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Variable coefficient fractional partial differential equations by Base–Chebyshev method

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Abstract

In this paper, the invariant subspace method is generalized and improved and is then used together with the Chebyshev polynomial to approximate the solution of the non-linear, mixed fractional partial differential equations $FPDEs$ with constant, non-constant coefficients. Some examples are given here to illustrate the efficiency of this method.

Keywords: fractional partial differential equation, Caputo fractional derivative, invariant subspace method, Chebyshev polynomial approximate.

1. Introduction

The last decades have shown that derivatives and integrals of arbitrary order are very convenient for describing properties of real materials. The new fractional-order models are more satisfying than the former integer-order ones, a natural phenomenon may depend not only on the time instant but also on the previous time history, which can be modeled by fractional calculus [3].

So motivated by these reasons, it is important to find efficient methods for solving fractional partial differential equations ($FPDEs$) such as invariant subspace method which gives the exact solution for a wide class of Caputo time and space and mixed ($FPDEs$) with constant coefficients [5, 6, 7, 8, 9].

But in the case of non-constant coefficients, we couple the invariant subspace method with the shifted Chebyshev polynomial of first kind ($CISM$), to get the approximate solution for such equations. Firstly the main idea of the invariant subspace method is the separate equation variables, to get a system of ordinary fractional differential equation which can be easy to solve.
To explain this method, let us state here the following operator form of $FPDEs$

$$\sum_{j=0}^{n} \lambda_j \frac{\partial^{\alpha+j}}{\partial\, t^{\alpha+j}} u(x,t) = N\Big(x, u, \frac{\partial^{\beta}}{\partial x^{\beta}} u, \frac{\partial^{\beta+1}}{\partial x^{\beta+1}} u, \cdots, \frac{\partial^{\beta+m}}{\partial x^{\beta+m}} u\Big) + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}} \Big(\frac{\partial^{\beta}}{\partial x^{\beta}} u\Big) \tag{1.1}$$

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$$a < \alpha \le a+1, \quad b < \beta \le b+1, \quad a, b \in N, \quad \lambda_j, \ \mu \in R$$

Solution of this equation by the invariant subspace method given in the following theorem

Theorem 1.1. *[5] Suppose $I_{n+1} = L\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$ is a finite- dimensional linear space, and it is invariant with respect to the operators $N[u]$ and $\frac{\partial^\beta}{\partial x^\beta} u$, then FPDE (1.1) has an exact solution as follows:*

$$u(x,t) = \sum_{i=0}^{n} k_i(t)\phi_i(x) \tag{1.2}$$

where $\{k_i(t)\}$ satisfies the following FDEs :

$$\sum_{j=0}^{m_1} \lambda_j \frac{d^{\alpha+j}}{dt^{\alpha+j}} k_i(t) = \psi_i + \mu \frac{d^\alpha \psi_{n+1+i}}{dt^\alpha}, \qquad i = 0, \cdots, n \tag{1.3}$$

where $\{\psi_0, \psi_1, \cdots, \psi_n\}$, $\{\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n+1}\}$ are the expansion coefficients of $N[u]$, $\frac{\partial^\beta}{\partial x^\beta} u$ respectively with respect to the base $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$, $\psi_i = \psi_i\big(k_0(t), k_1(t), \cdots, k_n(t)\big)$.

2. Analysis of the CISM

To explain the analytic view of this technique, we must given a simple argue of Chebyshev polynomials method for approximate a specific function. There are several kinds of Chebyshev polynomials [2] which have an important position in modern developments including orthogonal polynomial, polynomial approximation, numerical integration, and spectral methods for partial differential equations. In particular we shall focus on the first kind only among the fourth others.
It is well known that the $n-$ degree Chebyshev polynomial of first kind $T_n(x)$, which defined on $[-1, 1]$ by :

$$T_n(x) = \cos(n\theta), \qquad x = \cos\theta, \qquad \theta \in [0, \ \pi]$$

which have the following recurrence formula

$$T_{n+1}(x) = 2x \ T_n(x) - T_{n-1}(x), \qquad n = 1, 2, 3, \cdots \qquad \text{with } \ T_0(x) = 1, \quad T_1(x) = x$$

$T_n(x)$ has the following analytic form:

$$T_n(x) = \sum_{k=0}^{n/2} (-1)^k \ \frac{2^{n-2k-1} \ n(n-k-1)!}{(2k)!(2n-2)!} \ x^{n-2k}, \qquad n = 2, 3, \cdots$$

these polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$ i.e:

$$\int_{-1}^{1} \omega(x) \ T_n(x) \ T_m(x) \ dx = \begin{cases} \pi & n = m = 0 \\ \pi/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

If we shifted $T_n(x)$ which defined on $[-1, 1]$ to the interval $[0, 1]$, then we change the variable $x = 2t - 1$, and we get $T_n^*(t)$, the shifted Chebyshev polynomial defined by

$$T^*_{n+1}(t) = 2(2t-1) \ T^*_n(t) - T^*_{n-1}(t), \qquad n = 1, 2, 3, \cdots \qquad \text{with } \ T^*_0(t) = 1, \ \ T^*_1(t) = 2t - 1$$

which are orthogonal with respect to the weight function $\omega^*(t) = 1/\sqrt{t-t^2}$, that is:

$$\int_0^1 \omega^*(t)\ T_n^*(t)\ T_m^*(t)\ dt = \begin{cases} \pi & n=m=0 \\ \pi/2 & n=m\neq 0 \\ 0 & n\neq m \end{cases}$$

