Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1343-1352
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.19290.2070

Some properties of generalized fundamentally nonexpansive mappings in Banach spaces

Mohammad Moosaei[a]

[a] *Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran*

(Communicated by Hamid Khodaei)

Abstract

In this paper, we introduce a condition on mappings and show that the class of these mappings is broader than both the class of mappings satisfying condition (C) and the class of fundamentally nonexpansive mappings, and it is incomparable with the class of quasi-nonexpansive mappings and the class of mappings satisfying condition (L). Furthermore, we present some convergence theorems and fixed point theorems for mappings satisfying the condition in the setting of Banach spaces. Finally, an example is given to support the usefulness of our results.

Keywords: fixed point, fundamentally nonexpansive mappings, normal structure, Opial's condition, quasi-nonexpansive mappings, uniformly convex in every direction Banach spaces, weakly compact
2010 MSC: Primary 47H09, 47H10; Secondary 46TXX

1. Introduction

Let K be a nonempty subset of a Banach space $(X, \|.\|)$, and T be a mapping of K into X. We denote by $\overline{K}$ and $F(T)$ the closure of K in the norm topology and the fixed points set of T, i.e., $F(T) = \{x \in K : Tx = x\}$, respectively. The mapping T is called (1) *nonexpansive*(in short NE) if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$; (2) *quasi-nonexpansive* (in short QNE) if $F(T)$ is nonempty and $\|Tx - u\| \leqslant \|x - u\|$ for all $x \in K$ and $u \in F(T)$; (3) *fundamentally nonexpansive* (in short FNE), see [6], if $\|T^2x - Ty\| \leqslant \|Tx - y\|$ for all $x, y \in K$.

Recently, Suzuki [8] introduced a condition on mappings which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Suzuki's condition was called by him condition (C) which

*Corresponding author
Email address: moosaeimohammad@gmail.com and m.moosaei@basu.ac.ir (Mohammad Moosaei)

Received: December 2019 *Accepted:* September 2020

is as follows:
Let K be a nonempty subset of a Banach space X and $T : K \to X$ be a mapping. The mapping T is said to satisfy *condition (C)* (or it is said an *(C)-type* mapping) if

$$\frac{1}{2}\|x - Tx\| \leqslant \|x - y\| \text{ implies } \|Tx - Ty\| \leqslant \|x - y\|$$

for all $x, y \in K$. Furthermore, he presented some interesting fixed point theorems and convergence theorems for such mappings. In particular, he showed that if K is a nonempty weakly compact convex subset of a uniformly convex in every direction Banach space, then the fixed points set of self-mappings satisfying condition (C)on K is nonempty.

In this paper, we introduce a condition on mappings which is more general than both condition (C) and fundamental nonexpansiveness. We also give several convergence theorems and fixed point theorems for such mappings in the framework of Banach spaces. Hence it is organized as follows:
In Section 2, we recall some definitions, notations and results which will be employed in next sections. In Section 3, we introduce a new class of generalized nonexpansive mappings which it is called the class of generalized fundamentally nonexpansive mappings (in short GFNE) and show the class of such mappings is broader than both the class of mappings satisfying condition (C) and the class of fundamentally nonexpansive mappings; furthermore, we indicate that it is not comparable with both the class of quasi-nonexpansive mappings and the class of mappings satisfying condition (L). We also collect some properties of GFNE mappings. In Section 4, we present some convergence theorems and fixed point theorems for generalized fundamentally nonexpansive mappings. Specially, we prove that if range of a generalized fundamentally nonexpansive self-mapping is a weakly compact convex subset of a Banach space with normal structure, then its the fixed points set is nonempty and closed; moreover, it is convex if the Banach space is strictly convex. Finally, for illustration of our results, an example is given.

