



Numerical solutions of Abel integral equations via Touchard and Laguerre polynomials

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Abstract

In this article, two numerical methods based on Touchard and Laguerre polynomials were presented to solve Abel integral (AI) equations. Touchard and Laguerre matrices were utilized to transform Abel integral equations into an algebraic system of linear equations. Solve this system of these equations to obtain Touchard and Laguerre parameters. Four examples are given to demonstrate the presented methods. The solutions were compared with the solutions in the literature.

Keywords: Abel integral equation, Numerical solution, singular Volterra, Touchard polynomials, Laguerre polynomials.

1. Introduction

Abel problem is summarized as follows: Niles Abel is the Norwegian mathematician his study in 1823 to find a smooth curve vertical on x-y plane so that the particle slips down without friction from the highest known point to the origin point by the effect of gravity only without initial velocity [3, 12, 2]. In other word, (AI) equation occurs in several branches of scientific fields, for instance, microscopy, radio astronomy seismology, atomic scattering, electron emission, plasma diagnostics, optics, nuclear physics, physical electronics, etc.[3, 12, 2]. Abel established Abel integral equation

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[2, 6]. The standard forms of Abel integral equations of the second and first types are [2, 6, 4] respectively defined as follows:

$$P(\theta) = e(\theta) + \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}} P(\vartheta) d\vartheta, \quad (1.1)$$

$$e(\theta) = \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}} P(\vartheta) d\vartheta, \quad (1.2)$$

where c is a given real number value, $e(\theta), w(\theta)$ are known functions and $P(\theta)$ is an unknown function. In recent years, many scientists mathematicians and physicists have made great efforts aimed at finding the approximate numerical solutions for Abel integral equation, and among these methods: Taylor-Collocation Method [37], Lagrangian matrix approach [18], Chebyshev polynomials [27], Orthogonal polynomials [9], the product integration and Haar Wavelet approaches, have been used to find an approximate solutions of the fractional Volterra integral equations of the 2nd type [10], Hermite wavelet method [19], Babenko method and fractional and fractional integrals [15], Babenko approach and fractional calculus method [14], Fractional calculus method [11], Schwartz method [16], Mechanical quadrature technique [38], Collocation method [22], finally Adomian decomposition method, variation iteration method and homotopy analysis method [7]. The rest of this article is organized as follows: Touchard polynomials, approximation function for Touchard polynomials, Laguerre polynomials, approximation function for Laguerre polynomials, solutions Abel integral equation using Touchard polynomials, solution accuracy, numerical applications, Figures is presented, brief of conclusions. Also, the references.

2. Description of the Methods

2.1. Touchard Polynomials

Let us begin with the definition of the (TPs) [20, 32, 24, 34] that was derived by Jacques Touchard is a French mathematician (1885-1968), that is defined over $[0, 1]$ and consists of a binomial polynomial sequence. These polynomials later became known by his name and have the following formula:

$$g_n(\theta) = \sum_{b=0}^n \delta(n, b) \theta^b = \sum_{b=0}^n \binom{n}{b} \theta^b, \quad \binom{n}{b} = \frac{n!}{b!(n-b)!} \quad (2.1)$$

where n and b refer to the (TPs) degree and index respectively. The 1st six polynomials of the (TPs) are now as follows:

$$g_0(\theta) = 1, \quad g_1(\theta) = 1 + \theta, \quad g_2(\theta) = 1 + 2\theta + \theta^2, \quad g_3(\theta) = 1 + 3\theta + 3\theta^2 + \theta^3$$

$$g_4(\theta) = 1 + 4\theta + 6\theta^2 + 4\theta^3 + \theta^4, \quad g_5(\theta) = 1 + 5\theta + 10\theta^2 + 10\theta^3 + 5\theta^4 + \theta^5$$

2.1.1. Approximation Function for Touchard Polynomials

Assume that the following function is approximated using the (TPs):

$$P_n(\theta) = \mu_0 g_0(\theta) + \mu_1 g_1(\theta) + \dots + \mu_n g_n(\theta) = \sum_{b=0}^n \mu_b g_b(\theta), \quad 0 \leq \theta \leq 1 \quad (2.2)$$

the function $\{g_b(\theta)\}_{b=0}^n$ indicate the basis of the (TPs) of n th degree, according to Eq. (2.1), μ_b 's are the unknown parameters will be specified later and n represents any positive integer. Eq. (2.2) can

be described as follows:

$$P_n(\theta) = [g_0(\theta) \quad g_1(\theta) \quad \dots \quad g_n(\theta)] \cdot \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \tag{2.3}$$