Although the shifted Chebyshev polynomial of first kind has the following analytic form:

$$\begin{aligned} T_n^*(t) &= \sum_{k=0}^{n}(-1)^k\ \frac{2^{2n-2k}\ n(2n-k-1)!}{k!(2n-2)!}\ t^{n-k}, \qquad n=2,3,\cdots \quad \text{or} \\ &= n\ \sum_{k=0}^{n}(-1)^{n-k}\ \frac{2^{2k}\ (n+k-1)!}{(2k)!(n-k)!}\ t^{n-k}, \qquad n=2,3,\cdots \end{aligned}$$

A function $f(t)$, square integrable in $[0,1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$f(t) = \sum_{i=0}^{\infty} c_i\ T_i^*(t)$$

where the coefficients c_i are given by

$$c_i = \int_0^1 \omega^*(t)\ f(t)\ T_i^*(t)\ dt, \qquad i=0,1,2,\cdots$$

If we approximated $f(t)$ by $n-$order shifted first kind Chebyshev polynomials as:

$$f(t) \simeq \sum_{i=0}^{n}\ c_i\ T_i^*(t) = C^T\ \Phi(t)$$

such that C^T is the $(n+1)-$vector of constant, and $\Phi(t)$ is the $(n+1)\times(n+1)$ shifted first kind Chebyshev vector, then the fractional derivative $\alpha > 0$ of the shifted first kind Chebyshev polynomial has been expressed in the next theorem.

Theorem 2.1. *[2] Let $\Phi(t)$ be shifted first kind Chebyshev vector which defined by*

$$\Phi(t) = \begin{pmatrix} T_0^*(t) & T_1^*(t) & \cdots & T_n^*(t)\end{pmatrix}^T, \qquad \textit{then} \quad D^\alpha\ \Phi(t) = \Delta^\alpha\ \Phi(t)$$

where, Δ^α is $(m+1)\times(m+1)$ operational matrix of fractional derivative with respect to the Caputo sense and it is defined by

$$\Delta^\alpha = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ W_{0,0,i} & W_{0,1,i} & \cdots & W_{0,m,i} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=0}^{n-\lceil\alpha\rceil}\ W_{n-\lceil\alpha\rceil,0,i} & \sum_{i=0}^{n-\lceil\alpha\rceil}\ W_{n-\lceil\alpha\rceil,1,i} & \cdots & \sum_{i=0}^{n-\lceil\alpha\rceil}\ W_{n-\lceil\alpha\rceil,m,i} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=0}^{m}\ W_{m,0,i} & \sum_{i=0}^{m}\ W_{m,1,i} & \cdots & \sum_{i=0}^{m}\ W_{m,m,i} \end{pmatrix}$$

$$W_{n-\lceil\alpha\rceil,j,i} = \frac{c_j}{\sqrt{\pi}} \sum_{k=0}^{j} (-1)^{k+i} 2^{2(j+n-k-i)} \frac{n(2n-i-1)!j(2j-k-1)!\Gamma(n-i-\alpha+j-k+\frac{1}{2})}{i!(2n-2i)!k!(2j-2k)!\Gamma(n-i-\alpha+1)\Gamma(n-i-\alpha+k+1)}$$

$$\textit{where} \qquad c_j = \begin{cases} 1 & j=0 \\ 2 & j\neq 0 \end{cases}, \qquad n = \lceil\alpha\rceil \cdots m$$

Now, it is time to construct the solution of non-constant coefficients nonlinear fractional order partial differential equation to obtain an approximate solution of such equations by applying our technique in this section which conclude coupled the invariant subspace with the Chebyshev polynomial for solving $FPDE$ in the form:

$$\sum_{j=1}^{m} \lambda_j(t) D_t^{j\alpha}\ u = N\Big(x, u, D_x^{\beta}\ u, D_x^{2\beta}\ u, \cdots, D_x^{m_1\ \beta}\ u\Big) + \mu D_t^{\alpha}\ \Big(D_x^{\beta}\ u\Big) \tag{2.1}$$

Subject to the initial conditions

$$D_t^{j\alpha}\ u(x,0) = f_j(x) \tag{2.2}$$

where $u = u(x,t),\ N$ is a non–linear operator; $D_t^{j\alpha}\ u,\ j = 1,2,\cdots,m;\ m\ \in N$ and $D_x^{i\ \beta}\ u,\ i = 1,2,\cdots,m_1;\ m_1\ \in N$ are Caputo time derivatives and Caputo space derivatives, respectively.
$a < \alpha \leq a+1,\quad b < \beta \leq b+1,\ \ a,b\ \in\ N,\ \mu\ \in\ R.$
According to the invariant subspace method, which stated in Chapter 1, the exact solution of (2.1) has the form

