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Nonlinear fractional differential equations with advanced arguments

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Abstract

In this paper, we develop the existence and uniqueness theory of fractional differential equation involving Riemann-Liouville differential operator of order $0<\alpha<1$, with advanced argument and integral boundary conditions. We investigate the uniqueness of the solution by using Banach fixed point theorem, we apply the comparison result to obtain the existence and uniqueness of solution by monotone iterative technique also by using weakly coupled extremal solution for the nonlinear boundary value problem (BVP). As an application of this technique, existence and uniqueness results are obtained.

Keywords: Fractional differential equation, existence and uniqueness, monotone iterative technique, integral boundary conditions.
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1. Introduction and preliminaries

In this paper, we study the following nonlinear (BVP) for Riemann-Liouville fractional differential equation with advanced argument and integral boundary conditions:

$$\begin{cases} D_{0+}^{\alpha} x(t) = f\left(t, x(t), x(\theta(t))\right), \\ \quad x(0) = \lambda \int_0^T x(s)ds + r, \end{cases} \tag{1.1}$$

where $t \in J=[0,T](T>0)$, $f\left(t,x(t),x(\theta(t))\right) \in C\left(J\times\mathbb{R}^2,\mathbb{R},\right)$, $\theta \in C\left(J,J\right)$, $t\leq\theta(t)$, $\lambda, r\in\mathbb{R}$ and D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α $(0<\alpha<1)$.

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Recently, the fractional differential equations with advanced argument have been of great interest in the study of various problems in physics, mechanics, chemistry, engineering, economics (see [3, 8, 15] and references therein). Many people have paid more and more attention to study the existence and uniqueness of a solution of different problems in fractional differential equations with deviating argument (see [1, 4, 5, 6, 7]). However, the theory of nonlinear fractional differential equation with integral boundary value problem is still in the initial stage. The monotone iterative method combined with the technique of upper and lower solutions provides an effective mechanism to prove existence results for nonlinear differential equations. For details (see [10, 11, 12, 13]).

The paper is organized as follows: In Section 1, we present some useful definitions and lemmas and fundamental facts of fractional calculus. In Section 2, by applying Banach fixed point theorem with the corresponding weighted norm, we prove the uniqueness of solution for nonlinear BVP(1.1). In Sections 3, 4, we develop the monotone iterative technique for solving nonlinear BVP(1.1), and existence and uniqueness results is obtained. Two converging monotone sequences are obtained with the monotone iterative technique based on upper and lower solutions or weakly coupled ones. Those two converging monotone sequences will converge to the extremal solution or weakly coupled extremal solution of nonlinear BVP(1.1). Lastly, we illustrate our result with a suitable example.

We need to recall the definitions of Riemann-Liouville integral, derivative and some basic lemmas which will be used in further discussions. First, we introduce the Banach space $C_{1-\alpha}$ by $C_{1-\alpha}(J,\mathbb{R}) = \{x \in C(J,\mathbb{R}) : t^{1-\alpha}x(t) \in C(J,\mathbb{R})\}$ with the norm $\|x\|_{C_{1-\alpha}} = \max_{t\in J} |t^{1-\alpha}x(t)|$.

Definition 1.1. [9, 14] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function $x(t) \in C([0,T])$ is defined as

$$I_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

provided the integral exists. $\Gamma(\alpha)$ denotes Euler's Gamma function.

Definition 1.2. [9, 14] For function $I_{0^+}^{n-\alpha}x(t) \in AC^n[0,T]$ the Riemann-Liouville derivative of order α $(n-1 < \alpha \leq n)$ can be written as

$$D_{0^+}^{\alpha}x(t) = \left(\frac{d}{dt}\right)^n \left(I_{0^+}^{n-\alpha}x\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}x(s)ds,\ t>0.$$

Lemma 1.3. [9] Let $x(t) \in C^n[0,T], \alpha \in (n-1,n)$, $n \in \mathbb{N}$. Then for $t \in J$,

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0).$$

Lemma 1.4. [2] Let $m \in C_{1-\alpha}(J,\mathbb{R})$ and for any $t_1 \in (0,T]$, we have

$$m(t_1) = 0 \text{ and } m(t) \leq 0 \ \text{ for } \ 0 \leq t \leq t_1.$$

Then it follows that,

$$D_{0^+}^{\alpha}m(t_1) \geq 0.$$

Lemma 1.5. [17] (Lebesgue's dominated convergence theorem) Let E be a measurable set and let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in E, and for every $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ a.e. in E, where g is integrable on E. Then

$$\lim_{n\to\infty}\int_E f_n(x)dx = \int_E f(x)dx.$$

Lemma 1.6. Function $x(t) \in C_{1-\alpha}(J,\mathbb{R})$ is a solution of the nonlinear BVP(1.1) if and only if $x(t)$ is a solution of the integral equation

$$x(t) = \lambda \int_0^T x(s)ds + r + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(\theta(s)))ds. \tag{1.2}$$