Eq. (2.3) can be specified as:

$$P_n(\theta) = [1 \quad \theta \quad \theta^2 \quad \dots \quad \theta^n] \cdot \begin{bmatrix} \epsilon_{00} & \epsilon_{01} & \epsilon_{02} & \dots & \epsilon_{0n} \\ 0 & \epsilon_{11} & \epsilon_{12} & \dots & \epsilon_{1n} \\ 0 & 0 & \epsilon_{22} & \dots & \epsilon_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon_{nn} \end{bmatrix} \cdot \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \tag{2.4}$$

where $\{\epsilon_{kk}\}_{k=0}^n$ are the power base parameters used to obtain the (TPs) parameters and the matrix in Eq. (2.4) is certainly invertible. As a result, the operational matrices for $n = 1, 2,$ and 3 will be shown in Eqs. (2.5a), (2.5b) and (2.5c) respectively:

$$P_1(\theta) = [1 \quad \theta] \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mu_0 \\ \mu_1 \end{bmatrix} \tag{2.5a}$$

$$P_2(\theta) = [1 \quad \theta \quad \theta^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix} \tag{2.5b}$$

$$P_3(\theta) = [1 \quad \theta \quad \theta^2 \quad \theta^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \tag{2.5c}$$

2.2. Laguerre Polynomials

Let us describe the (LPs) [25, 33] that was derived by Adrien-Marie Laguerre in 1782, that is defined over $[0, \infty)$ and consists of a binomial polynomial sequence. These polynomials have the following formula:

$$k_n(\theta) = \sum_{b=0}^n (-1)^b \frac{1}{b!} \binom{n}{b} \theta^b = \sum_{b=0}^n \frac{(-1)^b}{(b!)^2 (n-b)!} \theta^b, \quad n = 0, 1, 2, \dots \text{ and } \theta \in [0, \infty). \tag{2.6}$$

where n and b refer to the (LPs) degree and index respectively. The 1st six polynomials of the (LPs) are as follows:

$$k_0(\theta) = 1, \quad k_1(\theta) = 1 - \theta, \quad k_2(\theta) = \frac{1}{2} (2 - 4\theta + \theta^2), \quad k_3(\theta) = \frac{1}{6} (6 - 18\theta + 9\theta^2 - \theta^3),$$

$$k_4(\theta) = \frac{1}{24} (24 - 96\theta + 72\theta^2 - 16\theta^3 + \theta^4), \quad k_5(\theta) = \frac{1}{120} (120 - 600\theta + 600\theta^2 - 200\theta^3 + 25\theta^4 - \theta^5).$$

2.2.1. Approximation Function for Laguerre polynomials

To obtain a numerical solution for Eqs.(1.1) and (1.2), Assume that the following function is approximated using the (LPs):

$$P_n(\theta) = \rho_0 k_0(\theta) + \rho_1 k_1(\theta) + \dots + \rho_n k_n(\theta) = \sum_{b=0}^n \rho_b k_b(\theta), \quad 0 \leq \theta < \infty \tag{2.7}$$

the function $\{k_b(\theta)\}_{b=0}^n$ denotes the basis of the (LPs) of nth degree, according to Eq. (2.6), ρ_b 's are the unknown parameters will be specified later and n represents any positive integer. Now Eq. (2.7) can be written as follows:

$$P_n(\theta) = [k_0(\theta) \quad k_1(\theta) \quad \dots \quad k_n(\theta)] \cdot \begin{bmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_n \end{bmatrix} \tag{2.8}$$