2. Preliminaries

In this section, we recall the needed definitions, notations and results. Throughout this paper, we suppose that $(X, \|.\|)$ is a Banach space, and K is a nonempty subset of X. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers, respectively. Let $\{x_n\}$ be a sequence in X and $x \in X$, we denote by $x_n \rightharpoonup x, x_n \to x$ the weak convergence and the strong convergence of the sequence $\{x_n\}$ to x, respectively.
The Banach X is said to satisfy the *Opial condition*, see [7], if whenever a sequence $\{x_n\}$ in X converges weakly to $x \in X$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in X$ with $x \neq y$.

Example 2.1. *All Hilbert spaces and $l^p (1 < p < \infty)$ have the Opial condition (see [7]).*

The Banach space X is called (1) *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ (see [1]); (2) *uniformly convex in every direction* (UCED, for short), see [8], if for any $\epsilon \in (0, 2]$ and $z \in X$ with $\|z\| = 1$, there exists $\delta = \delta(\epsilon, z) > 0$, whenever $x, y \in X$, $\|x\| \leqslant 1, \|y\| \leqslant 1$ and $x - y \in \{tz : t \in [-2, -\epsilon] \cup [\epsilon, 2]\}$, then $\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta$. It is evident that every uniformly convex in every direction Banach space is a strictly convex Banach space.

Let K be a nonempty bounded subset of a Banach space $(X, \|.\|)$. A point $x_0 \in K$ is called a *nondiametral point* of K if

$$sup\{\|x - x_0\| : x \in K\} < diam(K),$$

where $diam(K)$ denotes diameter of K (see [1, page 146] for more details).
A nonempty convex subset K of a Banach space X is said to have *normal structure* if for any convex bounded subset D of K with more than one point contains a nondiametral point. The Banach space X is said to have *normal structure* if each closed convex bounded subset K of X with at least two points has normal structure (see [1]).

Let $T : K \to K$ be a mapping. A sequence $\{x_n\}$ in K is called an *approximate fixed point sequence* (AFPS, for short) for T in K if $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, see [1].

We will need the following two definitions.
Let K be a nonempty closed convex and bounded subset of a Banach space X, and $T : K \to K$ be a mapping. The mapping T is said to have *property (A)* if for any nonempty closed and convex subset D of K with T-invariant (i.e., $T(D) \subseteq D$), T has an AFPS in D, see [2]. The mapping T is said to satisfy *condition (L)* (or it is said an *(L)-type* mapping), see [5], if the following hold:

(i) T has property (A).
(ii) For any AFPS $\{x_n\}$ of T in K,

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|$$

holds for every $x \in K$.

Using the part (ii) of the above definition, One can show that every mapping satisfies condition (L) whose the set of fixed points is nonempty, is a QNE mapping.

Let $T : K \to X$ be a mapping. The mapping T satisfies *condition (E_μ)* if there exists $\mu \geq 1$ such that

$$\|x - Ty\| \leq \mu \|x - Tx\| + \|x - y\|$$

holds for all $x, y \in K$. T is said to satisfy *condition (E)* on K if there exists $\mu \geq 1$ such that T satisfies condition (E_μ), see [4].

The following results were given in [3] and [5], respectively.

Lemma 2.1. *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space X such that*

$$x_{n+1} = \lambda y_n + (1 - \lambda)x_n \quad \text{and} \quad \|y_n - y_{n+1}\| \leq \|x_n - x_{n+1}\|$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 1)$. Then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

Theorem 2.2. *Let X be a Banach space with normal structure. Let K be a nonempty weakly compact and convex subset of X. Let $T : K \to K$ be a mapping satisfying condition (L), then T has a fixed point.*

3. Some properties of GFNE mappings

In this section, we introduce a new class of generalized nonexpansive mappings and study basic properties of its members.

Definition 3.1. *Let K be a nonempty subset of a Banach space X. We say that a mapping $T : K \to X$ is generalized fundamentally nonexpansive (in short GFNE) if*

$$\tfrac{1}{2}\,\|Tx - x\| \leqslant \|x - Ty\| \text{ implies } \|Tx - T^2y\| \leqslant \|x - Ty\|$$

holds for all $x, y \in K$.