Proof . Assume that $x(t)$ satisfies the nonlinear BVP(1.1). From the first equation of the nonlinear BVP(1.1) and Lemma (1.3), we have

$$x(t) = \lambda \int_0^T x(s)ds + r + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(\theta(s)))ds.$$

Conversely, assume that $x(t)$ satisfies (1.2). It is easy to check that $x(t) \in C_{1-\alpha}(J,\mathbb{R})$. Applying the operator D_{0+}^{α} to both sides of (1.2), we have

$$D_{0+}^{\alpha} x(t) = f(t, x(t), x(\theta(t))).$$

In addition, we can easily show that $x(0) = \lambda \int_0^T x(s)ds + r$. The proof is complete. □

Corollary 1.7. [16] Let $\{x_\epsilon(t)\}$ be a family of continuous functions defined on J, for each $\epsilon > 0$, which satisfies

$$\begin{cases} D_{0+}^{\alpha} x_\epsilon(t) = f(t, x_\epsilon(t), x_\epsilon(\theta(t))), \\ x_\epsilon(0) = \lambda \int_0^T x_\epsilon(s)ds + r, \end{cases} \tag{1.3}$$

where $|f(t, x_\epsilon(t), x_\epsilon(\theta(t)))| \leq M$ for $t \in J$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on J.

2. Uniqueness of solution of BVP(1.1)

In this section, we discuss the uniqueness of solution of the nonlinear BVP(1.1) for Riemann-Liouville fractional differential equation with advanced argument and integral boundary conditions.

Theorem 2.1. Assume that:
(H_1) $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\theta \in C(J,J)$, $t \leq \theta(t)$, $t \in J$,
(H_2) there exists nonnegative constants M, N such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq M|x_1-y_1| + N|x_2-y_2|, \ \forall t \in J, \ x_i, y_i \in \mathbb{R}, i = 1,2.$$

Then the nonlinear BVP(1.1) has a unique solution.

Proof . We define the operator $T : C_{1-\alpha}(J,\mathbb{R}) \to C_{1-\alpha}(J,\mathbb{R})$ as follows:

$$Tx(t) = \lambda \int_0^T x(s)ds + r + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(\theta(s)))ds.$$

Clearly, the operator T is well defined on $C_{1-\alpha}(J,\mathbb{R})$. Next, we show that T is a contraction operator. For convenience, let

$$\lambda < \frac{\Gamma(2\alpha) - \Gamma(\alpha)T^{\alpha}(M+N)}{T\Gamma(2\alpha)}. \tag{2.1}$$

Using assumption (H_2), for any $x, y \in C_{1-\alpha}(J,\mathbb{R})$, we have

$$\begin{aligned}
\|Tx - Ty\|_{C_{1-\alpha}} &= \max_{t\in J}\left|t^{1-\alpha}\left[(Tx)(t) - (Ty)(t)\right]\right| \\
&\le \max_{t\in J} t^{1-\alpha}\lambda \int_0^T |x(s) - y(s)|\, ds + \max_{t\in J} \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left|f\left(s, x(s), x(\theta(s))\right) - f\left(s, y(s), y(\theta(s))\right)\right| ds \\
&\le \lambda \int_0^T ds\, \|x - y\|_{C_{1-\alpha}} + \max_{t\in J} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left[\left|M\left(x(s) - y(s)\right)\right| + \left|N\left(x(\theta(s)) - y(\theta(s))\right)\right|\right] ds \\
&\le \left[\lambda T + \frac{\Gamma(\alpha) T^\alpha}{\Gamma(2\alpha)}\,(M+N)\right] \|x - y\|_{C_{1-\alpha}}.
\end{aligned}$$

According to (2.1) and the Banach fixed point theorem, the nonlinear BVP(1.1 has a unique solution. The proof is complete. □

Corollary 2.2. Suppose that M, N are constants and $h \in C_{1-\alpha}(J,\mathbb{R})$. The following linear problem

$$\begin{cases} D^\alpha_{0^+} x(t) + Mx(t) + Nx(\theta(t)) = h(t), \ t \in J, \ 0 < \alpha < 1, \\ x(0) = \lambda \int_0^T x(s)ds + r, \end{cases} \tag{2.2}$$

has a unique solution $x(t) \in C_{1-\alpha}(J,\mathbb{R})$.

