Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1633-1640
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/IJNAA.2021.5292

Solution of Riccati matrix differential equation using new approach of variational iteration method

Fadhel S.Fadhel[a*], Huda Omran Altaie[b]

[a] *Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Iraq.*

[b] *Department of Mathematics, College of Education For Pure Scineces, Ibn Al-Haitham, University of Baghdad, Iraq.*

(Communicated by Madjid Eshaghi Gordji)

Abstract

To obtain the approximate solution to Riccati matrix differential equations, a new variational iteration approach was proposed, which is suggested to improve the accuracy and increase the convergence rate of the approximate solutons to the exact solution. This technique was found to give very accurate results in a few number of iterations. In this paper, the modified approaches was derived to give modified solutions of proposed and used and the convergence analysis to the exact solution of the derived sequence of approximate solutions is also stated and proved. Two examples were also solved, which shows the reliability and applicability of the proposed approach.

Keywords: Riccati matrix differential equation, Variational iteration method, Differential equation, He's method, Exact solution, Approximate solutions.

1. Introduction

A well-known "differential equation" is the Riccati differential equation (RMDE) that has a vast range of engineering and scientific applications [18]. This matrix equations is labeled as RMDE after an Italian aristocrat mathematician Count Jacopo Francesco Riccati (1676-1754), [4]. Various approaches can be used to solve RMDE with constant coefficients analytically. Recently, "Adomian's decomposition method (ADM)", "He's variational iteration method (VIM)", "homotopy perturbation method (HPM)" were suggested for solving quadratic RMDE and other types of such equations.

An approximate analytical VIM was first proposed and included by He [12, 13, 14, 21] and many authors have proven it to be powerful and effective method for solving several kinds of problems [7, 10, 22, 17]. The VIM [1, 2, 3, 5, 15, 7, 8] is currently widely utilized by academics to solve a wide range of linear and nonlinear problems. This method provides an effective approach for evaluating

*Corresponding author: Fadhel S.Fadhel

Email addresses: fsf@sc.nahrainuniv.edu.iq (Fadhel S.Fadhel[a*]), dr.Huda_hm2019@yahoo.com (Huda Omran Altaie[b])

Received: April 2021 *Accepted:* June 2021

analytic approximate solutions and carrying out numeric simulations for real-life applications [8, 9, 10, 11, 12]. The VIM, which is based on the use of certain variation, which is called the restricted variations, and a correction functional, has found widespread use in solving nonlinear differential equations, ordinary and partial [7, 8, 9, 10, 11, 12]. This approach do not require an apperance of small parameters in the differential equation and produces the solution, or its approximation, as an evaluation of iterated approximated sequence. For the new variational iteration approach used in this work, it is not required for nonlinear terms to be differentiable with respect to the dependent variable and its derivatives. Recently, [1, 2, 16, 20, 6] applied the differential method to solve RMDEs.

In this article, we modify the VIM to find the approximate numerical solution to RMDE, and two examples are solved as an illustration to demonstrate the accuracy of the suggested new approach.

2. Preliminary Concepts for Solving RMDEs Using VIM

For the purpose of introducing, new approach using the VIM by modifying the linear operator of the RMDE. For this objective, we must first recall some basic concepts related to RMDEs. When "A", "B" and "C" are n n matrices with real entries such that "A", "C" are symmetric, then the RMDE considered in this paper is assumed of the form [13, 14]:

$$P' + PB + B^T P - PAP = -C(t), \quad 0 \leq t \leq 1 \tag{1}$$

The derivation approach starts by introducing the correctional functional related to the RMDE using the VIM, which is for all $n = 0,\ 1, \dots$:

$$P_{n+1}(t) = P_n(t) + \int_a^t \lambda(t,s)[P_n'(s) + P_n(s)B + B^T P_n(s) - P_n(s)AP_n(s) + C]ds \tag{2}$$

where λ is the general Lagrange multiplier and for simplification purpose, equation (1) may be rewritten as [15]:

$$P'(t) + Y(t, P(t)) = 0 \tag{3}$$

where $Y(t, P(t)) = P(t)B + B^T P(t) - P(t)AP(t) + C$ and define the nonlinear operator $\mathcal{A}$ related to the RMDE as:

$$\mathcal{A}. = \frac{d}{dt}. + .B + B^T. - .A. \tag{4}$$

Now, decomposing the nonlinear operator given in equation (4) into linear and nonlinear parts namely L and N, respectively, where:

$$L. = \frac{d}{dt}. + .B + B^T. \tag{5}$$

$$N. = -.A. \tag{6}$$

The approximate sequence of solutions obtained by applying the VIM in relation to the nonlinear RMDE will then take the form:

$$LP(t) + NP(t) = -C(t) \tag{7}$$

where P is an unknown function which have to be determined.

Therefore using the VIM, the correction functional takes the form:

$$P_{n+1}(t) = P_n(t) + \int_0^t \lambda(t,s)[L(P_n(s)) + N(\widetilde{P}_n(s)) + C(s)]ds \tag{8}$$

where $\widetilde{P}$ is assumed here as a restricted variation, i.e., its first variation $\partial \widetilde{P}_n$ equals zero.

3. New VIM Formulation for Solving RMDEs

Now, the new approach will be introduced next by modifying the linear and nonlinear operators L and N, respectively. This has been made by introducing any linear operator, say L_1, as follows:

$$L(P(t)) + L_1(P(t)) - L_1(P(t)) + N(P(t)) = -C(t)$$

Hence, the modified correction functional may be constructed based on the following new linear and nonlinear operators of P, which are abbreviated as $\overline{L}$ and $\overline{N}$ and defined respectively by:

$$\begin{aligned} \overline{L}(P(t)) &= L(P(t)) + L_1(P(t)) \\ \overline{N}(P(t)) &= -L_1(\widetilde{P}(t)) + N(\widetilde{P}(t)) \end{aligned}$$

Thus, for all $n = 0,\ 1,\ 2, \ldots$; the new correction functional with considering $\overline{L}$ and $\overline{N}$ in mind then has the form:

$$\begin{aligned} P_{n+1}(t) &= P_n(t) + \int_0^t \lambda(t,s)[L(P_n(s)) + L_1(P_n(s)) - L_1(\widetilde{P}_n(s)) + N(\widetilde{P}_n(s)) + C(s)]ds \\ &= P_n(t) + \int_0^t \lambda(t,s)[\overline{L}(P_n(s)) + \overline{N}(\widetilde{P}_n(s)) + C(s)]ds \end{aligned} \tag{9}$$

For simplicity and application of this approach, one may take $L_1(P) = P$ and hence in order to find the general Lagrange multiplier λ, we take the first variation δ with respect to $P_n(t)$ and considering $\delta\widetilde{P}_n(t) = 0$ and $\delta C(t) = 0$, which will yields to:

$$\delta P_{n+1}(t) = \delta P_n(t) + \int_0^t \lambda(t,s)[(P_n'(s)) + P_n(s) + F(s, P_n(s))]ds \tag{10}$$

where:

$$F(s, P_n(s)) = P_n(s)B + B^T P_n(s) - P_n(s) - P_n(s)AP_n(s) + C$$

which may be considered as the nonlinear term, i.e., F may be rewritten in terms of the restricted variation $\widetilde{P}_n(s)$ and since $\delta\widetilde{P}_n(s) = 0$. Therefore $\delta F(s, \widetilde{P}_n(s)) = 0$. Thus equations (10) will be reduced to:

$$\begin{aligned} \delta P_{n+1}(t) &= \delta P_n(t) + \int_0^t \lambda(t,s)\delta P_n'(s)ds + \int_0^t \lambda(t,s)\delta P_n(s)ds + \int_0^t \lambda(t,s)\delta F(s, \widetilde{P}_n(s))ds \\ &= \delta P_n(t) + \int_0^t \lambda(t,s)\delta P_n'(s)ds + \int_0^t \lambda(t,s)\delta P_n(s)ds \end{aligned} \tag{11}$$

Evaluating the first integral of equation (11) using integration by parts, we get:

$$\delta P_{n+1}(t) = [1 + \lambda(t,s)]P_n(s)|_{s=t} + \int_0^t [1 - \lambda'(t,s)]\delta P_n(s)ds \tag{12}$$