Example 3.1. *Let* $X = \mathbb{R}$ *and* $K = [0,1]$. *Define a mapping* $T : K \to K$ *as follows:*

$$Tx = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then $F(T) = \{\frac{1}{2}\}$. *We observe that*

$$\frac{1}{2} = \Big|T(\frac{51}{100}) - \frac{1}{2}\Big| > \Big|\frac{51}{100} - \frac{1}{2}\Big| = \frac{1}{100},$$

so T *is not QNE and hence it is no satisfying condition (L). We now show that the mapping* T *is GFNE. To see that, let* $(x,y) \in [0,1] \times [0,1]$ *and consider the following two cases.*
Case 1. *If* $(x,y) \in \big([0,\frac{1}{2}] \times [0,\frac{1}{2}]\big) \cup \big([0,\frac{1}{2}] \times (\frac{1}{2},1]\big) \cup \big((\frac{1}{2},1] \times (\frac{1}{2},1]\big) \cup$

$\big(\{1\} \times [0,\frac{1}{2}]\big)$, *then* $\big|Tx - T^2y\big| \leq \big|x - Ty\big|$ *holds.*

Case 2. *If* $(x,y) \in (\frac{1}{2},1) \times [0,\frac{1}{2}]$, *then* $\big|Tx - T^2y\big| > \big|x - Ty\big|$ *implies*

$$\tfrac{1}{2}\big|Tx - x\big| > \big|x - Ty\big|.$$

These mean that T *is GFNE. On the other hand, we have*

$$\big|T(0.51) - T^2(0.5)\big| > \big|0.51 - T(0.5)\big|.$$

Hence T *is not FNE. We also have*

$$\tfrac{1}{2}\big|T(0.5) - 0.5\big| \leq \big|0.5 - 0.6\big| \text{ and } \big|T(0.5) - T(0.6)\big| > \big|0.5 - 0.6\big|,$$

these mean that T *dose not satisfy condition (C). Briefly,* T *is not QNE, FNE and satisfying conditions (C) and (L), but it is GFNE.*

Example 3.2. *Let* $X = \mathbb{R}$ *and* $K = [0,1]$. *Define a mapping* $S : K \to K$ *as follows:*

$$Sx = \begin{cases} 0 & \text{if } x \neq 1 \\ 0.9 & \text{if } x = 1. \end{cases}$$

Then $F(S) = \{0\}$ *and* S *is QNE. Because* $\big|Sx - 0\big| \leqslant \big|x - 0\big|$ *holds, for all* $x \in K$. *Also, we have*

$$\tfrac{1}{2}\big|S(1) - 1\big| \leqslant \big|1 - S(1)\big| \text{ and } \big|S(1) - S^2(1)\big| > \big|1 - S(1)\big|.$$

Therefore, S *is not GFNE. Here, we show that* S *satisfies condition (L). Suppose* D *is a nonempty closed convex and* T*-invariant subset of* K, *then* $0 \in D$. *Thus there exists some* k *in* $\mathbb{N}$ *such that the sequence* $\{\frac{1}{n}\}_{n=k}^{\infty}$ *is an AFPS for* S *in* D. *It is obvious that the sequence* $\{\frac{1}{n}\}_{n=1}^{\infty}$ *is an AFPS for* S *in* K. *These imply that* S *satisfies condition (L). Briefly,* S *satisfies condition (L), but it is not GFNE.*

We now gather some basic properties for generalized fundamentally nonexpansive mappings. The following propositions are easy to verify.