Proof . It follows from Theorem 2.1. □

3. Monotone iterative technique of BVP(1.1)

In this section, we mainly investigate the existence and uniqueness of solution of the nonlinear BVP(1.1) for Riemann-Liouville fractional differential equation with advanced argument by the method of lower and upper solutions combined with monotone iterative technique. Now, we define the sector as follows:

$$[\upsilon_0, w_0] = \{x \in C_{1-\alpha}(J,\mathbb{R}) : \upsilon_0(t) \le x(t) \le w_0(t) \ \ \forall t \in J\}.$$

First, we prove the following comparison result which plays an important role in our research.

Lemma 3.1. Let $\alpha \in (0,1)$, $\theta(t) \in C\,(J,J)$ and $t \le \theta(t)$ on J. Suppose that $p \in C_{1-\alpha}(J,\mathbb{R})$ satisfies the inequalities

$$\begin{cases} D^\alpha_{0^+} p(t) \le -Mp(t) - Np(\theta(t)) \equiv Fp(t), \ t \in J \\ p(0) \le 0, \end{cases} \tag{3.1}$$

where M and N are constants. If

$$-T^\alpha\,(M+N)\,\Gamma(1-\alpha) < 1,$$

then $p(t) \le 0$ for all $t \in J$.

Proof . Put $p_\varepsilon(t) = p(t) - \varepsilon$, $\varepsilon > 0$. Then

$$\begin{aligned} D^\alpha_{0+} p_\varepsilon(t) &= D^\alpha_{0+} p(t) - D^\alpha_{0+}\varepsilon \\ &< -Mp_\varepsilon(t) - Np_\varepsilon(\theta(t)) + \varepsilon[-(M+N) - \frac{1}{t^\alpha\Gamma(1-\alpha)}] \\ &< Fp_\varepsilon(t), \end{aligned}$$

and

$$p_\varepsilon(0) = p(0) - \varepsilon < 0.$$

We prove that $p_\varepsilon(t) < 0$ on J. Assume that $p_\varepsilon(t) \nless 0$ on J. Thus there exists $t_1 \in (0, T]$ such that $p_\varepsilon(t_1) = 0$ and $p_\varepsilon(t) < 0$, $t \in (0, t_1)$. In view of Lemma 1.4 we have $D^\alpha_{0+} p_\varepsilon(t_1) \geq 0$. It follows that

$$0 < Fp_\varepsilon(t_1) = -Np_\varepsilon(\theta(t_1)).$$

If $N = 0$, then $0 < 0$, which is a contradiction. If $-N < 0$, then $p_\varepsilon(\theta(t_1)) < 0$, which is again a contradiction. This proves that $p_\varepsilon(t) < 0$ on J. So $p(t) - \varepsilon < 0$ on J. Taking $\varepsilon \to 0$, we get required result. □

Definition 3.2. A pair of functions $[v_0, w_0]$ in $C_{1-\alpha}(J, \mathbb{R})$ is called lower and upper solutions of the nonlinear BVP(1.1) for $\lambda = 1$ if

$$D^\alpha_{0+} v_0(t) \leq f(t, v_0(t), v_0(\theta(t))), \quad v_0(0) \leq \int_0^T v_0(s)ds + r,$$

$$D^\alpha_{0+} w_0(t) \geq f(t, w_0(t), w_0(\theta(t))), \quad w_0(0) \geq \int_0^T w_0(s)ds + r.$$

Theorem 3.3. Assume that:

(i) $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\theta \in C(J, J)$, $t \leq \theta(t)$, $t \in J$,

(ii) functions $v_0(t)$ and $w_0(t)$ in $C_{1-\alpha}(J, \mathbb{R})$ are lower and upper solutions of the nonlinear BVP(1.1) such that $v_0(t) \leq w_0(t)$ on J,

(iii) there exists nonnegative constants M, N such that function f satisfies the condition

$$f(t, x_1, x_2) - f(t, y_1, y_2) \geq -M(x_1 - y_1) - N(x_2 - y_2),$$

for $y_0(t) \leq y_1 \leq x_1 \leq w_0(t)$, $v_0(\theta(t)) \leq y_2 \leq x_2 \leq w_0(\theta(t))$. Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J, \mathbb{R})$ such that

$$\{v_n(t)\} \longrightarrow v(t) \text{ and } \{w_n(t)\} \longrightarrow w(t) \text{ as } n \longrightarrow \infty$$

where $v(t)$ and $w(t)$ are minimal and maximal solutions of the nonlinear BVP(1.1) respectively, and $v(t) \leq x(t) \leq w(t)$ on J.