$$u(x,t) = \sum_{i=0}^{n} k_i(t)\phi_i(x) \tag{2.3}$$

Where $\phi_i(x)$ are members of the invariant subspace $I_{n+1} = L\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}\ \textit{with}\ N[u]$ and $\frac{\partial^\beta}{\partial x^\beta}u$. Then, if we approximate the $k_i's$ functions by the shifted first kind of Chebyshev polynomials with order p, then we have

$$k_i(t) = \sum_{\omega=0}^{p} a_{i\omega}\ T_\omega(t) = A_i^T\ \Phi(t),$$

where A_i is a $p+1$ vector of constants and $\Phi(t)$ is a p Chebyshev function vector, so (2.3)

$$u(x,t) = \sum_{i=0}^{n} k_i(t)\phi_i(x) = \sum_{i=0}^{n}\sum_{\omega=0}^{p} a_{i\omega}\ T_\omega(t) = \sum_{i=0}^{n} A_i^T\ \Phi(t)\phi_i(x).$$

Then the left hand side of (2.1) became

$$\begin{aligned} \sum_{j=1}^{m} \lambda_j(t) D_t^{j\alpha}\ u(x,t) &= \sum_{j=1}^{m} \lambda_j(t) D_t^{j\alpha} \sum_{i=0}^{n} k_i(t)\phi_i(x) = \sum_{j=1}^{m} \lambda_j(t) \sum_{i=0}^{n} D_t^{j\alpha}\ A_i^T\ \Phi(t)\phi_i(x) \\ &= \sum_{i=0}^{n}\sum_{j=1}^{m} \lambda_j(t) A_i^T\ \Delta^{j\alpha}\ \Phi(t)\ \phi_i(x) \end{aligned} \tag{2.4}$$

where $\Delta^{j\alpha}$ is the approximate matrix operation of the $j\alpha$ fractional Caputo derivative.
Since there are $2n+2$ functions $\quad \psi_0, \psi_1, \cdots, \psi_n; \quad \psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n+1}$

$$\psi_i = \psi_i\Big(k_0(t), k_1(t), \cdots, k_n(t)\Big), \quad i = 0,1,2,\cdots,2n+2 \ \text{ such that}$$

$$N[u] = N\Big(\sum_{i=0}^{n} k_i(t)\phi_i(x)\Big) = \sum_{i=0}^{n} A_i\ \Phi(t)\ \phi_i(x) = \sum_{i=0}^{n} \psi_i\ \ \phi_i(x)$$
$$D_x^{\beta}\ u(x,t) = \sum_{i=0}^{n} \psi_{n+1+i}\ \ \phi_i(x)$$

where $\{\psi_0, \psi_1, \cdots, \psi_n\}$, $\{\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n+1}\}$are the expansion coefficients of $N[u], D_x^{\beta}\ u$ respectively with respect to $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$. Thus

$$\begin{aligned} N[u] + \mu D_t^{\alpha}\left(D_x^{\beta}\ u\right) &= \sum_{i=0}^{n} \psi_i\ \ \phi_i(x) + \mu D_t^{\alpha}\Big(\sum_{i=0}^{n} \psi_{n+1+i}\ \ \phi_i(x)\Big) \\ &= \sum_{i=0}^{n} A_i^T\ \Phi(t)\ \phi_i(x) + \mu \sum_{i=0}^{n}\ A_i^T\ \Delta^{\alpha}\ \Phi(t)\ \phi_i(x) \\ &= \left[\sum_{i=0}^{n} A_i^T\ \Phi(t)\ + \mu \sum_{i=0}^{n} A_i^T\ \Delta^{\alpha}\ \Phi(t)\right]\ \phi_i(x) \end{aligned} \tag{2.5}$$

Substitute (2.4) and (2.5) in (2.1), we get

$$\sum_{i=0}^{n}\left[\sum_{j=1}^{m} \lambda_j(t) A_i^T\ \Delta^{j\alpha}\ \Phi(t)\right]\ \phi_i(x) \ = \ \sum_{i=0}^{n}\left[A_i^T\ \Phi(t)\ + \mu A_i^T\ \Delta^{\alpha}\ \Phi(t)\right]\ \phi_i(x)$$

Since $\phi_i(x)$ are linearly independent, we get the following ordinary fractional differential system with variable coefficients

$$\sum_{j=1}^{m} \lambda_j(t) A_i\ \Delta^{j\alpha}\ \Phi(t) = A_i^T\ \Phi(t)\ + \mu A_i^T\ \Delta^{\alpha}\ \Phi(t) \qquad i = 0, 1, \cdots, n \tag{2.6}$$

Subject to the initial conditions which can be derives from (2.2).
To solve this algebraic system i.e "finding the $A_i's$" vector, we must construct $p+1$ algebraic equation, however these equations arises from the substitute the roots of the polynomial $T^*_{p-\lceil\alpha\rceil+1}(t)$ in (2.6) for each i, and $\lceil\alpha\rceil$ equations produced from the initial conditions.
For simplify our work, we'll applying the following Algorithm steps.

Algorithm Steps

Step 1 : Choose a suitable invariant space for our problem.

Step 2 : Specify the order of the approximate Chebyshev polynomial.

Step 3 : Derive the approximate formulation of coefficient variable $(FPDEs)$ by using invariant subspace shifted first kind Chebyshev method $(CISM)$ in (2.6).

Step 4 : Derive the initial conditions for the system in Step 3, by using (2.2).