Proposition 3.2. *Let K be a nonempty subset of a Banach space X and $T : K \to X$ be a mapping satisfying condition (C). Then T is a generalized fundamentally nonexpansive mapping.*

Proposition 3.3. *Let K be a nonempty subset of a Banach space X and $T : K \to X$ be a fundamentally nonexpansive mapping. Then T is a generalized fundamentally nonexpansive mapping.*

Remark 3.4. *Example 3.1 shows that the converse of Propositions 3.2 and 3.3 are not true in general. Therefore, the class of generalized fundamentally nonexpansive mappings is broader than both the class of mappings satisfying condition (C) and the class of fundamentally nonexpansive mappings; hence, it is broader than the class of nonexpansive mappings. Moreover, it shows that the class of (L)-type mappings is not larger than the class of generalized fundamentally nonexpansive mappings. By attention to Examples 3.1 and 3.2, we observe that the class of generalized fundamentally nonexpansive mappings and the class of quasi-nonexpansive mappings are not comparable, whereas both the class of (C)-type mappings with a fixed point and the class of fundamentally nonexpansive mappings with a fixed point are contained in the class of quasi-nonexpansive mappings (see [6] and [8], for more details). Furthermore, from Examples 3.1 and 3.2, we realize that the class of generalized fundamentally nonexpansive mappings and the class of (L)-type mappings are not comparable.*

The following lemmas are useful to prove our results.

Lemma 3.5. *Let K be a nonempty subset of a Banach space X and $T : K \to X$ be a generalized fundamentally nonexpansive mapping. Then for all $x, y \in K$, the following statements hold:*

(i) $\|Tx - T^2x\| \leqslant \|x - Tx\|$.
(ii) Either $\frac{1}{2}\|Tx - x\| \leqslant \|x - Ty\|$ *or* $\frac{1}{2}\|T^2x - Tx\| \leqslant \|Tx - Ty\|$.
(iii) Either $\|Tx - T^2y\| \leqslant \|x - Ty\|$ *or* $\|T^2x - T^2y\| \leqslant \|Tx - Ty\|$.

Proof . (i) Let $x \in K$. As $\frac{1}{2}\|Tx - x\| \leqslant \|x - Tx\|$, the generalized fundamental nonexpansiveness of T implies

$$\left\|Tx - T^2x\right\| \leqslant \|x - Tx\|.$$

(ii) Suppose, for contradiction, that

$$\tfrac{1}{2}\|Tx - x\| > \|x - Ty\| \text{ and } \tfrac{1}{2}\|T^2x - Tx\| > \|Tx - Ty\|,$$

whenever $x, y \in K$. This and (i) yield

$$\begin{aligned}\|x - Tx\| &\leqslant \|x - Ty\| + \|Tx - Ty\| \\ &< \frac{1}{2}\|x - Tx\| + \frac{1}{2}\left\|Tx - T^2x\right\| \\ &\leqslant \frac{1}{2}\|x - Tx\| + \frac{1}{2}\|x - Tx\| \\ &= \|x - Tx\|,\end{aligned}$$

which is a contradiction. (iii) Follows from (ii). Therefore, we get the desired result. □

Lemma 3.6. *Let K be a nonempty subset of a Banach space X and T from K into X be a generalized fundamentally nonexpansive mapping with $F(T) \neq \emptyset$. Then*

$$\|T^2x - u\| \leqslant \|Tx - u\|$$

holds for all $x \in K$ and $u \in F(T)$.

Proof . Fix $x \in K, u \in F(T)$. Since $\frac{1}{2}\|u - Tu\| \leqslant \|u - Tx\|$, the generalized fundamental nonexpansiveness of T implies that $\|T^2x - u\| \leqslant \|Tx - u\|$. □

Lemma 3.7. *Let K be a nonempty subset of a Banach space X and $T : K \to X$ be a generalized fundamentally nonexpansive mapping. Then*

$$\|x - T^2y\| \leqslant 3\|x - Tx\| + \|x - Ty\|$$

holds for each $x, y \in K$.

