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New common best proximity point theorems in complete metric space with weak P-property

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Abstract

In this paper, we define ϕ-contraction, after that; we obtained the common best proximity point results for different types of contractions in the setting of complete metric spaces by using weak p-property and proved the uniqueness of these points. Also, we presented an example to support our results.

Keywords: common best proximity point, weak P-property, ϕ-contraction.
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1. Introduction

The notability of fixed point theory comes from the fact that many problems in mathematics can be formulated in terms of the existence of a fixed point and it is often much easier to show that such points exist and then approximate them than to find them explicitly. In 1998, Kannan[6] presented the following fixed point theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping such that there exists $k < \frac{1}{2}$ satisfying*

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. Then T has a unique fixed point.

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The Kannan fixed point theorem and some of its results are investigated in [5]. In particular, we have the following theorem and Lemmas:

Theorem 1.2. *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be a continuous mapping such that*

$$d(Tx,Ty) \leq \alpha d(x,Tx) + \beta d(y,Ty)$$

for all $x, y \in X$ and $x \neq y$ where α, β are positive real numbers satisfying $\alpha + \beta < 1$. Then T has a unique fixed point.

Let $T : A \longrightarrow B$ be a non$-$self mapping where A, B are non-empty subsets of a metric space (X,d). Then T may not have a fixed point. In this case, $d(x,Tx) > 0$ and it is important that we find an element $x \in A$ such that $d(x,Tx)$ is minimum in some sense. For example the best approximation problem and best proximity problem are investigated in this regard.
An element $x \in A$ is said to be the best proximity point of T if $d(x,Tx) = d(A,B)$ where

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$

It is easy to check that if T is self-mapping then the best proximity problem reduces to fixed point problem. There are several various of contractions that guarantee the existence of a best proximity point. Suppose that $S : A \longrightarrow B$ and $T : A \longrightarrow B$ be non$-$self mappings. It is possible that there exists $x \in A$ such that $d(x,Sx) = d(A,B)$ and $d(x,Tx) = d(A,B)$ simultaneously. In this case we said that x is common best proximity point of the pair (S,T). Many mathematicians have studied the existence and uniqueness of common best proximity points. In this paper, we present conditions that guarantee the existence and uniqueness of common best proximity point.
Define A_0 and B_0 as following;

$$A_0 = \{x \in A;\ \ d(x,y) = d(A,B)\ for\ some\ \ y \in B\}$$
$$B_0 = \{y \in B;\ \ d(x,y) = d(A,B)\ for\ some\ \ x \in A\}$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are non-empty.
We recall that the pair (A,B) of non-empty subsets of a metric space (X,d) with $A_0 \neq \emptyset$ is said to have $P-$property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:

$$\Big(d(x_1,y_1) = d(A,B) \text{ and } d(x_2,y_2) = d(A,B)\Big) \Rightarrow d(x_1,x_2) = d(y_1,y_2).$$

Definition 1.3. *[11] Let (A,B) be a pair of non-empty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. We say that (A,B) has weak $P-$property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:*

$$\Big(d(x_1,y_1) = d(A,B)\ and\ d(x_2,y_2) = d(A,B)\Big) \Longrightarrow d(x_1,x_2) \leq d(y_1,y_2)$$

Definition 1.4. *[8] A function $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is called a comparison if it satisfies the following conditions:*

- *ϕ is increasing,*
- *the sequence $(\phi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to +\infty$, for all $t \in [0, +\infty)$.*

We recall that a self-mapping T on a metric space (X, d) is said to be $\phi-$contraction if

$$d(T(x), T(y)) \leq \phi(d(x, y))$$

for any $x, y \in X$; where ϕ is comparision function. If ϕ is comparison function then $\phi(t) < t$ for any $t \in (0, +\infty)$.