Step 5 : Compute the solution of the system formulating in Step 3. And then for origin problem.

Step 6 : Check the efficient and convergent of the numerical solution with the exact solution of ordinary partial differential equation by using different fractional orders of (2.6).

2.1. Illustrative Numerical Examples

In this section, we give some numerical examples to clear the applicability and accuracy of the proposed method.

Example 2.2. *Consider the following nonlinear fractional order partial differential equation with variable coefficients*

$$t^{\alpha}\ D_t^{\alpha}\ u = \left(D_x^{\beta}\ D_x^{\beta}\ u\right)^2 - u^2, \qquad 1 < \alpha \le 2,\ 0 < \beta \le 1 \tag{2.7}$$

$$u(x,0) = 0, \quad u_t(x,0) = E_{\beta}(x^{\beta}). \tag{2.8}$$

The exact solution is $u(x,t) = tE_{\beta}(x^{\beta})$

Solution:

Step1: Let $I_2 = \{1, E_{\beta}(x^{\beta})\}$ be an invariant subspace under the operator $N[u] = \left(D_x^{\beta}\ D_x^{\beta}\ u\right)^2 - u^2$, as for $u = a\ + b\ E_{\beta}(x^{\beta})\ \in I_2$, we have

$$N[u] = \left(bE_{\beta}(x^{\beta})\right)^2 - \left(a\ +\ bE_{\beta}(x^{\beta})\right)^2 = -a^2 - 2\ a\ b\ E_{\beta}(x^{\beta})\ \in\ I_2$$

Step 2: For the $p = 4$ order of the shifted Chebyshev polynomial of the first kind, the approximate solution of (2.7) has the form

$$u(x,t) = \sum_{i=0}^{1} k_i(t)\phi_i(x) = \sum_{i=0}^{1} A_i^T\ \Phi(t)\ \phi_i(x), \quad with$$

$$k_0(t) = A_0^T\ \Phi(t) = A^T\ \Phi(t), \ \ k_1(t) = A_1^T\ \Phi(t) = B^T\ \Phi(t)$$

$$\text{where}\ \ A^T = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix},\ B^T = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

and $\Phi^T(t) = \begin{pmatrix} 1 & 2t-1 & 8t^2-8t+1 & 32x^3-48x^2+18x-1 & 128x^4-256x^3+160x^2-32x+1 \end{pmatrix}$

Step3: According to the discussion in section 2, we have the following ordinary $FDEs$ with variable coefficients

$$t^{\alpha}\ A^T \Delta^{\alpha}\ \Phi(t) = -\left(A^T\ \Phi(t)\right)^2 \tag{2.9a}$$

$$t^{\alpha}\ B^T \Delta^{\alpha}\ \Phi(t) = -2\ A^T\ \Phi(t)\ B^T\ \Phi(t) \tag{2.9b}$$

Step4: Subject to the following initial conditions, which derived from (2.8)

$$\begin{aligned} u(x,0) &= k_0(0) + k_1(0)\ E_{\beta}(x^{\beta}) = 0 \Longrightarrow\ k_0(0) = k_1(0) = 0 \\ u_t(x,0) &= \acute{k}_0(0) + \acute{k}_1(0)\ E_{\beta}(x^{\beta}) = E_{\beta}(x^{\beta}) \Longrightarrow\ \acute{k}_0(0) = 0,\ \acute{k}_1(0) = 1 \end{aligned} \tag{2.10}$$

Step5: Operational matrix of fractional derivative of order $\alpha = 1.5$ in the Caputo sense. is

$$\Delta^{1.5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3.8312 & -0.7662 & 0.3284 & -0.1824 \\ 0 & -10.7273 & 6.3488 & -2.1649 & 1.1477 \\ 0 & -2.7731 & -23.1088 & 8.6907 & -4.1169 \end{pmatrix}$$

By using the first root $t_r = 0.5,\ 0.933,\ 0.067$ of the polynomial $T^*_{p+1-\lceil\alpha\rceil}(t) = T^*_3(t)$, together (2.10), Equation (2.9a) reads

$$\begin{aligned}
(a_0 - a_2 + a_4)^2 + 0.206a_2 + 6.714a_4 - 1.839a_3 &= 0\\
\Big(a_0 - 0.866a_1 + 0.5a_2 - 0.5a_4\Big)^2 - 0.063a_2 + 0.206a_3 - 0.123a_4 &= 0\\
\Big(a_0 - 0.866a_1 + 0.499a_2 - 0.50a_4\Big)^2 + 2.727a_2 - 6.029a_3 - 10.721a_4 &= 0\\
a_0 - a_1 + a_2 - a_3 + a_4 &= 0\\
2a_1 - 8a_2 + 18a_3 - 32a_4 &= 0
\end{aligned}$$

Hence, the solution of equation (2.9a) is $\quad k_0(t) = A^T\ \Phi(t) = 0.$
By the same manipulate, for $(2.9b)$ we have the following algebraic system

$$\begin{aligned}
0.2064b_2 - 1.8389b_3 + 6.7147b_4 &= 0\\
2.727b_2 - 6.0288b_3 + 10.7207b_4 &= 0\\
-0.0657b_2 - 1.8389b_3 - 0.1143b_4 &= 0\\
b_0 - b_1 + b_2 - b_3 + b_4 &= 0\\
2b_1 - 8b_2 + 18b_3 - 32b_4 &= 1
\end{aligned}$$