Proof . Let $x, y \in K$. By Lemma 3.5 (iii), we have either

$$\|Tx - T^2y\| \leqslant \|x - Ty\| \text{ or } \|T^2x - T^2y\| \leqslant \|Tx - Ty\|.$$

We consider the following two cases:
Case 1. Assume that $\|Tx - T^2y\| \leqslant \|x - Ty\|$ holds. Then we obtain

$$\begin{aligned}\left\|x - T^2y\right\| &\leqslant \|x - Tx\| + \left\|Tx - T^2y\right\| \\ &\leqslant \|x - Tx\| + \|x - Ty\| \\ &\leqslant 3\|x - Tx\| + \|x - Ty\|.\end{aligned}$$

Case 2. Suppose that $\|T^2x - T^2y\| \leqslant \|Tx - Ty\|$ holds. This and Lemma 3.5 (i) imply

$$\begin{aligned}\left\|x - T^2y\right\| &\leqslant \|x - Tx\| + \left\|Tx - T^2x\right\| + \left\|T^2x - T^2y\right\| \\ &\leqslant 2\|x - Tx\| + \|Tx - Ty\| \\ &\leqslant 3\|x - Tx\| + \|x - Ty\|.\end{aligned}$$

Therefore, we get the demand result in both cases. □

The structure of the fixed points set of a generalized fundamentally nonexpansive mapping is studied in the following proposition.

Proposition 3.8. *Let K be a nonempty subset of a Banach space $(X, \|.\|)$, and $T : K \to K$ be a generalized fundamentally nonexpansive mapping with $F(T) \neq \emptyset$. Then the following statements hold:*

(i) If the closure of $F(T)$ is included in $T(K)$, then $F(T)$ is closed. In particular, if $T(K)$ is closed, then $F(T)$ is closed too.
(ii) If X is strictly convex and $T(K)$ is convex, then $F(T)$ is convex too.

Proof . (i) Let $u \in \overline{F(T)}$, then there is a sequence $\{x_n\}$ in $F(T)$ such that $x_n \to u$ as $n \to \infty$. Lemma 3.6 implies that the inequality

$$\|x_n - Tu\| \leq \|x_n - u\|$$

holds for each $n \in \mathbb{N}$. Because $x_n \to u$ as $n \to \infty$, the above inequality implies that u is a fixed point of T. Therefore, $F(T)$ is closed.

(ii) Let $u, v \in F(T)$ with $u \neq v$. Fix $\lambda \in (0, 1)$, and put $z = \lambda u + (1 - \lambda)v$. Since $T(K)$ is convex, $z \in T(K)$. Hence there exists some $x \in K$ such that $z = Tx$. As $u \in F(T)$, Lemma 3.2 implies that $\|T^2x-u\| \leqslant \|u-Tx\|$. So $\|Tz-u\| \leqslant \|u-z\|$. Similarly, we obtain $\|Tz-v\| \leqslant \|v-z\|$. Therefore, we have

$$\begin{aligned}\|u-v\| &\leqslant \|u-Tz\| + \|v-Tz\| \\ &\leqslant \|u-z\| + \|v-z\| \\ &= (1-\lambda)\|u-v\| + \lambda\|u-v\| \\ &= \|u-v\|,\end{aligned}$$

which yields

$$\begin{aligned}\|u-v\| &= \|u-Tz\| + \|v-Tz\| \\ &= \|u-z\| + \|v-z\|.\end{aligned} \tag{3.1}$$

Since X is strictly convex, it follows that there exists some $\mu \in (0,\infty)$ such that $Tz-v = \mu(u-Tz)$. This implies

$$Tz = \frac{\mu}{1+\mu}u + \frac{1}{1+\mu}v. \tag{3.2}$$

On the other hand, since $\|u-Tz\| \leqslant \|u-z\|$, equality (3.1) implies

$$\|u-Tz\| = \|u-z\|. \tag{3.3}$$

From this and equality (3.2), we get

$$\left\|\tfrac{\mu}{1+\mu}u + \tfrac{1}{1+\mu}u - \tfrac{\mu}{1+\mu}u - \tfrac{1}{1+\mu}v\right\| = \|\lambda u + (1-\lambda)u - \lambda u - (1-\lambda)v\|,$$

which implies $\lambda = \frac{\mu}{1+\mu}$. From this and (3.2), we obtain

$$Tz = \lambda u + (1-\lambda)v = z.$$

Consequently, $F(T)$ is convex. □

Example 3.3. *Set $X = \mathbb{R}$ and $K = [0,1] \cup \{1.1\}$. Define a mapping $T : K \to K$ by*

$$Tx = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x \in \{1, 1.1\}. \end{cases}$$