Proof . For any $\eta \in C_{1-\alpha}(J,\mathbb{R})$ such that $\eta \in [\upsilon_0, w_0]$, We consider the following linear problem:

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f\left(t, \eta(t), \eta(\theta(t))\right) + M[\eta(t) - x(t)] + N[\eta(\theta(t)) - x(\theta(t))], \\ \quad x(0) = \int_0^T x(s)ds + r. \end{cases} \tag{3.2}$$

Obviously, by Corollary 2.2, the linear problem (3.2) has a unique solution $x(t)$.
We next define the iterates as follows:

$$\begin{cases} D_{0^+}^{\alpha} \upsilon_{n+1}(t) = f\left(t, \upsilon_n(t), \upsilon_n(\theta(t))\right) - M\left[\upsilon_{n+1}(t) - \upsilon_n(t)\right] - N\left[\upsilon_{n+1}(\theta(t)) - \upsilon_n(\theta(t))\right], \\ \quad \upsilon_{n+1}(0) = \int_0^T \upsilon_n(s)ds + r, \end{cases} \tag{3.3}$$

and

$$\begin{cases} D_{0^+}^{\alpha} w_{n+1}(t) = f\left(t, w_n(t), w_n(\theta(t))\right) - M\left[w_{n+1}(t) - w_n(t)\right] - N\left[w_{n+1}(\theta(t)) - w_n(\theta(t))\right], \\ \quad w_{n+1}(0) = \int_0^T w_n(s)ds + r, \end{cases} \tag{3.4}$$

Obviously, the above arguments imply the existence of the unique solutions $\upsilon_{n+1}(t)$ and $w_{n+1}(t)$ of the problems (3.3), (3.4). By putting $n = 0$ in the problems (3.3), (3.4), we get the existence of solutions $\upsilon_1(t)$ and $w_1(t)$. We show that $\upsilon_0(t) \le \upsilon_1(t) \le w_1(t) \le w_0(t)$. For this, consider $p(t) = \upsilon_1(t) - \upsilon_0(t)$ on J, and $\upsilon_0(t)$ is the lower solution of the nonlinear BVP(1.1). Then we have

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} \upsilon_1(t) - D_{0^+}^{\alpha} \upsilon_0(t) \\ &\ge f\left(t, \upsilon_0(t), \upsilon_0(\theta(t))\right) - f\left(t, \upsilon_0(t), \upsilon_0(\theta(t))\right) - \\ &\qquad M\left[\upsilon_1(t) - \upsilon_0(t)\right] - N\left[\upsilon_1(\theta(t)) - \upsilon_0(\theta(t))\right] \\ &\ge -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = \upsilon_1(0) - \upsilon_0(0) \ge \int_0^T \upsilon_0(s)ds + r - \int_0^T \upsilon_0(s)ds - r \;\; = \;\; 0.$$

By Lemma 3.1, we get $p(t) \ge 0$, implies that $\upsilon_1(t) \ge \upsilon_0(t)$ on J. Similarly, we can prove $w_1 \le w_0$ and $\upsilon_1(t) \le w_1(t)$ on J. Thus $\upsilon_0(t) \le \upsilon_1(t) \le w_1(t) \le w_0(t)$. Assume that for some $k > 1$, $\upsilon_{k-1}(t) \le \upsilon_k(t) \le w_k(t) \le w_{k-1}(t)$ on J. We claim that $\upsilon_k(t) \le \upsilon_{k+1}(t) \le w_{k+1}(t) \le w_k(t)$ on J.
To prove the claim, set $p(t) = \upsilon_{k+1}(t) - \upsilon_k(t)$, we have

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} \upsilon_{k+1}(t) - D_{0^+}^{\alpha} \upsilon_k(t) \\ &= f\left(t, \upsilon_k(t), \upsilon_k(\theta(t))\right) - M\left[\upsilon_{k+1}(t) - \upsilon_k(t)\right] - N\left[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))\right] - \\ &\quad f\left(t, \upsilon_{k-1}(t), \upsilon_{k-1}(\theta(t))\right) + M\left[\upsilon_k(t) - \upsilon_{k-1}(t)\right] + N\left[\upsilon_k(\theta(t)) - \upsilon_{k-1}(\theta(t))\right] \\ &\ge -M\left[\upsilon_{k+1}(t) - \upsilon_k(t)\right] - N\left[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))\right] \\ &\ge -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$\begin{aligned} p(0) &= \upsilon_{k+1}(0) - \upsilon_k(0) = \int_0^T \upsilon_k(s)ds + r - \int_0^T \upsilon_{k-1}(s)ds - r \\ &\ge \int_0^T \left[\upsilon_k(s) - \upsilon_k(s)\right] ds = 0. \end{aligned}$$