So, Eq. (2.8) is converted into:

$$P_n(\theta) = [1 \quad \theta \quad \theta^2 \quad \dots \quad \theta^n] \cdot \begin{bmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \dots & \sigma_{0n} \\ 0 & \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ 0 & 0 & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_{nn} \end{bmatrix} \cdot \begin{bmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ rh\theta_n \end{bmatrix} \tag{2.9}$$

where $\{\rho_{kk}\}_{k=0}^n$ are the power base parameters used to obtain the (LPs) parameters and the matrix in Eq. (2.9) is certainly invertible. As a result, the operational matrices for n =1, 2, and 3 will be shown in Eqs. (2.10a), (2.10b) and (2.10c) respectively:

$$P_1(\theta) = [1 \quad \theta] \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} \tag{2.10a}$$

$$P_2(\theta) = [1 \quad \theta \quad \theta^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{bmatrix} \tag{2.10b}$$

$$P_3(\theta) = [1 \quad \theta \quad \theta^2 \quad \theta^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & -1/6 \end{bmatrix} \cdot \begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} \tag{2.10c}$$

3. Solutions Abel Integral Equation via Touchard Polynomials

Let us we're using the (TPs) to find approximate solutions to Eq. (1.1) and let by using Eq. (2.2)

$$P(\theta) \cong P_n(\theta) = \sum_{b=0}^n \mu_b g_b(\theta) \tag{3.1}$$

Now, substituting the Eq.(3.1) into the Eq.(1.1) yields:

$$\sum_{b=0}^n \mu_b g_b(\theta) = e(\theta) + \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}} \sum_{b=0}^n \mu_b g_b(\vartheta) d\vartheta \tag{3.2}$$

when Eq. (2.3) is used, Eq. (3.2) becomes:

$$[g_0(\theta) \quad g_1(\theta) \quad \dots \quad g_n(\theta)] \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = e(\theta) + \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}} [g_0(\vartheta) \quad g_1(\vartheta) \quad \dots \quad g_n(\vartheta)] \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} d\theta \tag{3.3}$$

and by using Eq. (2.4), then Eq. (3.3) converts to the following:

$$\begin{aligned}
 & [1 \quad \theta \quad \theta^2 \quad \dots \quad \theta^n] \begin{bmatrix} \epsilon_{00} & \epsilon_{01} & \epsilon_{02} & \dots & \epsilon_{0n} \\ 0 & \epsilon_{11} & \epsilon_{12} & \dots & \epsilon_{1n} \\ 0 & 0 & \epsilon_{22} & \dots & \epsilon_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon_{nn} \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = e(\theta) + \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}}. \\
 & [1 \quad \vartheta \quad \vartheta^2 \quad \dots \quad \vartheta^n] \begin{bmatrix} \epsilon_{00} & \epsilon_{01} & \epsilon_{02} & \dots & \epsilon_{0n} \\ 0 & \epsilon_{11} & \epsilon_{12} & \dots & \epsilon_{1n} \\ 0 & 0 & \epsilon_{22} & \dots & \epsilon_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon_{nn} \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} d\vartheta
 \end{aligned} \tag{3.4}$$

so, after finding the integrations of the Eq. (3.4) the unknown parameters of the (TPs) are discovered by choosing $\{\theta_\alpha\}_{\alpha=0}^n$ in the interval $[0,1]$. As a result, Eq. (3.4) becomes a system of $(n+1)$ algebraic linear equations with $(n+1)$ unknown parameters. This scheme can be solved via ‘‘Gauss elimination method’’ to obtain the parameters $\{\mu_b\}_{b=0}^n$ that are uniquely determined. Finally, plug these values into the Eq. (2.2) getting an approximate solution to n selecting.

Remark 3.1. For the Eq. (1.2), the same technique may be used. Also for the Eqs. (1.1) and (1.2), the same techniques may be used when applied the (LPs) method.

4. Solution Accuracy

The accuracy [33, 13, 8] of the proposed methods is checked.