Then T is GFNE, but it dose not satisfy condition (L). To see this, put

$$M = \big([0,1) \times [0,1)\big) \cup \big([0,1) \times \{1, 1.1\}\big) \cup \big(\{1, 1.1\} \times \{1, 1.1\}\big)$$

and $N = \{1, 1.1\} \times [0,1)$. Let $x, y \in K$ be arbitrary. If $(x,y) \in M$, then

$$|Tx - T^2y| \leq |x - Ty|.$$

If $(x,y) \in N$, then

$$\frac{1}{2}|x - Tx| \leq |x - Ty| \text{ implies } |Tx - T^2y| \leq |x - Ty|.$$

These imply that T is GFNE. Because $1 \in F(T)$ and $|1 - T(1.1)| \nleq ||1 - 1.1|$, T is not QNE, hence it dose not satisfy condition (L). It is clear that $F(T)$ is nonempty and convex but not closed, as $\overline{F(T)} \nsubseteq T(K)$.

Proposition 3.9. *Let K be a nonempty subset of a Banach space X and $T : K \to K$ be a generalized fundamentally nonexpansive mapping with $F(T) \neq \emptyset$. Assume also $T(K)$ is convex. Fix $x_1 \in K$ and define a sequence $\{Tx_n\}$ in $T(K)$ by x_1 and*

$$Tx_{n+1} = \lambda T^2 x_n + (1-\lambda)Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $\lambda \in (0,1)$. Then the sequence $\{\|Tx_n - u\|\}$ is convergent for all $u \in F(T)$.

Proof . Fix $u \in F(T)$. By Lemma 3.6, we have

$$\begin{aligned}\|Tx_{n+1} - u\| &= \left\|\lambda(T^2x_n - u) + (1-\lambda)(Tx_n - u)\right\| \\ &\leqslant \lambda \left\|T^2x_n - u\right\| + (1-\lambda)\|Tx_n - u\| \\ &\leqslant \lambda \|Tx_n - u\| + (1-\lambda)\|Tx_n - u\| \\ &= \|Tx_n - u\|\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $\{\|Tx_n - u\|\}$ is a bounded decreasing sequence. This completes the proof. □

4. Convergence theorems and fixed point theorems

In this section, we give some convergence theorems and fixed point theorems for generalized fundamentally nonexpansive mappings.

The following lemmas are useful to prove our main results.

Lemma 4.1. *Let K be a nonempty subset of a Banach space $(X, \|.\|)$, and let $T : K \to K$ be a a generalized fundamentally nonexpansive mapping whose the range is convex. Fix x_1 in K and define a sequence $\{Tx_n\}$ in $T(K)$ as follows:*

$$Tx_{n+1} = \lambda T^2 x_n + (1-\lambda)Tx_n \quad \text{for all } n \in \mathbb{N}, \tag{4.1}$$

where $\lambda \in [\frac{1}{2}, 1)$. It is an AFPS for T if the sequence $\{Tx_n\}$ is bounded.