By Lemma 3.1, we get $p(t) \geq 0$, implies that $\upsilon_{k+1}(t) \geq \upsilon_k(t)$ on J. Similarly, we can prove that $w_{k+1}(t) \leq w_k(t)$ and $\upsilon_{k+1}(t) \leq w_{k+1}(t)$ on J. By the principle of mathematical induction, we have

$$\upsilon_0 \leq \upsilon_1 \leq \upsilon_2 \leq \cdots \leq \upsilon_k \leq w_k \leq \cdots \leq w_2 \leq w_1 \leq w_0 \text{ on } J. \tag{3.5}$$

Obviously, the sequences $\{\upsilon_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded. We observe that $\{D^{\alpha}_{0+}\upsilon_n\}$ and $\{D^{\alpha}_{0+}w_n\}$ are also uniformly bounded on J, in view of the relations (3.3) and (3.4). Then using Corollary 1.7, we can conclude that sequences $\{\upsilon_n(t)\}$,$\{w_n(t)\}$ are equicontinuous. Hence by the Ascoli-Arzela theorem, the sequences $\{\upsilon_n(t)\}$ and $\{w_n(t)\}$ converge uniformly to υ and w, respectively on J. Using corresponding fractional Volterra integral equations

$$\begin{aligned} \upsilon_{n+1}(t) &= \upsilon_{n+1}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f\left(s, \upsilon_n(s), \upsilon_n(\theta(s))\right) - \right. \\ &\qquad \left. M\left[\upsilon_{n+1}(s) - \upsilon_n(s)\right] - N\left[\upsilon_{n+1}(\theta(s)) - \upsilon_n(\theta(s))\right]\right] ds \\ w_{n+1}(t) &= w_{n+1}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f\left(s, w_n(s), w_n(\theta(s))\right) - \right. \\ &\qquad \left. M\left[w_{n+1}(s) - w_n(s)\right] - N\left[w_{n+1}(\theta(s)) - w_n(\theta(s))\right]\right] ds \end{aligned}$$

By Lebesgue's dominated convergence Lemma 1.5 as $n \longrightarrow \infty$, it follows that $\upsilon(t)$ and $w(t)$ are solutions of the linear problem (3.2).

Now, we prove that $\upsilon(t)$ and $w(t)$ are the minimal and maximal solutions of the nonlinear BVP(1.1). Let $x(t)$ be any solution of the nonlinear BVP(1.1) different from $\upsilon(t)$ and $w(t)$, so that there exists k such that $\upsilon_k(t) \leq x(t) \leq w_k(t)$ on J. Set $p(t) = x(t) - \upsilon_{k+1}(t)$, we have

$$\begin{aligned} D^{\alpha}_{0+}p(t) &= D^{\alpha}_{0+}x(t) - D^{\alpha}_{0+}\upsilon_{k+1}(t) \\ &= f\left(t, x(t), x(\theta(t))\right) - f\left(t, \upsilon_k(t), \upsilon_k(\theta(t))\right) + \\ &\qquad M\left[\upsilon_{k+1}(t) - \upsilon_k(t)\right] + N\left[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))\right] \\ &\geq -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = x(0) - \upsilon_{k+1}(0) = \int_0^T \left[x(s) - \upsilon_k(s)\right] ds \geq 0.$$

By Lemma 3.1, we get $p(t) \geq 0$, implies that $x(t) \geq \upsilon_{k+1}(t)$ for all k on J. Similarly we can prove $x(t) \leq w_{k+1}(t)$ for all k on J. Since $\upsilon_0(t) \leq x(t) \leq x_0(t)$ on J.

By induction it follows that $\upsilon_k(t) \leq x(t)$ and $x(t) \leq w_k(t)$ for all k. Thus $\upsilon_k(t) \leq x(t) \leq w_k(t)$ on J. Taking limit as $k \longrightarrow \infty$, we get $\upsilon(t) \leq x(t) \leq w(t)$ on J.