4.1. For Abel integral equation by using Touchard polynomials

Since the following is the form of Eq. (3.2):

$$\sum_{b=0}^n \mu_b g_b(\theta) = e(\theta) + \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}} \sum_{b=0}^n \mu_b g_b(\vartheta) d\vartheta, \tag{4.1}$$

also the following is the form of Eq. (2.2):

$$P_n(\theta) = \sum_{b=0}^n \mu_b g_b(\theta)$$

And the unknown Touchard parameters $\{\mu_b\}_{b=0}^n$ were calculated by using Eq. (3.4). Also, Eq. (3.1) has the formula:

$$P(\theta) \cong P_n(\theta) = \sum_{b=0}^n \mu_b g_b(\theta), \tag{4.2}$$

since Eq.(4.2) is an approximation of Eq. (3.4) and is substituted into Eq (4.1). Now, let $\theta = \theta_\alpha$ belong to the interval $[0,1]$, such that $\alpha = 0, 1, 2, \dots, n$, so, the function of error:

$$Ef(\theta_\alpha) = \left| \sum_{b=0}^n \mu_b g_b(\theta_\alpha) - e(\theta_\alpha) - \int_c^{\theta_\alpha} \frac{1}{\sqrt{\theta_\alpha - \vartheta}} \sum_{b=0}^n \mu_b g_b(\vartheta) d\vartheta \right| \cong 0, \text{ then}$$

$Ef(\theta_\alpha) \leq \epsilon$, for each θ_α in the interval $[0,1]$ and $\epsilon > 0$

Hence, the difference for the function of error $Ef(\theta_\alpha)$ at each θ_α may be less than any positive integer number $\epsilon > 0$.

The error function $Ef(\theta)$ can then be calculated using the following equation:

$$Ef(\theta) = \sum_{b=0}^n \mu_b g_b(\theta) - e(\theta) - \int_c^\theta \frac{1}{\sqrt{\theta - \vartheta}} \sum_{b=0}^n \mu_b g_b(\vartheta) d\vartheta,$$

thus, $Ef(\theta) \leq \epsilon$.

Remark 4.1. *The same steps can be applied for Eq. (1.2) and the same steps are also applied if the (LPs) is used.*

5. Numerical Applications

In this part, four numerical examples have been given to illustrate the ability and effectiveness of the methods used to find the solutions by the codes written in matlab2018b. The convergence of solutions in the graphs was done.

Example 5.1. *The following Abel integral equation of the 1st type.*

$$\frac{2\sqrt{\theta}}{105} (105 - 56\theta^2 - 48\theta^3) = \int_0^\theta \frac{1}{\sqrt{\theta - \vartheta}} P(\vartheta) d\vartheta, \quad 0 \leq \theta \leq 1$$

$P(\theta) = \theta^3 - \theta^2 + 1$ is the exact solution. By applying the presented methods for $n = 3$, using (TPs) and (LPs) polynomials and selecting the values $\theta_0 = 0.1$, $\theta_1 = 0.2$ and $\theta_3 = 0.3$ in the interval $[0,1]$, solving the algebraic system by Gauss elimination method in matlab2018b, we obtained Touchard and Laguerre parameters, substituting these parameters into Eqs.(2.2) and (2.7), we have the approximate solution $P_3(\theta)$ as follows:

$$P_3(\theta) = (-1) g_0(\theta) + (5) g_1(\theta) + (-4) g_2(\theta) + (1) g_3(\theta) = \theta^3 - \theta^2 + 1, \text{ and}$$

$$P_3(\theta) = (5) k_0(\theta) + (-14) k_1(\theta) + (16) k_2(\theta) + (-6) k_3(\theta) = \theta^3 - \theta^2 + 1,$$

The comparison shows that the presented methods yield the same results as the analytical process of this example. By using the (LPs) method, [9, 5, 35, 26, 36], the exact solution was obtained for $n = 3$. Besides, [30] also found the exact solution with $n = 4$. [31] also obtained the exact solution. Moreover, [25, 17] found the absolute errors of order $1.0e - 16$ and $1.0e - 07$ by using the Laguerre and Bernstein polynomials for $n = 10$, respectively. Furthermore, [21] obtained the absolute error of order $1.0e - 07$ for $n = 20$ and $x = 1$. As a result, our techniques are more precise than [25, 17] and are similar to the others that have the same accuracy. The comparison with the exact solution is shown in Figure 1.