Proof . By the assumption, we have

$$\tfrac{1}{2}\|T^2x_n - Tx_n\| \leqslant \lambda \|T^2x_n - Tx_n\| = \|Tx_{n+1} - Tx_n\|$$

for all $n \in \mathbb{N}$. Since T is generalized fundamentally nonexpansive, it follows that

$$\|T^2x_{n+1} - T^2x_n\| \leqslant \|Tx_{n+1} - Tx_n\| \text{ for all } n \in \mathbb{N}.$$

The above inequality along with Lemma 2.1 implies $\lim\limits_{n\to\infty} \|Tx_n - T^2x_n\| = 0$. Therefore, the proof of the lemma is completed. □

Lemma 4.2. *Let K be a nonempty subset of a Banach space $(X, \|.\|)$ and $T : K \to X$ be a generalized fundamentally nonexpansive mapping. If $\{x_n\}$ is an AFPS for T in K, then the following statements hold:*

(i) If $\{x_n\}$ converges strongly to some $u \in T(K)$, then u is a fixed point of T.
(ii) If $\{x_n\}$ converges weakly to some $u \in T(K)$ and X has Opial condition, then u is a fixed point of T.

Proof . (i) According to Lemma 3.7, we get

$$\|x_n - Tu\| \leqslant 3\|x_n - Tx_n\| + \|x_n - u\| \text{ for all } n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$, we obtain $Tu = u$.
(ii) As $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, In view of the above inequality, we have

$$\liminf_{n\to\infty} \|x_n - Tu\| \leqslant \liminf_{n\to\infty} \|x_n - u\|.$$

From this and the Opial condition, we conclude that $Tu = u$. □

Theorem 4.3. *Let K be a nonempty subset of a Banach space $(X, \|.\|)$, and let $T : K \to K$ be a generalized fundamentally nonexpansive mapping whose the range is convex. Assume $\{Tx_n\}$ is the sequence defined by (4.1). If $T(K)$ is compact, then the sequence $\{Tx_n\}$ converges strongly to a fixed point of T.*

Proof . As $T(K)$ is compact, there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\{Tx_{n_k}\}$ converges strongly to some $u \in T(K)$. By Lemma 4.1, $\{Tx_{n_k}\}$ is an AFPS for T. So using Lemma 4.2 (i), we deduce that u is a fixed point of T. This along with Proposition 3.9 implies the sequence $\{\|Tx_n - u\|\}$ is convergent. Suppose $\lim_{n\to\infty} \|Tx_n - u\| = r$. Since $\lim_{k\to\infty} \|Tx_{n_k} - u\| = 0$, consequently, $r = 0$. Therefore, the sequence $\{Tx_n\}$ converges strongly to u. This completes the proof. □

Theorem 4.4. *Let K be a nonempty subset of a Banach space $(X, \|.\|$ with Opial condition, and let $T : K \to K$ be a generalized fundamentally nonexpansive mapping whose the range is convex. Assume that $\{Tx_n\}$ is the sequence defined by (4.1), and $T(K)$ is weakly compact. Then the sequence $\{Tx_n\}$ converges weakly to a fixed point of T.*

Proof . The weak compactness of $T(K)$ yields that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\{Tx_{n_k}\}$ converges weakly to some $u \in T(K)$. According to Lemma 4.1, $\{Tx_{n_k}\}$ is an AFPS for T. Applying Lemma 4.2(ii), it follows that u is a fixed point of T. We next show that the sequence $\{Tx_n\}$ converges weakly to u. Suppose, for contradiction, that there is a subsequence $\{Tx_{m_j}\}$ of $\{Tx_n\}$ such that converges weakly to some $v \in T(K)$ with $u \neq v$. Similarly, one can prove that v is a fixed point of T. By the Opial condition and Proposition 3.9, we have

$$\begin{aligned}
\lim_{n\to\infty} \|Tx_n - u\| &= \liminf_{k\to\infty} \|Tx_{n_k} - u\| \\
&< \liminf_{k\to\infty} \|Tx_{n_k} - v\| \\
&= \liminf_{j\to\infty} \left\|Tx_{m_j} - v\right\| \\
&< \liminf_{j\to\infty} \left\|Tx_{m_j} - u\right\| \\
&= \lim_{n\to\infty} \|Tx_n - u\|,
\end{aligned}$$

which is a contradiction. Therefore, $Tx_n \rightharpoonup u$ as $n \to \infty$, and the proof is complete. □