Lemma 1.5. *[8] Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$. If $(x_n)_{n\in\mathbb{N}}$ is not Cauchy sequence then there exist $\epsilon > 0$ and sequences $(n(k))$ and $(m(k))$ of positive integers such that the following sequences tend to ϵ as $k \to +\infty$:*

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)}), \quad d(x_{m(k)}, x_{n(k)+1}), \quad d(x_{m(k)-1}, x_{n(k)}), \\ &d(x_{m(k)-1}, x_{n(k)+1}), \quad d(x_{m(k)+1}, x_{n(k)+1}), \quad d(x_{m(k)+1}, x_{n(k)}). \end{aligned}$$

Lemma 1.6. *[7] Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d). Suppose that $T : A \longrightarrow B$ be a mapping such that A_0 is non-empty. Then $T(A_0) \subset B_0$.*

Lemma 1.7. *[7] Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d). Suppose that the following conditions are satisfied:*

(1) $A_0 \neq \emptyset$,

(2) The pair (A, B) has weak $P-$property.

Then, the set B_0 is closed.

2. Main results

In our main theorems, we prove the existence and uniqueness of a common best proximity point for contracive non$-$self mappings in class of weak $P-$property.

Theorem 2.1. *Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d) and $S, T : A \longrightarrow B$ be two mappings. Assume that the following conditions are satisfied:*

(1) $A_0 \neq \emptyset$,

(2) The pair (A, B) has weak $P-$property,

(3) $d(Sx, Ty) \leq \phi(d(x, y))$ where ϕ is comparison function.

Then there exists a common best proximity point x to the pair (S, T).

Proof . Since $A_0 \neq \emptyset$ we can choose $x_0 \in A_0$ and fix it. By Lemma 1.6 $Sx_0 \in S(A_0) \subset B_0$ and then by definition of A_0 we can find $x_1 \in A_0$ such that $d(x_1, Sx_0) = d(A, B)$. Since $Tx_1 \in T(A_0) \subset B_0$, we can find $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Considering that $x_2 \in A_0$ and $S(A_0) \subset B_0$ we can find $x_3 \in A_0$ such that $d(x_3, Sx_2) = d(A, B)$. In this way we can find $x_4 \in A_0$ such that

$d(x_4, Tx_3) = d(A, B)$ as $Tx_3 \in T(A_0) \subset B_0$. By continuing this process we can get the sequence $(x_n)_{n\in\mathbb{N}}$ in A_0 such that for any $n \in \mathbb{N}$

$$d(x_{2n}, Tx_{2n-1}) = d(A, B) \tag{2.1}$$

$$d(x_{2n-1}, Sx_{2n-2}) = d(A, B). \tag{2.2}$$

Since (A, B) has weak P$-$property and by condition (2) we can show that

$$d(x_{2n}, x_{2n-1}) \leq d(Tx_{2n-1}, Sx_{2n-2}) = d(Sx_{2n-2}, Tx_{2n-1})$$

and

$$d(x_{2n+1}, x_{2n}) \leq d(Sx_{2n}, Tx_{2n-1}) = d(Tx_{2n-1}, Sx_{2n}).$$

By condition (2)

$$d(x_{2n+2}, x_{2n+1}) \leq d(Sx_{2n}, Tx_{2n+1}) \leq \phi(d(x_{2n}, x_{2n+1})),$$

$$d(x_{2n+1}, x_{2n}) \leq d(Sx_{2n}, Tx_{2n-1}) \leq \phi(d(x_{2n}, x_{2n-1})).$$

Thus we have

$$d(x_{n+1}, x_n) \leq \phi\big(d(x_n, x_{n-1})\big) \qquad \text{for all } n \in \mathbb{N}, \tag{2.3}$$

so,

$$d(x_n, x_{n-1}) \leq \phi\big(d(x_{n-1}, x_{n-2})\big) \qquad \text{for all } n \in \mathbb{N},$$

hence,

$$\phi\big(d(x_n, x_{n-1})\big) \leq \phi^2\big(d(x_{n-1}, x_{n-2})\big) \qquad \text{for all } n \in \mathbb{N},$$

and so

$$d(x_{n+1}, x_n) \leq \phi^2\big(d(x_{n-1}, x_{n-2})\big) \qquad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Again by inequality (2.3) we have

$$d(x_{n-1}, x_{n-2}) \leq \phi\big(d(x_{n-2}, x_{n-3})\big) \qquad \text{for all } n \in \mathbb{N},$$

so

$$\phi\big(d(x_{n-1}, x_{n-2})\big) \leq \phi^2\big(d(x_{n-2}, x_{n-3})\big) \qquad \text{for all } n \in \mathbb{N},$$

hence

$$\phi^2\big(d(x_{n-1}, x_{n-2})\big) \leq \phi^3\big(d(x_{n-2}, x_{n-3})\big) \qquad \text{for all } n \in \mathbb{N},$$

and so by inequality (2.4) we get

$$d(x_{n+1}, x_n) \leq \phi^3\big(d(x_{n-2}, x_{n-3})\big) \qquad \text{for all } n \in \mathbb{N}.$$