Using calculus of variation's theory, the first variation $\delta P_{n+1}(t) = 0$, the Euler-Lagrange necessary condition for extremizing functional (12), we get as a result the following differential equation:

$$1 + \lambda'(t,s) = 0 \tag{13}$$

with the natural boundary condition:

$$1 + \lambda(t,s)|_{s=t} = 0 \tag{14}$$

Equations (13) and (14) represent an initial value problem which may be solved to find λ and it is easily found to be $\lambda(t,s) = s - t - 1$. Therefore, the final iterative solution equation is found after substituting λ back into equation (2), which is:

$$P_{n+1}(t) = P_n(t) + \int_0^t \lambda(s-t-1)[P_n'(s) + P_n(s)B + B^T P_n(s) - P_n(s)AP_n(s) + C]ds \tag{15}$$

4. Convergence Analysis

Convergence to the problem exact solution of the sequence of approximate solutions obtained from equation (15) can be proved, as it is given in the next theorem:

Theorem 4.1. *Let respectively $P,\ P_n \in \mathcal{C}^1[0,1]$, for all $n = 0,\ 1, \ldots$; to be the exact and approximate solutions of equation (1). Suppose that $e_n(t) = P_n(t) - P(t)$ for all $t \in [0,1]$ and the nonlinear operator $NP = -PAP$ satisfies Lipschitz constant condition with constant ℓ, such that $\ell < 2\|B\|$, then the sequence $\{P_n(t)\}$, $n = 0,\ 1, \ldots$ of approximate solutions converge to the exact solution $P(t)$, for all $t \in [0,1]$.*

Proof . Since upon using the VIM, equation (15) give the approximate-numerical solution of equation (1) and if P is the exact solution, then P satisfies also the VIM. Hence, the solution, we have:

$$P(t) = P(t) + \int_0^t (s-t-1)[P'(s) + P(s)B + B^T P(s) - P(s)AP(s) + C]ds \tag{16}$$

Therefore subtracting equation (15) from equation (16) give:

$$\begin{aligned} P_{n+1}(t) - P_n(t) &= P_n(t) - P(t) + \int_0^t (s-t-1)[P_n'(s) - P'(s) + (P_n(s) - P(s)) \cdot B \\ &\quad + B^T(P_n(s) - P(s)) - P_n(s)AP_n(s) + P(s)AP(s))]ds \end{aligned} \tag{17}$$

and since the error function e_n as defined by $e_n(t) = P_n(t) - P(t)$, then equation (17) may be rewritten in terms of e_n as:

$$\begin{aligned} e_{n+1}(t) &= e_n(t) + \int_0^t (s-t-1)e_n'(s)ds + \int_0^t (s-t-1)e_n(s)Bds \\ &\quad + \int_0^t (s-t-1)B^T e_n(s)ds - \int_0^t (s-t-1)[P_n(s)AP_n(s) - P(s)AP(s)]ds \end{aligned}$$

Since $t, s \in [0,1]$, hence the supremum value of $s - t - 1 \leq 1$, and therefore:

$$\begin{aligned} e_{n+1}(t) &\leq e_n(t) + \int_0^t e_n'(s)ds + \int_0^t e_n(s)Bds + \int_0^t B^T e_n(s)ds - \int_0^t P_n(s)AP_n(s)ds \\ &\quad - \int_0^t P(s)AP(s)ds = e_n(t) - e_n(t) + e_n(0) + \int_0^t e_n(s)Bds + \int_0^t B^T e_n(s)ds \\ &\quad - \int_0^t [P_n(s)AP_n(s) - P(s)AP(s)]ds \end{aligned} \tag{18}$$

