional linear PDEs*, J. Geom. Phys. 133 (2018) 10–16 .

[34] Y. Yıldırım and E. Yaşar, *A (2+ 1)-dimensional breaking soliton equation: solutions and conservation laws*, Chaos, Solitons Fract. 107 (2018) 146–155.

[35] G. Yildiz and D. Daghan, *Solution of the (2+ 1) dimensional breaking Soliton equation by using two different methods*, J. Engin. Tech. Appl. Sci. 1 (2016) 13–18.

[36] V. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP. 5 (1972) 908–914.

[37] Y. Zhou, M. Wang and Y. Wang, *Periodic wave solutions to a coupled KdV equations with variable coefficients*, Phys. Lett. A. 1 (2003) 31–36.