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On the dynamical behavior of an eco-epidemiological model

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Abstract

The aim of this article is to study the dynamical behavior of an eco-epidemiological model. A prey-predator model comprising infectious disease in prey species and stage structure in predator species is suggested and studied. Presumed that the prey species growing logistically in the absence of predator and the ferocity process happened by Lotka-Volterra functional response. The existence, uniqueness, and boundedness of the solution of the model are investigated. The stability constraints of all equilibrium points are determined. The constraints of persistence of the model are established. The local bifurcation near every equilibrium point is analyzed. The global dynamics of the model are investigated numerically and confronted with the obtained outcomes.

Keywords: Prey-predator, Disease, Stage-structure, Stability, Bifurcation

1. Introduction

The eco-epidemiological model is important in both applied mathematics as well as theoretical ecology. May and Anderson [13], were the former who connect epidemiology and ecology and suggested a prey-predator model with infectious diseases in the prey species. Infectious diseases play a role in eco-epidemiological models. As an outcome, several mathematical models have been advanced. Most papers have deal with prey-predator models with the disease in the prey see ([10], [17], [18], [1]). Further, in present years, eco-epidemiological systems with the disease in predator have become the most interesting part of research among all mathematical models ([15], [6], [2], [14]). On the other hand, the effect of disease in both prey and predator species is considered too, see for example [9, 3, 4, 11].

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In the naturalist world, the species have a lifetime history that contains at least two stages immature and mature. All stages have different behavioral properties. Many researcher have studied of stage-structured models[16, 20]. Naji and Majeed [16] proposed a time-delayed prey-predator model involving stage-structure in the predator. Furthermore, Savitri and Abadi [19] proposed and studied the dynamical behaviors of a prey-predator model with stage-structure for prey.

In this article, the dynamical behavior of a prey-predator model with infectious disease in prey and stage-structured predator is proposed. Presume that the infectious disease for prey of type SI, and the predator splits into two stages immature and mature. The individuals in the immature stage cannot reproduce and hunt, they rely on their parents (mature). The stability analysis and persistence of the suggested model are investigated in part (3). In part (4), we study the bifurcation analysis are interested. Numerical analysis outcomes of the model are introduced in part (5). Finally, in part (6) we ending with a concise discussion and conclusion.

2. Formulation of Mathematical Model

The dynamics of the prey-predator model involving disease and stage-structure in the prey and predator, respectively is formulated mathematically. It is assumed that $X(T)$ represents the population density of prey at time T and divides into two types: $S(T)$ and $I(T)$, where $S(T)$ represents the susceptible prey at time T, $I(T)$ is the infected prey at time T. Let $Y(T)$ represents the population density of predator at time T, which split into two stages: mature with density at time T represented by $Y_1(T)$ and immature with density at time T represented by $Y_2(T)$. So, the formulation of the model mathematically depended on the following hypotheses.

1. The prey growth logistically with intrinsic growth rate $r > 0$ and carrying capacity $K > 0$.
2. the susceptible prey $S(T)$ becomes infected by contact with infected prey according to infection rate $\alpha > 0$. Moreover, it is assumed that the disease causes death with a disease death rate denoted by $D_1 > 0$.
3. It's assumed that the immature predator grows exponentially depending on their parents with a growth rate $\gamma > 0$, whereas portion from it grow up to become mature with grown−up rate $\mu > 0$. Moreover both the populations facing natural death with death rates $D_2 > 0$ and $D_3 > 0$ for $Y_1(T)$ and $Y_2(T)$ respectively.
4. It is assumed that $Y_1(T)$ feeding on $S(T)$ and $I(T)$ using Lotka-Volterra type with maximum attack rates $\beta_1 > 0$ and $\beta_2 > 0$ respectively, and conversion rates $e_1, e_2 \in (0, 1)$.