It is clear that from the initial condition, we have $e_n(0) = P_n(0) - P(0) = 0$ and upon taking the supremum norm of inequality (18), we get:

$$\begin{aligned} \|e_{n+1}(t)\| &\leq \int_0^t \|e_n(s)\|\ \|B\|ds + \int_0^t \|B^T\|\ \|e_n(s)\|ds + \int_0^t \|P_n(s)AP_n(s) - P(s)AP(s)\|ds \\ &\leq \|B\| \int_0^t \|e_n(s)\|ds + \|B^T\| \int_0^t \|e_n(s)\|ds + \ell \int_0^t \|P_n(s) - P(s)\|ds \end{aligned}$$

and so:

$$\begin{aligned}\|e_{n+1}(t)\| &\leq \|B\|\int_0^t \|e_n(s)\|ds + \|B^T\|\int_0^t \|e_n(s)\|ds + \ell\int_0^t \|e_n(s)\|ds \\ &= (2\|B\|+\ell)\int_0^t \|e_n(s)\|ds\end{aligned}$$

Now, based on mathematical induction, when $n=0$, implies:

$$\begin{aligned}\|e_1(t)\| &\leq (2\|B\|+\ell)\int_0^t \|e_0(s)\|ds \\ &\leq (2\|B\|+\ell)\sup|e_0|\int_0^t ds \\ &\leq (2\|B\|+\ell)\cdot t\cdot \sup|e_0(t)|\end{aligned}$$

While when $n=1$, implies to:

$$\begin{aligned}\|e_2(t)\| &\leq (2\|B\|+\ell)\int_0^t \|e_1(s)\|ds \\ &\leq \frac{(2\|B\|+\ell)^2}{2}\, t^2\ \sup|e_0|\int_0^t ds\end{aligned}$$

So on, similarly for any natural number n, the following inequality is derived:

$$\begin{aligned}\|e_{n+1}(t)\| &\leq (2\|B\|+\ell)\int_0^t \|e_n(s)\|ds \\ &\leq \frac{(2\|B\|+\ell)^n}{n!}\, t^n\ \sup|e_0|\int_0^t ds\end{aligned}$$

Since $l<2\|B\|$, then $\dfrac{(2\|B\|+\ell)^n}{n!}\to 0$ as $n\to\infty$ and therefore, $\|e_n\|\to 0$ as $n\to\infty$, i.e., $P_n\to P$ as $n\to\infty$, for all $t\in[0,1]$. □

5. Numerical Results

Here, we will introduce a novel approximated method based on a new style of variational iteration formula used in previous section.

Example 5.1. *Consider the scalar RDE:*

$$y'(t)-1+y^2(t)-t^2=0, \quad y=1, \quad t\in[0,1] \tag{19}$$

with the exact solution $y(t)=t+\dfrac{e^{-t^2}}{1+\int_0^t e^{-u^2}du}$.

The new technique of VIM for equation (19) upon applying equation (15) is:

$$y_{n+1}(t)=y_n(t)+\int_0^t (s-t-1)[y_n'(s)-1+y_n^2(s)-s^2]ds$$

Starting with $y_0(t) = 1$, then:

$$\begin{aligned} y_1(t) &= y_0(t) + \int_0^t (s-t-1)[y_0'(s) - 1 + y_0^2(s) - s^2]ds = 1 + \frac{t^3}{3} + \frac{t^4}{12} \\ y_2(t) &= y_1(t) + \int_0^t (s-t-1)[y_1'(s) - 1 + y_1^2(s) - s^2]ds \\ &= \frac{t^3}{3} + \frac{t^9}{648} - \frac{t^8}{112} - \frac{t^7}{63} - \frac{t^6}{180} - \frac{t^5}{12} - \frac{t^4}{6} - \frac{t^{10}}{12960} + 1 \end{aligned}$$

By the same technique, we obtain $y_3(t),\ y_4(t), \ldots,\ y_{10}(t)$. Table (5.1) shows the last two approximate solutions absolute error with the exact solution.