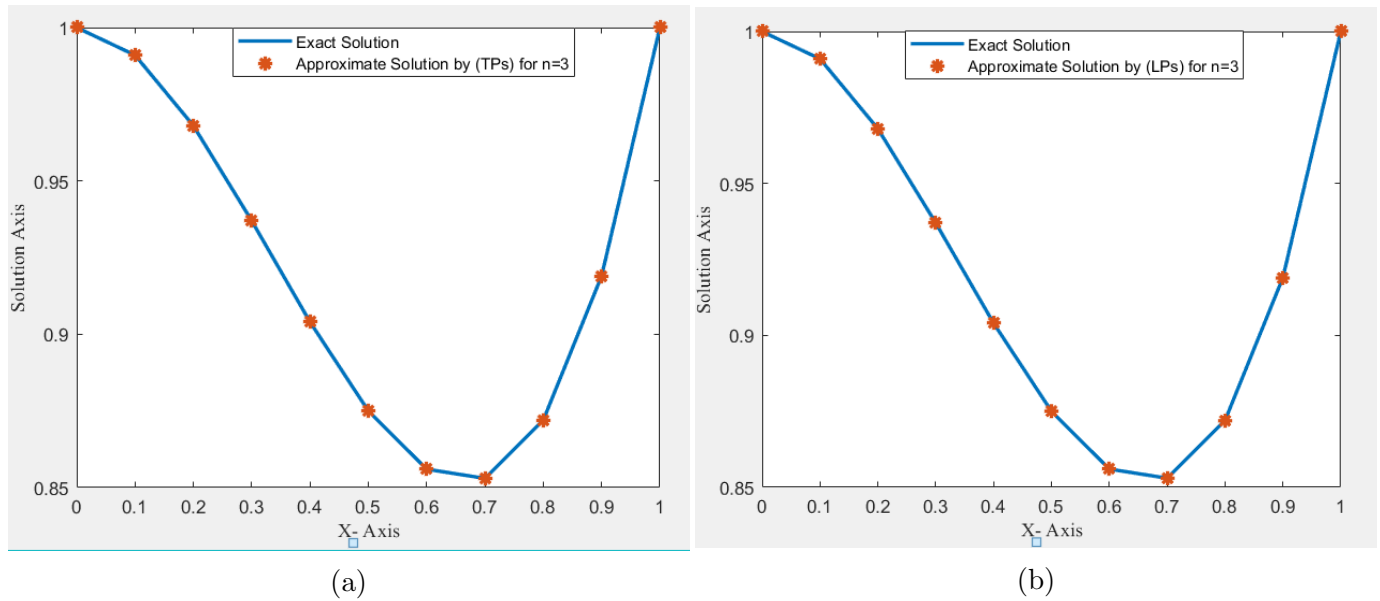


Figure 1: (a) For n=3, compare the (TPs) to the exact solution, (b) For n=3, compare the (LPs) to the exact solution

Example 5.2. The following Abel integral equation of the 1st type.

$$\frac{4}{3}\theta^{\frac{3}{2}} - \frac{32}{35}\theta^{\frac{7}{2}} - \int_0^\theta \frac{1}{\sqrt{\theta - \vartheta}} P(\vartheta) d\vartheta, \quad 0 \leq \theta \leq 1$$

$P(\theta) = \theta - \theta^3$ is the exact solution. By applying the presented methods in Eq.(3.4) for $n = 3$ and selecting the values $\theta_0 = 0.1, \theta_1 = 0.2$ and $\theta_3 = 0.3$ in the given interval and solving the algebraic system by Gauss elimination method in matlab2018b, we have the approximate solutions using the (TPs) and the (LPs) are respectively as follows:

$$P_3(\theta) = (0) g_0(\theta) + (-2) g_1(\theta) + (3) g_2(\theta) + (-1) g_3(\theta) = \theta - \theta^3, \text{ and}$$

$$P_3(\theta) = (-5) k_0(\theta) + (17) k_1(\theta) + (18) k_2(\theta) + (6) k_3(\theta) = \theta - \theta^3,$$

The comparison shows that the presented methods yield the same results as the analytical process of this example. By using Lagrangian matrix approach,[18] obtained the absolute errors of order $1.0e - 34$ and $1.0e - 33$ for $n = 4$ and 5 respectively by using scheme1 and of order $1.0e - 41$ for $n = 4$ and 5 by using scheme2. Furthermore,[29] found the absolute error of order $1.0e - 9$ for $n = 6$. Moreover, [1] obtained the exact solution $n \geq 3$. Therefore, our methods are similar to [1] and more accurate than the others. The comparison with the exact solution is shown in Figure 2.