The functions $\upsilon(t)$ and $w(t)$ are the minimal and maximal solutions to the nonlinear BVP(1.1). The proof is complete. □

Next, we obtain the uniqueness of solution of the nonlinear BVP(1.1) as follows:

Theorem 3.4. Assume that:

(i) all the conditions of the Theorem 3.3 hold,

(ii) there exists nonnegative constants M, N such that function f satisfies the condition

$$f(t,x_1,x_2)-f(t,y_1,y_2)\le M\left(x_1-y_1\right)+N\left(x_2-y_2\right), \tag{3.6}$$

for $\upsilon_0(t)\le y_1\le x_1\le w_0(t)$, $\upsilon_0(\theta(t))\le y_2\le x_2\le w_0(\theta(t))$. Then the nonlinear BVP(1.1) has a unique solution.

Proof . Since $\upsilon(t)\le w(t)$, it is sufficient to prove $\upsilon(t)\ge w(t)$. Consider $p(t)=w(t)-\upsilon(t)$, then

$$\begin{aligned} D^{\alpha}_{0^+}p(t) &= D^{\alpha}_{0^+}w(t)-D^{\alpha}_{0^+}\upsilon(t)\\ &= f\left(t,w(t),w(\theta(t))\right)-f\left(t,\upsilon(t),\upsilon(\theta(t))\right)\\ &\le Mp(t)+Np(\theta(t)), \end{aligned}$$

and

$$p(0) = w(0)-\upsilon(0)\le 0.$$

By Lemma 3.1, we get $p(t)\le 0$, implies that $w(t)\le\upsilon(t)$, which means $w(t)=\upsilon(t)$ is a unique solution of the nonlinear BVP(1.1). The proof is complete. □

4. Weakly coupled lower and upper solutions of BVP(1.1)

In this section, we investigate the existence and uniqueness of solution of the nonlinear BVP(1.1) by weakly coupled lower and upper solutions.

Definition 4.1. A pair of functions $[\upsilon_0,w_0]$ in $C_{1-\alpha}(J,\mathbb{R})$ is called weakly coupled lower and upper solutions of the nonlinear BVP(1.1) for $\lambda=-1$ if

$$D^{\alpha}_{0^+}\upsilon_0(t)\le f\left(t,\upsilon_0(t),\upsilon_0(\theta(t))\right),\quad \upsilon_0(0)\le-\int_0^T w_0(s)ds+r,$$

$$D^{\alpha}_{0^+}w_0(t)\ge f\left(t,w_0(t),w_0(\theta(t))\right),\quad w_0(0)\ge-\int_0^T \upsilon_0(s)ds+r.$$

Theorem 4.2. Assume that:

(i) $f\in C\left(J\times\mathbb{R}^2,\mathbb{R}\right),\ \theta\in C\left(J,J\right),\ t\le\theta(t),\ t\in J,$

(ii) functions $\upsilon_0(t)$ and $w_0(t)$ in $C_{1-\alpha}(J,\mathbb{R})$ are weakly coupled lower and upper solutions of the nonlinear BVP(1.1) such that $\upsilon_0(t)\le w_0(t)$ on J,

(iii) there exists nonnegative constants M, N such that function f satisfies the condition

$$f(t,x_1,x_2)-f(t,y_1,y_2)\ge -M(x_1-y_1)-N(x_2-y_2),$$

for $\upsilon_0(t)\le y_1\le x_1\le w_0(t)$, $\upsilon_0(\theta(t))\le y_2\le x_2\le w_0(\theta(t))$. Then there exists monotone sequences $\{\upsilon_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J,\mathbb{R})$ such that

$$\{\upsilon_n(t)\}\longrightarrow \upsilon(t)\text{ and }\{w_n(t)\}\longrightarrow w(t)\text{ as }n\longrightarrow\infty$$

where $\upsilon(t)$ and $w(t)$ are minimal and maximal solutions of the nonlinear BVP(1.1), respectively and $\upsilon(t)\le x(t)\le w(t)$ on J.

Proof . We consider the following linear problem:

$$\begin{cases} D_{0^+}^{\alpha} x(t) = -Mx(t) - Nx(\theta(t)) + h(t) \\ \quad x(0) = -\int_0^T x(s)ds + r, \end{cases} \tag{4.1}$$

where $h(t) = f(t, \eta(t), \eta(\theta(t))) - M\eta(t) - N\eta(\theta(t))$ and $\eta \in C_{1-\alpha}(J, \mathbb{R})$.
The unique of solution of the linear problem (4.1) can be proved as in Corollary 2.2.
Define the iterates as follows:

$$\begin{cases} D_{0^+}^{\alpha} \upsilon_{n+1}(t) = f(t, \upsilon_n(t), \upsilon_n(\theta(t))) - M[\upsilon_{n+1}(t) - \upsilon_n(t)] - N[\upsilon_{n+1}(\theta(t)) - \upsilon_n(\theta(t))], \\ \quad \upsilon_{n+1}(0) = -\int_0^T w_n(s)ds + r, \end{cases} \tag{4.2}$$

and

$$\begin{cases} D_{0^+}^{\alpha} w_{n+1}(t) = f(t, w_n(t), w_n(\theta(t))) - M[w_{n+1}(t) - w_n(t)] - N[w_{n+1}(\theta(t)) - w_n(\theta(t))], \\ \quad w_{n+1}(0) = -\int_0^T \upsilon_n(s)ds + r, \end{cases} \tag{4.3}$$