By continuing this process we can get the following inequation

$$d(x_{n+1}, x_n) \leq \phi^n(d(x_1, x_0)) \qquad \text{for all } n \in \mathbb{N}.$$

Therefore $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. By Lemma 1.5 we prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $(x_n)_{n\in\mathbb{N}}$ is not Cauchy. Then Lemma 1.5 implies that there exists $\epsilon > 0$ and two sequences $\big(n(k)\big)$ and $\big(m(k)\big)$ of positive integers such that

$$n(k) > m(k) > k,\ d(x_{m(k)}, x_{n(k)-1}) < \epsilon \ \text{ and } \ d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{2.5}$$

We claim that

$$d(x_{m(k)+1}, x_{n(k)}) \leq \phi\big(d(x_{m(k)}, x_{n(k)-1})\big). \tag{2.6}$$

To prove this claim we consider four possible cases for $m(k)$ and $n(k)$.

The first case: Let $m(k)$ and $n(k)$ be odd. Then by equations (2.1) and (2.2) we have

$$d(x_{m(k)+1}, Tx_{m(k)}) = d(A, B),$$

$$d(x_{n(k)}, Sx_{n(k)-1}) = d(A, B)$$

and so by weak $P-$property of (A, B) we get

$$d(x_{m(k)+1}, x_{n(k)}) \leq d(Tx_{m(k)}, Sx_{n(k)-1}).$$

Hence by condition (2) we have

$$d(x_{m(k)+1}, x_{n(k)}) \leq \phi\big(d(x_{m(k)}, x_{n(k)-1})\big).$$

The scond case: Let $m(k)$ and $n(k)$ be even. with the same way as the first case, we can prove that

$$d(x_{m(k)+1}, x_{n(k)}) \leq \phi\big(d(x_{m(k)}, x_{n(k)-1})\big).$$

The third case: Let $m(k)$ be even and $n(k)$ be odd. Then by equations (2.1) and (2.2) we have

$$d(x_{m(k)+1}, Sx_{m(k)}) = d(A, B),$$

$$d(x_{n(k)}, Sx_{n(k)-1}) = d(A, B)$$

and so by weak P-property of (A, B) we get

$$d(x_{m(k)+1}, x_{n(k)}) \leq d(Sx_{m(k)}, Sx_{n(k)-1}).$$

By using condition (2) we can easily prove that

$$d(x_{m(k)+1}, x_{n(k)}) \leq \phi\big(d(x_{m(k)}, x_{n(k)-1})\big).$$

The fourth case: Let $m(k)$ be odd and $n(k)$ be even. Then with the same way as the third case, we can prove that

$$d(x_{m(k)+1}, x_{n(k)}) \leq \phi\big(d(x_{m(k)}, x_{n(k)-1})\big).$$

So the claim is established. Thus by (2.5) and (2.6) we get

$$d(x_{m(k)+1}, x_{n(k)}) \leq \phi(d(x_{m(k)}, x_{n(k)-1})) \leq \phi(\epsilon) < \epsilon. \tag{2.7}$$

Letting $k \to +\infty$ in inequation (2.7), we get

$$\epsilon = \lim_{k \to +\infty} d(x_{m(k)+1}, x_{n(k)}) \leq \phi(\epsilon) < \epsilon.$$

This is a contradiction. Hence (x_n) is a Cauchy sequence in A. As A is a closed subset of a complete metric space so A is complete. Therefore there exists $x \in A$ such that $x_n \to x$ when $n \to +\infty$. We prove that $d(Sx_n, Sx) \to 0$ and $d(Tx_n, Tx) \to 0$ as $n, m \to +\infty$.