Example 5.3. The following Abel (the weakly singular Volterra) integral equation of the 2nd type.

$$P(\theta) = \theta + \frac{4}{3}\theta^{\frac{3}{2}} - \int_0^\theta \frac{1}{\sqrt{\theta - \vartheta}} P(\vartheta) d\vartheta, \quad 0 \leq \theta \leq 1$$

$P(\theta) = \theta$ is the exact solution. By applying the presented methods in Eq.(3.4) for $n = 1$ and selecting the values $\theta_0 = 0.1$ and $\theta_1 = 0.2$ in the given interval and solving the algebraic system by Gauss elimination method in matlab2018b, we have the approximate solutions using the (TPs) and the (LPs) are respectively as follows:

$$P_1(\theta) = (-1) g_0(\theta) + (1) g_1(\theta) = \theta, \text{ and}$$

$$P_1(\theta) = (1) k_0(\theta) + (-1) k_1(\theta) = \theta,$$

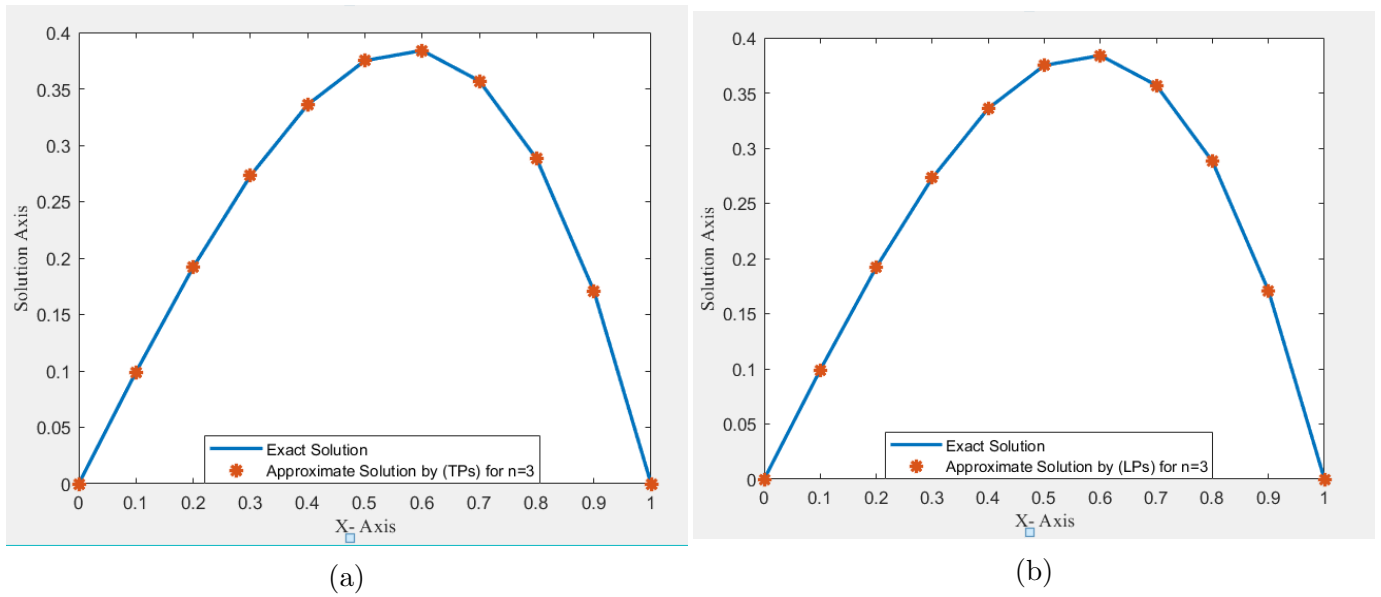


Figure 2: (a) For $n=3$, compare the (TPs) to the exact solution, (b) For $n=3$, compare the (LPs) to the exact solution

The comparison shows that the presented methods yield the same results as the analytical process of this example. [9] the exact solution was obtained for $n = 1$, also, [15, 1] obtained the exact solutions respectively. In addition to these, [23] found the maximum absolute error and relative error of nearly $5.0e-03$ and $5.0e-01$, respectively for $n = 18$. Moreover, [28] found approximate solution nearly $1.0e-10$ absolute error for $n = 14$. Therefore, our methods are better than [23], and are similar to the others that have the same precise. Figure 3 shows the comparison with the exact solution.