Solving this system yields $B^T = \begin{pmatrix}1/2 & 1/2 & 0 & 0 & 0\end{pmatrix}$, Consequently, the approximate solution of (2.9b) is

$$\begin{aligned}
k_1(t) &= B^T\ \Phi(t) = b_0(1) + b_1\Big(2t-1\Big) + b_2\Big(8t^2 - 8t + 1\Big) + b_3\Big(32x^3 - 48x^2 + 18x - 1\Big)\\
&+ b_4\Big(128x^4 - 256x^3 + 160x^2 - 32x + 1\Big) = 1/2 + t - 1/2 = t.
\end{aligned}$$

Also, for other values of $\alpha \in (1, 2]$, and other order of Chebyshev polynomials (p), we have the same solution.
Finally, the approximate solution of the original equation (2.7) obtain by $CISM$ is given by

$$u(x,t) = k_0(t) + k_1(t)E_\beta(x^\beta) = A^T\ \Phi(t) + B^T\ \Phi(t)\ E_\beta(x^\beta) = t\ E_\beta(x^\beta)$$

which is exact solution in this case.

Example 2.3. *Consider the following nolinear fractional partial differential equation with variable coefficients*

$$\Big(1 - t^\alpha\Big)D_t^\alpha u = \Big(D_x^\beta u\Big)^2 - \frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\ u\ \Big(D_x^\beta\ D_x^\beta u\Big) \qquad 1 < \alpha \leq 2,\ \ 0 < \beta \leq 1. \tag{2.11}$$

$$u(x,0) = 0,\ \ u_t(x,0) = x^{2\beta}$$

The exact solution is $u(x,t) = t\ x^{2\beta}$

Solution:

Step1: By consider $I_3 = L\{1,\ x^{\beta},\ x^{2\beta}\}$ be an invariant subspace under the nonlinear operator

$$\begin{aligned} N[u] &= \Big(D_x^{\beta} u\Big)^2 - \frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\ u\ D_x^{\beta}\ D_x^{\beta} u \\ &= \Big(\Gamma(1+\beta)\ b + \frac{\Gamma(1+2\beta)}{\Gamma(1+\beta)} x^{\beta}\Big)^2 - \frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\Big(a\ + b\ x^{\beta}\ + c\ x^{2\beta}\Big)\Gamma(1+2\beta)\ c \\ &= \Big(\Gamma(1+\beta)^2\ b^2 - \frac{\Gamma^2(1+2\beta)}{\Gamma^2(1+\beta)} a\ c\Big) + \Big(2\Gamma(1+2\beta) - \frac{2\Gamma^2(1+2\beta)}{\Gamma^2(1+\beta)}\Big)\ b\ c\ x^{\beta} \qquad \in\ I_3 \end{aligned}$$

$$\text{whenever}\ \ u = a + b\ x^{\beta}\ + c\ x^{2\beta}\ \in\ I_3$$

Step2: For the $p = 4$, order of the shifted Chebyshev polynomial of the first kind, the approximate solution of (2.11) has the form

$$u(x,t) = \sum_{i=0}^{2} k_i(t)\phi_i(x) = \sum_{i=0}^{2} A_i^T\ \Phi(t)\ \phi_i(x)$$

with

$$k_0(t) = A_0^T\ \Phi(t) = A^T\ \Phi(t),\ \ k_1(t) = A_1^T\ \Phi(t) = B^T\ \Phi(t)$$

$$\text{where}\ \ A^T = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix},\ B^T = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

and $\Phi^T(t) = \begin{pmatrix} 1 & 2t-1 & 8t^2-8t+1 & 32x^3-48x^2+18x-1 & 128x^4-256x^3+160x^2-32x+1 \end{pmatrix}$

Step3: According to the discussion in section 2, we have the following ordinary $FDEs$ with variable coefficients

$$\Big(1-t^{\alpha}\Big)A^T\Delta^{\alpha}\ \Phi(t) = \Gamma^2(1+\beta)\ \Big(B^T\ \Phi(t)\Big)^2 - \frac{\Gamma^2(1+2\beta)}{\Gamma^2(1+\beta)} A^T\ \Phi(t)\ C^T\ \Phi(t) \tag{2.12a}$$

$$\Big(1-t^{\alpha}\Big)B^T\Delta^{\alpha}\ \Phi(t)) = \Big[2\Gamma(1+\beta) - \frac{\Gamma^2(1+2\beta)}{\Gamma^2(1+\beta)}\Big]\ B^T\ \Phi(t)\ C^T\ \Phi(t) \tag{2.12b}$$

$$\Big(1-t^{\alpha}\Big)C^T\Delta^{\alpha}\ \Phi(t) = 0 \tag{2.12c}$$

Step4: Subject to

$$\begin{aligned} u(x,0) = 0 \Longrightarrow k_0(0) + k_1(0)\ x^{\beta} + k_2(0)\ x^{2\beta} = 0\ \Longrightarrow k_0(0) = k_1(0) = k_2(0) = 0 \\ u_t(x,0) = x^{2\beta} \Longrightarrow \acute{k}_0(0) + \acute{k}_1(0)\ x^{\beta} + \acute{k}_2(0)\ x^{2\beta} = x^{2\beta} \Longrightarrow \acute{k}_0(0) = \acute{k}_1(0) = 0,\ \acute{k}_2(0) = 1 \end{aligned} \tag{2.13}$$