Theorem 4.5. *Let K be a nonempty subset of a Banach space X with normal structure, and $T : K \to K$ be a generalized fundamentally nonexpansive mapping. If $T(K)$ is weakly compact and convex, then the fixed points set of T is nonempty and closed. Moreover, if X is strictly convex, then the fixed points set of T is convex as well.*

Proof . Using Lemma 3.7, Lemma 4.1 and [5, Proposition 3.5], we deduce that T satisfies condition (L) on $T(K)$. It follows from Theorem 2.2 that the fixed points set of T is nonempty, and by the part (i) of Proposition 3.8 we conclude that $F(T)$ is closed. Also, if X is strictly convex, the part (ii) of Proposition 3.8 implies the convexity of $F(T)$. $\square$

An immediate conclusion of the preceding theorem and Theorem 3.3.6 of [1] is the following.

Theorem 4.6. *Let K be a nonempty subset of a UCED Banach space X, and let $T : K \to K$ be a generalized fundamentally nonexpansive mapping. If $T(K)$ is a weakly compact convex subset of X, then the fixed points set of T is nonempty, closed and convex.*

The following example supports the usefulness of our results.

Example 4.1. *Let $X = \mathbb{R}$ and $K = [0, 1] \cup (\frac{3}{2}, 2]$. Define a mapping $T : K \to K$ by*

$$Tx = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{if } \frac{3}{2} < x \leqslant 2 \end{cases}$$

for all $x \in K$. Then $T(K) = [0, 1]$ is a weakly compact convex subset of the UCED Banach space X, but K is not weakly compact and convex. We also observe that $F(T) = [0, 1]$ and it is closed and convex.

We now show that T is GFNE. In order to see this, let $(x, y) \in K \times K$ and consider the following two cases:

Case1. *If $(x, y) \in \big([0, 1] \times [0, 1]\big) \cup \big([0, 1] \times (\frac{3}{2}, 2]\big) \cup \big((\frac{3}{2}, 2] \times (\frac{3}{2}, 2]\big)$, then*

$$\big|Tx - T^2y\big| \leqslant \big|x - Ty\big|.$$

Case 2. *If $(x, y) \in (\frac{3}{2}, 2] \times [0, 1]$, then we have the following implication holds.*

$$\tfrac{1}{2}\big|x - Tx\big| \leqslant \big|x - Ty\big| \text{ implies } \big|Tx - T^2y\big| \leqslant \big|x - Ty\big|.$$

These imply that T is a FNGE mapping. Since $1 \in F(T)$ and $1 = \big|1 - T(\frac{3}{2})\big| \nleqslant \big|1 - \frac{3}{2}\big| = \frac{1}{2}$, T is not QNE, T is FNGE, but it is neither QNE nor satisfying condition (L). We also observe that the domain of T is neither closed nor convex.

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Springer, New York, 2009.

[2] S. Dhompongsa and N. Nanan, *Fixed point theorems by ways of ultra-asymptotic centers*, Abst. Appl. Anal. 2011 (2011) Article ID 826851.

[3] K. Goebel and W. A. Kirk, *Iteration processes for nonexpansive mappings*, Contemp. Math. 21 (1983) 115–123.

[4] J. Garća-Falset, E. Llorens-Fuster and T. Suzuki, *Fixed point theory for a class of generalized of nonexpansive mappings*, J. Math. Anal. Appl. 375 (2011) 115–123.

[5] E. Llorens-Fuster and E. Moreno-Gálvez, *The fixed point theory for some generalized nonexpansive mappings*, Abst. Appl. Anal. 2011 (2011) Article ID 435686.

[6] M. Moosaei, *On fixed point of fundamentally nonexpansive mappings in Banach spaces*, Int. J. Nonlinear Anal. Appl. 7(1) (2016) 219–224.

[7] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1976) 591–597.

[8] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. 340 (2008) 1088–1095.