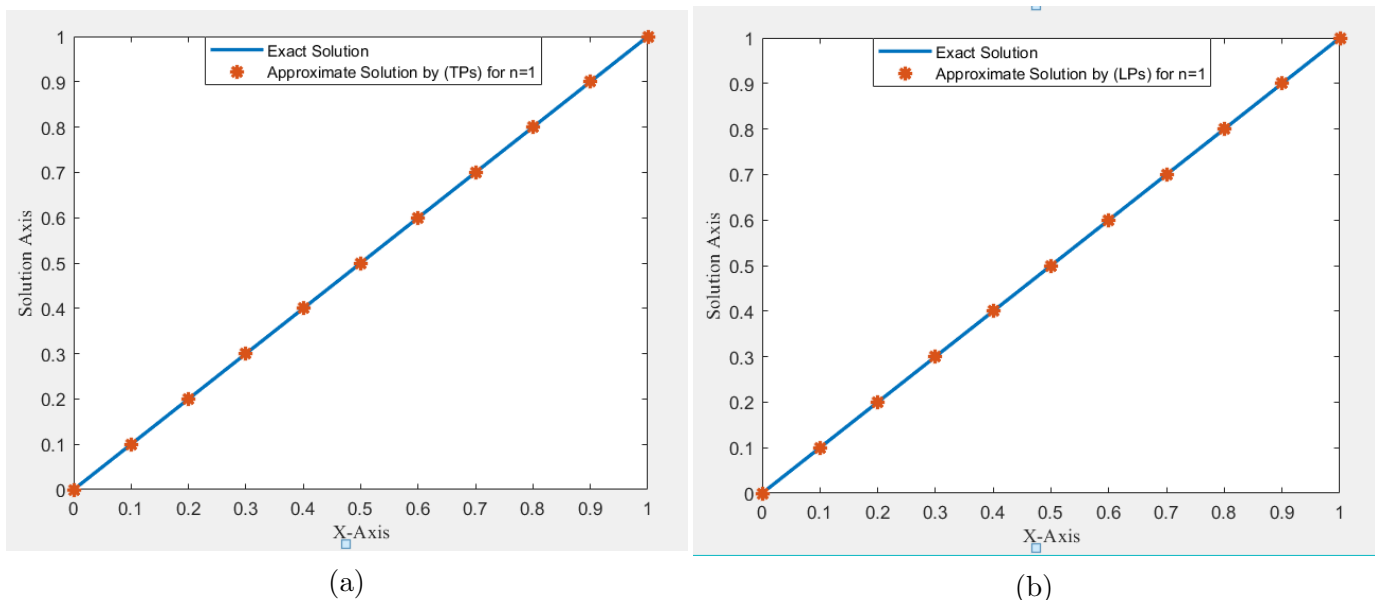


Figure 3: (a) For $n=1$, compare the (TPs) to the exact solution, (b) For $n=1$, compare the (LPs) to the exact solution

Example 5.4. *The following Abel (the weakly singular Volterra) integral equation of the 2nd type.*

$$P(\theta) = \theta^2 + \frac{16}{15}\theta^{\frac{5}{2}} - \int_0^\theta \frac{1}{\sqrt{\theta - \vartheta}} P(\vartheta) d\vartheta, \quad 0 \leq \theta \leq 1$$

P(θ) = θ² is the exact solution. By applying the presented methods in Eq. (3.4) for n = 2 and selecting the values θ₀ = 0.1, θ₁ = 0.2 and θ₂ = 0.3 in the given interval and solving the algebraic system by Gauss elimination method in matlab2018b, we obtained the following Touchard and Laguerre parameters, substituting these parameters into Eqs. (2.2), and (2.7), we have the approximate solutions are respectively as follows:

$$P_2(\theta) = (1) g_0(\theta) + (-2) g_1(\theta) + (1) g_2(\theta) = \theta^2, \text{ and}$$

$$P_2(\theta) = (2) k_0(\theta) + (-4) k_1(\theta) + (2) k_2(\theta) = \theta^2$$