Obviously, the above arguments imply the existence of the unique solutions $\upsilon_{n+1}(t)$ and $w_{n+1}(t)$ for the problems (4.2), (4.3). By setting $n = 0$ in the problems (4.2), (4.3), we get the existence of solutions $\upsilon_1(t)$ and $w_1(t)$. We show that $\upsilon_0(t) \leq \upsilon_1(t) \leq w_1(t) \leq w_0(t)$. For this, consider $p(t) = \upsilon_1(t) - \upsilon_0(t)$ on J, and $\upsilon_0(t)$ is the lower solution of the nonlinear BVP(1.1). Then

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} \upsilon_1(t) - D_{0^+}^{\alpha} \upsilon_0(t) \\ &\geq -M[\upsilon_1(t) - \upsilon_0(t)] - N[\upsilon_1(\theta(t)) - \upsilon_0(\theta(t))] \\ &\geq -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = \upsilon_1(0) - \upsilon_0(0) \geq 0.$$

By Lemma 3.1, we get $p(t) \geq 0$, implies that $\upsilon_1(t) \geq \upsilon_0(t)$ on J. Similarly, we can prove $w_1 \leq w_0$ and $\upsilon_1(t) \leq w_1(t)$ on J. Thus $\upsilon_0(t) \leq \upsilon_1(t) \leq w_1(t) \leq w_0(t)$. Assume that for some $k > 1$, $\upsilon_{k-1}(t) \leq \upsilon_k(t) \leq w_k(t) \leq w_{k-1}(t)$ on J. We claim that $\upsilon_k(t) \leq \upsilon_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on J. To prove the claim, set $p(t) = \upsilon_{k+1}(t) - \upsilon_k(t)$, we have

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} \upsilon_{k+1}(t) - D_{0^+}^{\alpha} \upsilon_k(t) \\ &\geq -M[\upsilon_{k+1}(t) - \upsilon_k(t)] - N[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))] \\ &\geq -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = \upsilon_{k+1}(0) - \upsilon_k(0) = \int_0^T w_k(s)ds - \int_0^T w_{k-1}(s)ds \geq 0.$$

By Lemma 3.1, we get $p(t) \geq 0$, implies that $\upsilon_{k+1}(t) \geq \upsilon_k(t)$ on J. Similarly, we can prove that $\upsilon_{k+1}(t) \leq w_{k+1}(t)$ and $w_{k+1}(t) \leq w_k(t)$ on J. By the principle of mathematical induction, we have

$$\upsilon_0 \leq \upsilon_1 \leq \upsilon_2 \leq \cdots \leq \upsilon_k \leq w_k \leq \cdots \leq w_2 \leq w_1 \leq w_0 \text{ on } J. \tag{4.4}$$

Obviously, the sequences $\{\upsilon_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded. We observe that $\{D_{0^+}^{\alpha} \upsilon_n\}$ and $\{D_{0^+}^{\alpha} w_n\}$ are uniformly bounded on J, in view of the relations (4.2) & (4.3).

Then using Corollary 1.7, we can conclude the equicontinuous of the sequences $\{\upsilon_n(t)\}, \{w_n(t)\}$. Hence by the Ascoli-Arzela theorem, the sequences $\{\upsilon_n(t)\}$ and $\{w_n(t)\}$ converge uniformly to υ and w, respectively on J. Using corresponding fractional Volterra integral equations

$$\begin{aligned}\upsilon_{n+1}(t) &= \upsilon_{n+1}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f\left(s, \upsilon_n(s), \upsilon_n(\theta(s))\right) - \right. \\ &\qquad \left. M\left[\upsilon_{n+1}(s) - \upsilon_n(s)\right] - N\left[\upsilon_{n+1}(\theta(s)) - \upsilon_n(\theta(s))\right]\right] ds\end{aligned}$$

$$\begin{aligned}w_{n+1}(t) &= w_{n+1}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f\left(s, w_n(s), w_n(\theta(s))\right) - \right. \\ &\qquad \left. M\left[w_{n+1}(s) - w_n(s)\right] - N\left[w_{n+1}(\theta(s)) - w_n(\theta(s))\right]\right] ds\end{aligned}$$

By Lebesgue's dominated convergence Lemma 1.5 as $n \longrightarrow \infty$, it follows that $\upsilon(t)$ and $w(t)$ are solutions of the linear problem (4.1).