$$d(Sx_n, Sx) \leq d(Sx_n, Tx_m) + d(Tx_m, Sx) \leq \phi(d(x_n, x_m)) + \phi(d(x_m, x)). \tag{2.8}$$

Take $n, m \to +\infty$ in equation (2.8) we get that $d(Sx_n, Sx) \to 0$. With a similar process we can show that $d(Tx_n, Tx) \to 0$. Letting $n \to +\infty$ in (2.1) and (2.2) we get that $d(x, Sx) = d(x, Tx) = d(A, B)$ i.e. x is a common best proximity point for the pair of mappings (S, T) from the pair of (A, B). Now we prove that x is unique. Let y be another element in A such that

$$d(y, Sy) = d(y, Ty) = d(A, B).$$

Since

$$d(x, Sx) = d(x, Tx) = d(A, B),$$
$$d(y, Sy) = d(y, Ty) = d(A, B)$$

and by using weak P-property of the pair (A, B) we get the three following inequations

(1)
$$d(x, y) \leq d(Sx, Ty) \leq \phi(d(x, y)),$$

(2)
$$d(x, y) \leq d(Sx, Sy) \leq d(Sx, Ty) + d(Ty, Sy) \leq \phi(d(x, y)) + \phi(d(y, y)) = \phi(d(x, y)),$$

(3)
$$d(x, y) \leq d(Tx, Ty) \leq d(Tx, Sy) + d(Sy, Ty) \leq \phi(d(x, y)) + \phi(d(y, y)) = \phi(d(x, y)).$$

Since ϕ is a comparison functions, we have $\phi(t) \leq t$. So each of the above inequations implies that $d(x, y) = 0$. This completes the proof. □

Now,we state an example illustrates our result.

Example 2.2. *Consider* $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$, $A = \{0, \frac{1}{2}, \frac{1}{4}, ...\}$ *and* $B = \{0, 1, \frac{1}{3}, \frac{1}{5}, ...\}$. *Let* $d : X \times X \longrightarrow [0, +\infty)$ *be a metric on* X *defined by*

$$d(x, y) = \begin{cases} 0 & x = y \\ \max\{x, y\} & x \neq y \end{cases}.$$

Here, $A_0 = \{0\}$, $B_0 = \{0\}$, $d(A, B) = 0$ *and the pair* (A, B) *has weak* $p-$*property. Suppose that* $S, T : A \to B$ *and* $\phi : [0, +\infty) \to [0, +\infty)$ *are defined by*

$$S(x) = \frac{x}{1 - x}; \quad T(x) = \frac{x}{1 + x};$$

$$\phi(x) = \begin{cases} \dfrac{x}{1 - x} & x \in [0, 1) \\ 0 & otherwise \end{cases}$$

It is easy to check that ϕ *is a comparison function. We prove that the pair* (S, T) *is satisfied in condition* (2) *in Theorem 2.1 with comparison function* ϕ. *Let* x *and* y *belong to* A. *We split the checking of condition* (2) *in Theorem 2.1 for the pair* (S, T), *into five cases.*

The first case: If $x = \dfrac{1}{n}$, $y = \dfrac{1}{m}$ *(n and m are even) and* $n < m$ *then*

$$d\big(S(\frac{1}{n}), T(\frac{1}{m})\big) = d(\frac{1}{n-1}, \frac{1}{m-1}),$$
$$\phi\big(d(\frac{1}{n}, \frac{1}{m})\big) = \phi(\frac{1}{n}) = \frac{1}{n-1}.$$

So,

$$d\big(S(\frac{1}{n}),T(\frac{1}{m})\big)=\phi\big(d(\frac{1}{n},\frac{1}{m})\big).$$

The second case: Let $x=\frac{1}{n}$, $y=\frac{1}{m}$ *(n and m are even) and* $n<m$. *Since* n *and* m *are even,* $n-m\geq 2$.
If $n-m=2$ *then*

$$d\big(S(\frac{1}{n}),T(\frac{1}{m})\big)=d(\frac{1}{n-1},\frac{1}{m+1})=0$$
$$\phi\big(d(\frac{1}{n},\frac{1}{m})\big)=\phi(\frac{1}{m})=\frac{1}{m-1}.$$