The comparison shows that the proposed methods yield the same results as the analytical process of this example. [3] for n = 6 obtained the maximum absolute error of order 1.0e-08, 1.0e-13 and 1.0e-14 for n = 8, 16 and 32 respectively. [12] the exact solution was obtained for n = 2. Besides, [37] also found the exact solution with n = 5. [36] also obtained the exact solution for n = 3 and k = 0. Besides, [1] found the exact solution for n ≥ 2. Moreover, [21] obtained the absolute error of order 1.0e-07 for n = 20 and x = 1. Moreover, [28] found the maximum absolute error of order 1.0e-11 for n = 14. Therefore, our methods are better than [12, 37, 36], and are similar to the others that have the same accuracy.

The comparison with the exact solution is shown in Figure 4.

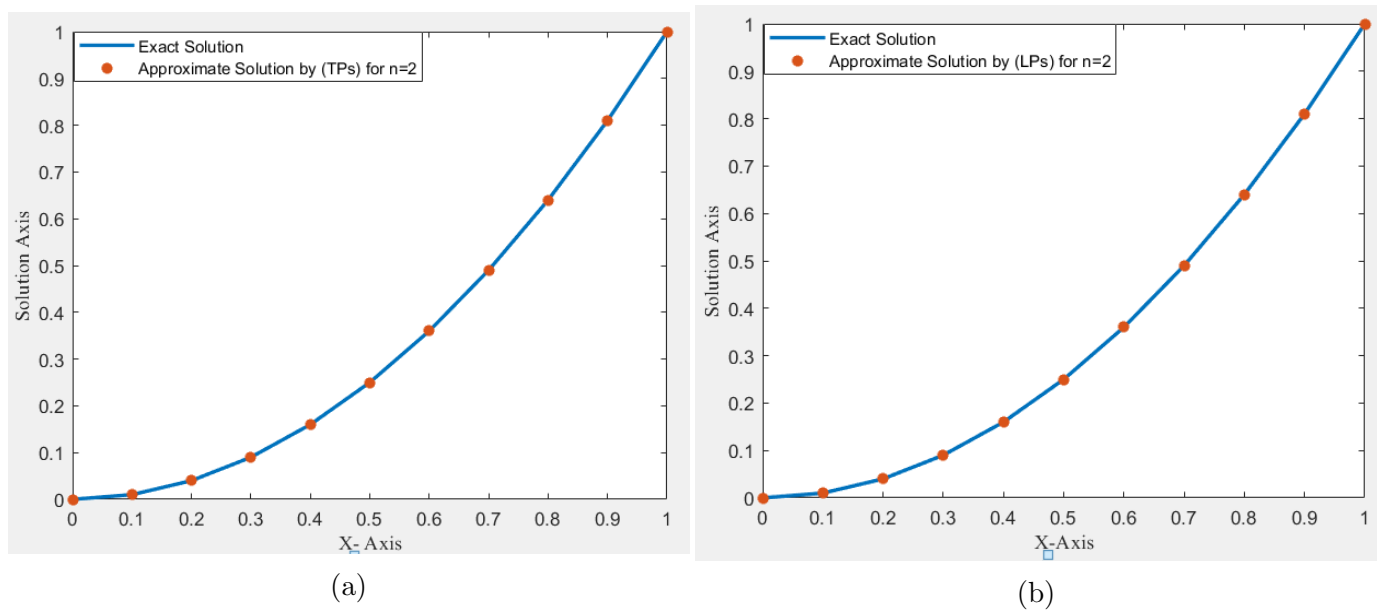


Figure 4: (a) For n=2, compare the (TPs) to the exact solution, (b) For n=2, compare the (LPs) to the exact solution

6. Conclusion

In this paper, Touchard and Laguerre polynomials for solving Abel integral equations of the first and second types were used. Four numerical examples have been solved by using the code written

in matlab2018b. Then, all of the solutions have been compared to those found in the literature. Obtained results show the efficiency and accuracy of the applied methods. As the result, we obtained the same exact solutions when the four numerical examples were solved.

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