Now, we prove that $\upsilon(t)$ and $w(t)$ are the minimal and maximal solutions of nonlinear BVP(1.1). Let $x(t)$ be any solution of the nonlinear BVP(1.1) different from $\upsilon(t)$ and $w(t)$, so that there exists k such that $\upsilon_k(t) \leq x(t) \leq w_k(t)$ on J. Set $p(t) = x(t) - \upsilon_{k+1}(t)$. we have

$$\begin{aligned}D^\alpha_{0+} p(t) &= D^\alpha_{0+} x(t) - D^\alpha_{0+} \upsilon_{k+1}(t) \\ &\geq -M\left[x(t) - \upsilon_{k+1}(t)\right] - N\left[x(\theta(t)) - \upsilon_{k+1}(\theta(t))\right] \\ &\geq -Mp(t) - Np(\theta(t)),\end{aligned}$$

and

$$p(0) = x(0) - \upsilon_{k+1}(0) = \int_0^T \left[x(s) - w_k(s)\right] ds \geq 0.$$

By Lemma 3.1, we get $p(t) \geq 0$, implies that $x(t) \geq \upsilon_{k+1}(t)$ for all k on J. Similarly we can prove $x(t) \leq w_{k+1}(t)$ for all k on J. Since $\upsilon_0(t) \leq x(t) \leq x_0(t)$ on J. By induction it follows that $\upsilon_k(t) \leq x(t)$ and $x(t) \leq w_k(t)$ for all k. Thus $\upsilon_k(t) \leq x(t) \leq w_k(t)$ on J. Taking limit as $k \longrightarrow \infty$, it follows that $\upsilon(t) \leq x(t) \leq w(t)$ on J. The functions $\upsilon(t)$ and $w(t)$ are the minimal and maximal solutions to the nonlinear BVP(1.1). The proof is complete. □

Next, we obtain the uniqueness of solutions of the nonlinear BVP(1.1) as follows:

Theorem 4.3. Assume that:

(i) all the conditions of the Theorem 4.2 hold,

(iii) there exists nonnegative constants M, N such that function f satisfies the condition

$$f(t, x_1, x_2) - f(t, y_1, y_2) \leq M\left(x_1 - y_1\right) + N\left(x_2 - y_2\right),$$

for $\upsilon_0(t) \leq y_1 \leq x_1 \leq w_0(t)$, $\upsilon_0(\theta(t)) \leq y_2 \leq x_2 \leq w_0(\theta(t))$.

Then the nonlinear BVP(1.1) has a unique solution.

Proof . This can be proved as in Theorem 3.4. □

5. An example

In the section, we illustrate our result with the following example.

Example 5.1. *Consider the fractional differential equation:*

$$\begin{cases} D^{\alpha}_{0^+}x(t) = f\left(t, x(t), x(\theta(t))\right), \ t \in [0,1], \\ \quad x(0) = \lambda \int_0^T x(s)ds + r, \end{cases} \tag{5.1}$$

where $\alpha = \frac{1}{2}, T = 1, \theta(t) = t^{\gamma}, 0 < \gamma < 1, \lambda = \frac{1}{4}, r = 1$ *and* $f\left(t, x(t), x(t^{\gamma})\right) = t + \frac{t^2+1}{30}x(t) + \frac{t^4+1}{15}x(t^{\gamma})$. *Obviously,* $f\left(t, x(t), x(t^{\gamma})\right)$ *satisfies Lipschitz condition and there exist constants* $M = \frac{1}{60}, N = \frac{1}{30}$ *such that*

$$|f(t, x_1(t), x_2(t^{\gamma})) - f(t, y_1(t), y_2(t^{\gamma}))| \ \leq \ \frac{1}{60}\left|x_1(t) - y_1(t)\right| + \frac{1}{30}\left|x_2(t^{\gamma}) - y_2(t^{\gamma})\right|, for\ t \in J.$$

Furthermore, we find that

$$\lambda < 1 - \frac{1}{20}\sqrt{\pi} = 0.911377.$$

Inequality (2.1) holds. It shows that the condition (H_2) *of Theorem 2.1 holds, we conclude that the problem (5.1) has a unique solution.*

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