Step5: On the other hand, operational matrix of fractional derivative of order $\alpha = 1.6$ in the Caputo sense is

$$\Delta^{1.6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3.501 & -0.8753 & 0.4119 & -0.2434 \\ 0 & -7.8774 & 6.7962 & -2.682 & 1.5227 \\ 0 & -14.0483 & -23.1921 & 10.4168 & -5.4133 \end{pmatrix}$$

Then, by using $t = 0.5,\ 0.933,\ 0.06699$ the roots of the polynomial $T^*_{p+1-\lceil\alpha\rceil}(t) = T^*_3(t)$, together with equations (2.13), and (2.12c) gives

$$\begin{aligned} 0.4235c_2 - 3.5339c_3 + 11.914c_4 &= 0 \\ 0.2852c_2 - 0.4396c_3 - 2.2112c_4 &= 0 \\ -3.3035c_2 + 9.3333c_3 + 3.2343c_4 &= 0 \\ c_0 - c_1 + c_2 - c_3 + c_4 &= 0 \\ 2c_1 - 8c_2 + 18c_3 - 32c_4 &= 1 \end{aligned}$$

Solving this system to get $C^T = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \end{pmatrix}$, and the approximate solution of (2.12c) is

$$\begin{aligned} C^T\Phi(t) &= c_0(1) + c_1(2t-1) + c_2(8t^2 - 8t + 1) + c_3(32x^3 - 48x^2 + 18x - 1) \\ &+ c_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) = \frac{1}{2} + \frac{1}{2}(2t-1) = t \end{aligned}$$

For $\beta = 0.6$, (2.12b) together with (2.13) yields

$$\begin{aligned} 0.5567b_2 - 0.1332b_0 - 3.5339b_3 + 11.7807b_4 &= 0 \\ 0.161b_2 - 0.2486b_0 - 0.4396b_3 - 2.0868b_4 - 0.2153b_1 &= 0 \\ 3.3125b_2 - 0.0179b_0 + 9.3333b_3 + 3.2432_4 + 0.0155b_1 &= 0 \\ b_0 - b_1 + b_2 - b_3 + b_4 &= 0 \\ 2b_1 - 8b_2 + 18b_3 - 32b_4 &= 0. \end{aligned}$$

This gives the zero vector B. So, the solution of (2.12b) is zero, i.e $B^T\Phi(t) = 0$.
By the same way, we can have A is the zero vector. And the solution of (2.12a) is $A^T\phi(t) = 0$.
Finally, the solution of the original equation (2.11) is

$$u(x,t) = c_0(t) + c_1(t)\ x^\beta + c_2(t)\ x^{2\beta} = A^T\ \Phi(t) + B^T\ \Phi(t)\ x^\beta + C^T\ \Phi(t)\ x^{2\beta} = t\ x^{2\beta}$$

which is the exact solution.

Example 2.4. *Consider the following non-linear fractional partial differential equation with variable coefficients*

$$t^{2\alpha}\left(D_t^\alpha - D_t^{2\alpha}\right)u = \left(D_x^\beta\ u\right)^2 - 2u\ D_x^\beta\ u, \quad 0 < \alpha,\ \beta \le 1. \tag{2.14}$$

Subject to

$$u(x,0) = E_\beta(2x^\beta), \quad u_t(x,0) = E_\beta(2x^\beta) \qquad e^{2x+t} \tag{2.15}$$

Solution:
Step1: By consider $I_2 = L\{1,\ E_\beta(2x^\beta)\}$ be an invariant subspace under the nonlinear operator

$$N[u] = \left(D_x^\beta\ u\right)^2 - 2u\ D_x^\beta,$$

since for $u = a + b\ E_\beta(2x^\beta) \quad \in \ I_2$

$$\begin{aligned} N[u] &= \left(D_x^\beta\ u\right)^2 - 2u\ D_x^\beta = \left(2\ b\ E_\beta(2x^\beta)\right)^2 - 2\left(a + b\ E_\beta(2x^\beta)\right)(2\ b\ E_\beta(2x^\beta)) \\ &= -4\ a\ b\ E_\beta(2x^\beta) \qquad \in \ I_3 \end{aligned}$$

Step2: For the $p = 2$, order of the shifted Chebyshev polynomial of the first kind, the approximate solution of (2.14) has the form

$$u(x,t) = \sum_{i=0}^{1} k_i(t)\phi_i(x) = \sum_{i=0}^{1} A_i^T\ \Phi(t)\ \phi_i(x)$$

with

$$k_0(t) = A_0^T\ \Phi(t) = A^T\ \Phi(t), \quad k_1(t) = A_1^T\ \Phi(t) = B^T\ \Phi(t)$$

where $A^T = \begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix}$, $B^T = \begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix}$ and $\Phi^T(t) = \begin{pmatrix} 1 & 2t-1 & 8t^2 - 8t + 1 \end{pmatrix}$