Hence

$$d\big(S(\frac{1}{n}),T(\frac{1}{m})\big)\leq\phi\big(d(\frac{1}{n},\frac{1}{m})\big).$$

If $n-m\geq 2$, *then*

$$d\big(S(\frac{1}{n}),T(\frac{1}{m})\big)=d(\frac{1}{n-1},\frac{1}{m+1})=\frac{1}{m+1}$$
$$\phi\big(d(\frac{1}{n},\frac{1}{m})\big)=\phi(\frac{1}{m})=\frac{1}{m-1}.$$

So

$$d\big(S(\frac{1}{n}),T(\frac{1}{m})\big)\leq\phi\big(d(\frac{1}{n},\frac{1}{m})\big).$$

The third case: If $x=y=0$ *then*

$$d\big(S(0),T(0)\big)=d(0,0)=0$$
$$\phi\big(d(0,0)\big)=\phi(0)=0.$$

So

$$d\big(S(0),T(0)\big)=\phi\big(d(0,0)\big).$$

The fourth case: If $x=0$ *and* $y=\frac{1}{m}$ *then,*

$$d\big(S(0),T(\frac{1}{m})\big)=d(0,\frac{1}{m+1})=0$$
$$\phi\big(d(0,\frac{1}{m})\big)=\phi(0)=0.$$

So

$$d\big(S(0),T(\frac{1}{m})\big)=\phi\big(d(0,\frac{1}{m})\big).$$

The fifth case: If $x=\frac{1}{n}$ *and* $y=0$ *then,*

$$d\big(S(\frac{1}{n}),T(0)\big)=d(\frac{1}{n-1},0)=0$$
$$\phi\big(d(\frac{1}{n},0)\big)=\phi(0)=0.$$

So

$$d\big(S(\frac{1}{n}), T(0)\big) = \phi\big(d(\frac{1}{n-1}, 0)\big).$$

Therefore, the pair (S,T) is satisfied in condition (2) in Theorem 2.1 with comparison function ϕ. Obviously, the functions S and T are continuous. Hence, by using Theorem 2.1 we conclude that (S,T) has unique common best proximity point in A, $x^* = 0$.

Definition 2.3. *Let (A,B) be a pair of non-empty subsets of a metric space (X,d), $\rho(A_0)$ be the power set of a set A_0 and $T : A \longrightarrow B$ be a non−self mapping. Consider the set−valued mapping $P^T_{A_0} : T(\overline{A_0}) \longrightarrow \rho(A_0)$ which defined by*

$$P^T_{A_0}(y) = \{x \in A_0; d(x,y) = d(A,B)\}.$$

If for any $y \in T(\overline{A_0})$; $P^T_{A_0}(y)$ is non-empty then we say that T is proximinal and if for any $y \in T(\overline{A_0})$; $P^T_{A_0}(y)$ is a singleton then we say that T is Chebyshev.

Theorem 2.4. *Let (A,B) be a pair of non-empty closed subsets of a complete metric space (X,d) and let $S : A \longrightarrow B$ and $T : A \longrightarrow B$ be continuous mappings. Suppose that the following conditions are true:*

(1) The pair (A, B) has weak P-property,

(2) $A_0 \neq \emptyset$,

(3) The following inequalities are satisfied

- $d(Sx,Ty) \leq \alpha d(x,Sx) + \beta d(y,Ty) - d(A,B)$
- $d(Sx,Sy) \leq \alpha d(x,Sx) + \beta d(y,Sy) - d(A,B)$
- $d(Tx,Ty) \leq \alpha d(x,Tx) + \beta d(y,Ty) - d(A,B)$

in which α and β are positive real numbers such that $\alpha + \beta < 1$.

Then the pair (S,T) has only one common best proximity point.