Table (5.1)
The absolute error of the 9^{th} and 10^{th} approximate solutions

t	Exact solution	The absolute error with y_9	The absolute error with y_{10}
0.1	1.00031731	2.11×10^{-16}	2.11045×10^{-16}
0.2	1.002419825	3.39×10^{-13}	1.68754×10^{-15}
0.3	1.007794588	4.8642×10^{-11}	3.39817×10^{-11}
0.4	1.017650879	1.6154×10^{-9}	1.51098×10^{-10}
0.5	1.032957576	2.5009×10^{-8}	2.93965×10^{-9}
0.6	1.05446681	2.4036×10^{-7}	3.4122×10^{-7}
0.7	1.082727481	1.6713×10^{-6}	2.78951×10^{-7}
0.8	1.118092545	9.2128×10^{-6}	1.77355×10^{-6}
0.9	1.160723973	4.2708×10^{-5}	9.34812×10^{-4}
1	1.2106	0.00017329	4.26587×10^{-5}

Example 5.2. *Let a system of RMDE with variable coefficients be given by:*

$$P' + P\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^T P - P\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} P = -\begin{pmatrix} 1-4t & 2t+2t^2 \\ 4t^2 & t^2-3 \end{pmatrix} C(t) \tag{20}$$

where $0 \le t \le 1$*. If* $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$*, then applying the new VIM given by equation (15) we get the results presented in Tables (5.2)-(5.5) for* $P_{11},\ P_{12},\ P_{21}$ *and* P_{22}*, and their absolute errors in comparison with the corresponding exact solution* $P = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$

Table (5.2)
Results of P_{11} for the 9^{th} approximation

t	Exact solution	P_{11}	Absolute error
0.1	0.1	0.1	3.16543×10^{-17}
0.2	0.2	0.143	6.53436×10^{-13}
0.3	0.3	0.28888859769	3.76514×10^{-10}
0.4	0.4	0.38888987632	5.78643×10^{-9}
0.5	0.5	0.6999976538	6.20612×10^{-8}
0.6	0.6	0.71000031623	2.06341×10^{-7}
0.7	0.7	0.70000032456	2.01127×10^{-6}
0.8	0.8	0.60000087635	2.154063×10^{-5}
0.9	0.9	0.78472213613	2.24651×10^{-4}

Table (5.3)
Results of P_{12} for the 9^{th} approximation

t	Exact solution	P_{12}	Absolute error
0.1	1	1	1.87654×10^{-16}
0.2	1	1	$4.85439\times10_{14}$
0.3	1	1	2.3843×10^{-11}
0.4	1	0.9999999876	5.1493×10^{-10}
0.5	1	0.9999997854	2.01233×10^{-8}
0.6	1	0.8877777785	8.53473×10^{-7}
0.7	1	0.9999999876	2.00065×10^{-6}
0.8	1	0.8889994532	2.03221×10^{-6}
0.9	1	0.9999913514	1.03276×10^{-4}

Table (5.4)
Results of P_{21} for the 9^{th} approximation

t	Exact solution	P_{21}	Absolute error
0.1	0	1.321855×10^{-16}	1.431874×10^{-15}
0.2	0	5.243575×10^{-14}	3.458554×10^{-13}
0.3	0	8.234127×10^{-12}	7.226473×10^{-11}
0.4	0	1.963543×10^{-9}	1.763236×10^{-9}
0.5	0	9.245987×10^{-9}	9.438406×10^{-9}
0.6	0	7.769893×10^{-8}	7.154987×10^{-8}
0.7	0	0.000009	1.005464×10^{-7}
0.8	0	0.000046	2.161240×10^{-6}
0.9	0	0.000076	2.107563×10^{-5}

Table (5.5)
Results of P_{22} for the 9^{th} approximation

t	Exact solution	P_{22}	Absolute error
0.1	0.1	0.1	4.07843×10^{-16}
0.2	0.2	0.221	7.56370×10^{-13}
0.3	0.3	0.2800000651	1.05543×10^{-11}
0.4	0.4	0.3120000009	4.94422×10^{-9}
0.5	0.5	0.4900000879	7.87224×10^{-9}
0.6	0.6	0.5400009863	2.01401×10^{-8}
0.7	0.7	0.4897276832	1.13569×10^{-7}
0.8	0.8	0.6889975438	3.12137×10^{-7}
0.9	0.9	0.700150863	1.1456×10^{-5}

6. Conclusion:

In this paper, solution of RDE's using a new technique of VIM were established by deriving the approximate solutions and prove its convergence o the exact solution. The results convergence to the exact solution has been shown to be quite fast and the acuracy has also been enhanced. Two numerical examples are considered and solved using the new VIM in which the obtained results are more accurate than those obtained in [5].