Step 3: According to the discussion in section 2, we have the following ordinary $FDEs$ with variable coefficients

$$A^T\ \Delta^\alpha\ \Phi(t) + \ A^T\ \Delta^{2\alpha}\ \Phi(t) \ = \ 0 \tag{2.16a}$$

$$B^T\ \Delta^\alpha\ \Phi(t) + \ B^T\ \Delta^{2\alpha}\ \Phi(t) \ + \ 4\ A^T\ \ \Phi(t)\ B^T\ \Phi(t) \ = \ 0 \tag{2.16b}$$

Case 1: $\alpha \ \in \ (0, 0.5]$
Step4: Subject to the following initial conditions which can be derive from (2.15)

$$\begin{aligned} u(x,0) &= k_0(0) + k_1(0)\ E_\beta(x^\beta) = E_\beta(x^\beta) \Longrightarrow k_0(0) = 0, \quad k_1(0) = 1 \\ u_t(x,0) &= \dot{k}_0(0) + \dot{k}_1(0)\ E_\beta(x^\beta) = E_\beta(x^\beta) \Longrightarrow \dot{k}_0(0) = 0, \quad \dot{k}_1(0) = 1 \end{aligned} \tag{2.17}$$

Step 5: Operational matrix of fractional derivative of order $\alpha = 0.35$ in the Caputo sense. is

$$\Delta^{0.35} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.0238 & -0.1352 \\ 0 & -1.3611 & 1.0278 \end{pmatrix}, \quad \Delta^{0.7} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.7528 & -0.2291 \\ 0 & 0.48 & 1.2338 \end{pmatrix}$$

By using $t = 0.8536,\ 0.1464$ the roots of the polynomial $T^*_{p+1-\lceil 2\alpha \rceil}(t) = T^*_2(t)$, then
For $(2.16a)$, with (2.17) we have

$$\begin{aligned} 0.1917\ a_1\ + 1.3021\ a_2 &= 0 \\ 1.302\ a_2 - 0.1917 a_1 &= 0 \\ a_0\ -\ a_1 +\ a_2 &= 0 \end{aligned}$$

Solving this system yields A is a zero vector.
For (2.16b), we have the following algebraic system

$$\begin{aligned} 0.192\ b_1\ - 1.302\ b_2 &= 0 \\ -0.192\ b_1\ + 1.302\ b_2 &= 0 \\ b_0\ -\ b_1 +\ a_2 &= 1 \end{aligned}$$

Solving this system to get $B^T = \begin{pmatrix}1 & 0 & 0\end{pmatrix}$, and the solution of equation (2.16b) is $B^T\ \Phi(t) = 1$
So, in this case the approximate solution of the original equation (2.14) is as following

$$u(x,t) = k_0(t) + k_1(t)\ E_\beta(2x^\beta) = A^T\ \Phi(t)\ +\ B^T\ \Phi(t) = E_\beta(2x^\beta)$$

Case 2: $\alpha \in (0.5,\ 1]$
Operational matrix of fractional derivative of order $\alpha = 0.6$ in the Caputo sense. is

$$\Delta^{1.2} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 4.1507 & -0.2965\end{pmatrix},\quad \Delta^{0.6} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0.8753 & -0.2188\\ 0 & -0.2188 & 1.2614\end{pmatrix}$$

By using $t = 0.5$, the root of the polynomial $T^*_{p+1-\lceil 2\alpha\rceil}(t) = T^*_1(t)$, then for(2.16$a$), together (2.17), we have the same solution in case 1.
For (2.16b), the following algebraic system

$$\begin{aligned} 0.219\ b_1\ - 1.558\ b_2 &= 0\\ 2b_1 - 8b_2 &= 1\\ b_0 - b_1 + b_2 &= 1 \end{aligned}$$

gives $B^T = \begin{pmatrix}1.981 & 1.141 & 0.16\end{pmatrix}$, thus the solution of (2.16b) is $B^T\ \Phi(t) = 1 + t + 1.282\ t^2$.
So, the approximate solution of (2.14) in this case is

$$\begin{aligned} u(x,t) = k_0(t)\ +\ k_1(t)\ E_\beta(2x^\beta) &= \Big(1 + t + 1.282\ t^2\Big)E_\beta(2x^\beta), \quad and\\ for\ \ \alpha = 0.95, \quad u(x,t) = k_0(t)\ +\ k_1(t)\ E_\beta(2x^\beta) &= \Big(1 + t + 0.528\ t^2\Big)E_\beta(2x^\beta),\\ for\ \ \alpha = 0.75, \quad u(x,t) = k_0(t)\ +\ k_1(t)\ E_\beta(2x^\beta) &= \Big(1 + t + 0.851\ t^2\Big)E_\beta(2x^\beta). \end{aligned}$$

Exact solution and some of approximate solutions for (2.14) are plotted in Figure 1.

3. Conclusion

The technique CISM which applies in this work for solving some linear and nonlinear space, time, and mixed fractional partial differential equations is a sufficient and important tool for which it sometimes gives the exact solution for such equations dependent on $\alpha-$ fractional order of Caputo derivative. On the other hand, the approximated solutions resulting from which done by Mathcad and Maple programs are very accurate.