Proof . By Lemma 1.7, B_0 is closed as $A_0 \neq \emptyset$ and the pair (A,B) has weak P−property. By Lemma 1.6 $S(\overline{A_0}) \subset B_0$ and $T(\overline{A_0}) \subset B_0$. Since the pair (A,B) has weak P−property, the mapping S is Chebyshev. So we have a well defined mapping $P^S_{A_0} : S(\overline{A_0}) \longrightarrow \rho(A_0)$ defined by $P^S_{A_0}(y) = x$ where $x \in A_0$ and $d(x,y) = d(A,B)$. Since the pair (A,B) has weak P−property, then $d(P^S_{A_0}(Sx), S(x)) = d(A,B)$ and $d\big(P^S_{A_0}(Sx), S(x)\big) = d(A,B)$ imply

$$\begin{aligned}&d\big(P^S_{A_0}(Sx), P^S_{A_0}(Sy)\big) \leq d(Sx,Sy) \leq \alpha d(x,Sx) + \beta d(y,Ty) - d(A,B) \leq \\ &\alpha d\big(x, P^S_{A_0}(Sx)\big) + \alpha d\big(P^S_{A_0}(Sx), Sx\big) + \beta d\big(y, P^S_{A_0}(Sy)\big) + \beta d(P^S_{A_0}(Sy), Sy) - d(A,B) \leq \\ &\alpha d\big(x, P^S_{A_0}(Sx)\big) + \beta d\big(y, P^S_{A_0}(Sy)\big)\end{aligned}$$

for every $x, y \in \overline{A_0}$. Therefore, by Theorem 1.2 the self-mapping $P^S_{A_0} \circ S : \overline{A_0} \to \overline{A_0}$ has a unique fixed point x^*_1. By a similar process we can show that the self−mapping $P^T_{A_0} \circ T : \overline{A_0} \to \overline{A_0}$ has a unique fixed point x^*_2. So

$$P^S_{A_0} \circ S(x_1^*) = x_1^* \Longrightarrow P^S_{A_0}(S(x_1^*)) = x_1^*$$
$$P^T_{A_0} \circ T(x_2^*) = x_2^* \Longrightarrow P^T_{A_0}(T(x_2^*)) = x_2^*.$$

Hence,

$$d(x_1^*, S(x_1^*)) = d(A, B)$$

and

$$d(x_2^*, T(x_2^*)) = d(A, B).$$

Claim: $x_1^* = x_2^*$. Suppose that $x_1^* \neq x_2^*$. By weak P-property

$$\begin{aligned} d(x_1^*, x_2^*) &\leq d(Sx_1^*, Tx_2^*) \leq \alpha d(x_1^*, Sx_1^*) + \beta d(x_2^*, Tx_2^*) - d(A, B) \\ &\leq \alpha d(A, B) + \beta d(A, B) - d(A, B) < 0. \end{aligned}$$

This is a contradiction. So $d(x_1^*, x_2^*) = 0$ and hence $x_1^* = x_2^* = x^*$ (call). Therefore x^* is a common best proximity point.Now remain that x^* is unique. If not then there exists another common best proximity point y^*(call) for the pair of none self-mapping (S, T). Then

$$d\big(x^*, S(x^*)\big) = d\big(x^*, T(x^*)\big) = d(A, B)$$

and

$$d\big(y^*, S(y^*)\big) = d\big(y^*, T(y^*)\big) = d(A, B).$$

Therefore, by using weak P-property of the pair (A, B) we get the three following inequations

(1)
$$d(x^*, y^*) \leq d(Sx^*, Ty^*) \leq \alpha d(x^*, Sx^*) + \beta d(y^*, Ty^*) - d(A, B) < 0,$$

(2)
$$d(x^*, y^*) \leq d(Sx^*, Sy^*) \leq d(Sx^*, Ty^*) + d(Ty^*, Sy^*) \leq \Big(\alpha d(x^*, Sx^*) + \beta d(y^*, Ty^*) - d(A, B)\Big) + \Big(\alpha d(y^*, Ty^*) + \beta d(y^*, Sy^*) - d(A, B)\Big) < 0,$$

(3)
$$d(x^*, y^*) \leq d(Tx^*, Ty^*) \leq d(Tx^*, Sy^*) + d(Sy^*, Ty^*) \leq \Big(\alpha d(x^*, Tx^*) + \beta d(y^*, Sy^*) - d(A, B)\Big) + \Big(\alpha d(y^*, Sy^*) + \beta d(y^*, Ty^*) - d(A, B)\Big) < 0.$$

So in any of three cases we have a contradiction. So $x^* = y^*$. Hence, the result holds. □

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