References

[1] T. A. Abassy, *Modified variational iteration method (nonlinear homogeneousinitial value problem),* Computer and Mathematics with Application, 59(2) (2010) 912-918.

[2] R. Abazari, *Solution of Riccati types matrix differential equations using matrix differential transform method,* J. Appl. Math. Informatics, 27(5-6) (2009) 1133-1143.

[3] S. Abbasbandy, *A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials,* Journal of Computational and Applied Mathematics, 207(1) (2007) 59-63.

[4] S. Abbasbandy, *Homotopy perturbation method for quadratic Riccati differential equationand comparison with Adomians decomposition method,* Applied Mathematics and Computation, 172 (2006) 485-490.

[5] K. H. Al-jizani, N. A. Ahmad and F. S. Fadhel, *Variational Iteration Method for Solving Riccati Matrix Differential Equations,* Indonesian Journal of Electrical Engineering and Computer Science, 5(3) (2017) 673-683.

[6] H. Aminikhah, *Approximate analytical solution for quadraticRiccati differential equation,* Iran J. Numer. Anal. Optim., 3(2) (2013) 21-31.

[7] B. Batiha, *A new efficient method for solving quadratic Riccati differential equation,* International Journal of Applied and Mathematical Research, 4(1) (2015) 24-29.

[8] R. Bellman and K. Cook, *Differential difference equations,* Academic Press, Inc., New York, 1963.

[9] E. Davison and M M.aki, *The numerical solution of the matrix Ricattidifferential equation,* 18(1) (1973) 71-73.

[10] F. Geng, A modified variational iteration method for solving Riccati differential equations, Computers and Mathematics with Applications, 60(7) (2010) 1868-1872.

[11] A.Ghorbani and S. Momani, *An effective variation iteration algorithm for solving Riccati differential equations*, Applied Mathematics Letters, 23(8) (2010) 922-927.

[12] J. H. He, *Variational iteration method for delay differential equations,* Communications in Nonlinear Science and Numerical Simulation, 2(4) (1997)235-236.

[13] J. H. He, *Variational iteration method - a kind of non-linear analytical technique: some examples,* International Journal of Non-Linear Mechanics, 34(4) 1999,pp.699-708.

[14] J. H.He, *Variational iteration method; some recent Results and new interpretations,* Journal of Computational and Applied Mathematics, 207(1) (2007) 3-17.

[15] K. H. Mohammedali, N. A. Ahmad and F. S. Fadhel, *He's Variational Iteration Method for Solving Riccati Matrix Delay Differential Equations of Variable Coefficients,* The 4th International Conference on Mathematical Sciences. AIP Conference Proceedings, 1830, 020029, (2017) 020029-1 020029-10.

[16] S. Momani and S. Abuasad, *Application of He's variational iteration method to Helmholtz equation,* Chaos Solitons Fractals, 27(5) (2006) 1119-1123.

[17] S. Riaz, M. Rafiq and O. Ahmad, *Non standard finite difference method for quadratic Riccati differential equation,* Pakistan, Punjab University J. Math., 47(2), 2015.

[18] M. R. Scott, *Invariant imbedding and its applications to ordinary differential equations,* Addison-Wesley, 1973.

[19] Sh. Sharma and A. J. Obaid, *Mathematical modelling, analysis and design of fuzzy logic controller for the control of ventilation systems using MATLAB fuzzy logic toolbox,* Journal of Interdisciplinary Mathematics, 23(4) (2020) 843-849, DOI: 10.1080/09720502.2020.1727611.

[20] A. M. Wazwaz, *The variational iteration method for exact solutions of Laplace equation,* Physics letter A, 363(4) (2007) 260-262.

[21] A. M. Wazwaz, *The variational iteration method for analytic treatment of linear and nonlinear ODEs,* Applied Mathematics and Computation, 212(1) (2009) 120-134.

[22] A. M. Wazwaz, *The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations,* International Journal of Computer Mathematics, 87(5) (2010) 1131-1141.