References

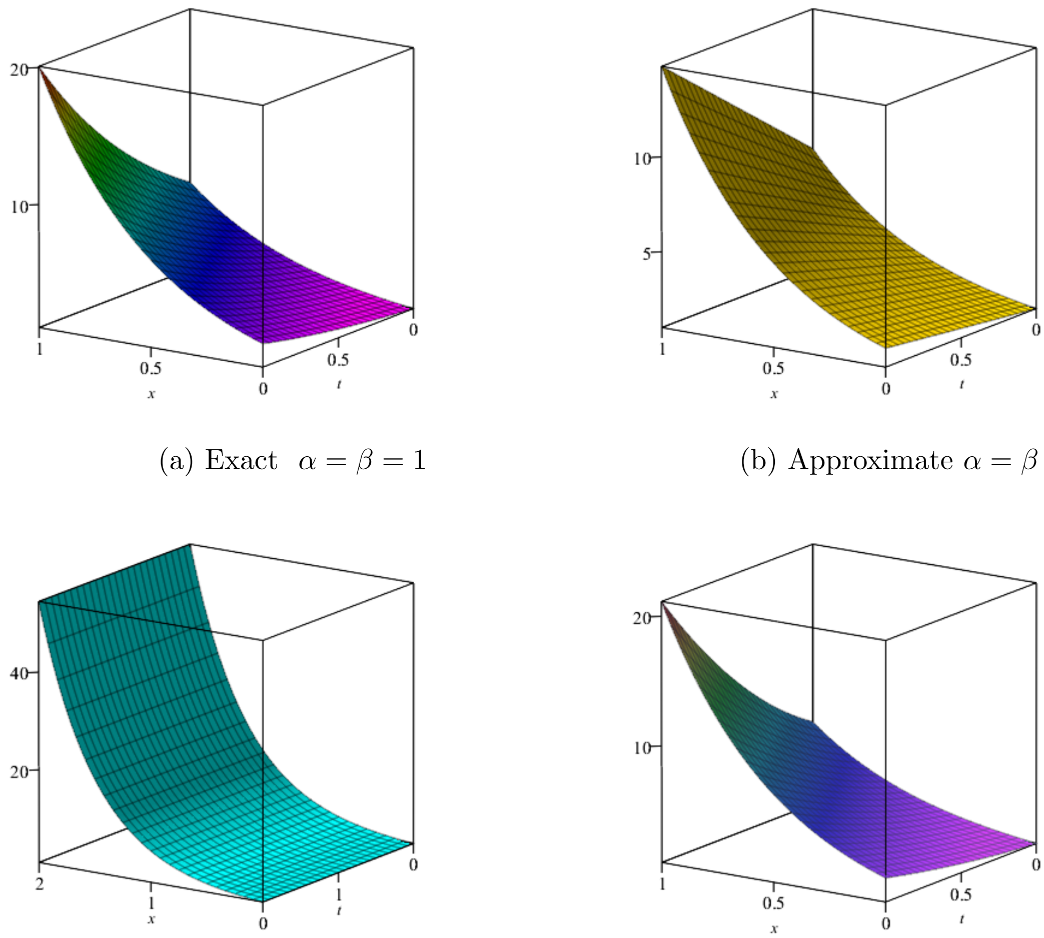
[1] S. Choudhary and G. Daftardar, *Invariant subspace method: A tool for solving fractional partial differential equations*, J. Fract. Calc. Appl. Anal. 20 (2017) 1–18.

[2] A. Elhameed and Y. Youssri, *Chebyshev polynomial solutions for fractional differential equations*, J. Comput. Appl. Math. 37 (2018) 2897–2921.

[3] M. Francesco, M. Antonio and P. Gianni, *Fractional Differential Equations*, Int. J. Diff. Eq. 2010 (2009) 1–29.

[4] P. Harris and R. Garra, *Analytic solution of nonlinear fractional burgers-type equation by invariant subspace method*, Nonlinear Stud. 20 (2031) 1–12.

[5] Jun Jiang, Yuqiang Feng and Shougui Li, *Exact solutions to the fractional differential equations with mixed partial derivatives*, Axioms 3 (2018) 55–76.



(a) Exact $\alpha = \beta = 1$

(b) Approximate $\alpha = \beta = 1$

(c) $\alpha = 0.3, \quad \beta = 1$

(d) $\alpha = 1, \quad \beta = 0.9$

Figure 1: Exact solution and some of it's approximate solutions for equation (2.14) with different values of $\alpha, \quad \beta$ obtained by $(CISM)$

[6] A. Ouhadan and H. Elkinani, *Invariant subspace method and fractional modified kuramoto-sivashinsky equation*, arXiv, (2015) 1–12.

[7] R. Shadevan and T. Bakkyaraj, *Invariant subspace method and exact solutions of certain nonlinear time fractional partial defferential equations*, Fract. Calc. Appl. Anal. 18 (2015) 146–162.

[8] R. Shadevan and P. Prakash, *Exact solution of certain time fractional nonlinear partial differential equations*, Nonlinear Dyn. 85 (2016) 659—673.

[9] S. Svirshchevskii, *Invariant linear spaces and exact solutions of nonlinear evolution equations*, J. Nonlinear Math. Phys. 3 (2013) 164–169.