By the suppositions, the model is

$$\begin{aligned}
\frac{\mathrm{dS}}{\mathrm{dT}} &= rS\left(1-\frac{S+I}{K}\right) - \alpha SI - \beta_1 SY_1,\\
\frac{\mathrm{dI}}{\mathrm{dT}} &= \alpha SI - \beta_2 IY_1 - D_1 I,\\
\frac{dY_1}{\mathrm{dT}} &= e_1\beta_1 SY_1 + e_2\beta_2 IY_1 + \mu Y_2 - D_2 Y_1,\\
\frac{dY_2}{\mathrm{dT}} &= \gamma Y_1 - \mu Y_2 - D_3 Y_2.
\end{aligned} \tag{2.1}$$

The state space $\mathbb{R}^4_+ = \{(S, I, Y_1, Y_2) \in \mathbb{R}^4 : S \geq 0, I \geq 0, Y_1 \geq 0, Y_2 \geq 0\}$.

Now, the system (2.1) contains 12 parameters. we simplify system (2.1) and reduced to 9 using the dimensionless as following:

$$\begin{gathered} t=\mathrm{rT}\ ,\ S=sK\ ,\ I=iK\ ,\ Y_1=\frac{\mathrm{r}\ y_1}{\beta_1}\ ,\ Y_2=\frac{r^2\ y_2}{\mu\ \beta_1},\ \alpha_1=\frac{\alpha\ \mathrm{K}}{r}\ ,\ \beta=\frac{\beta_2}{\beta_1}, \\ d_1=\frac{D_1}{r}\ ,\theta_1=\frac{e_1\ \beta_1 K}{r},\ \theta_2=\frac{e_2\ \beta_2 K}{r},\ d_2=\frac{D_2}{r},\ \gamma_1=\frac{\gamma\ \mu}{r^{\ 2}}\ ,\ \sigma=\frac{\mu}{r},\ d_3=\frac{D_3}{r} \end{gathered} \tag{2.2}$$

Now, the system (2.1) reduces to the dimensionless system as following:

$$\begin{aligned} \frac{\mathrm{ds}}{\mathrm{dt}} &= s\left(1-(s+i)\right)-\alpha_1 \mathrm{si}-sy_1=f_1\left(s,i,y_1,y_2\right), \\ \frac{\mathrm{di}}{\mathrm{dt}} &= \alpha_1 \mathrm{si}-\beta i y_1-d_1 i=f_2\left(s,i,y_1,y_2\right), \\ \frac{dy_1}{\mathrm{dt}} &= \theta_1 s y_1+\theta_2 i y_1+y_2-d_2 y_1=f_3\left(s,i,y_1,y_2\right), \\ \frac{dy_2}{\mathrm{dt}} &= \gamma_1 y_1-\sigma y_2-d_3 y_2=f_4\left(s,i,y_1,y_2\right). \end{aligned} \tag{2.3}$$

Theorem 2.1. *Every solutions of system (2.3) initiating in $\mathbb{R}_+^4$ are bounded if*

$$d_2>\gamma_1. \tag{2.4}$$

Proof . *Let $\frac{ds}{dt}\le s\left(1-s\right)$.*
By using the comparison theory,

$$s\left(t\right)\le\frac{1}{1+ce^{-t}}\qquad\forall\ t\ge 0\ \ ,\ \ s\left(0\right)=s_0\ \textit{and}\ c=\frac{1}{s_0}-1.$$

Hence, $s\left(t\right)\le 1$, as $t\longrightarrow\infty$. Let

$$w\left(t\right)=s\left(t\right)+i\left(t\right)+y_1\left(t\right)+y_2\left(t\right) \tag{2.5}$$

Now,

$$\frac{dw}{dt}+\rho w\le 2\ ,\quad \textit{where}\quad \rho=min\left\{1,d_1,d_2-\gamma_1,(\sigma+d_3)-1\right\}$$

Grownwall lemma [7] applied in above inequality and get

$$w\left(t\right)\le w_0 e^{-\rho t}+\frac{2}{\rho}\left(1-e^{-\rho t}\right).$$

Hence, as $t\to\infty$, every solutions that initiate in $\mathbb{R}_+^4$ confined in region Ω,

$$\Omega=\left\{(s,i,y_1,y_2)\in\mathbb{R}_+^4\ ;w\le\frac{2}{\rho}\right\}.$$

□

3. Local Stability, Global stability and Persistence

in system (2.3), every equilibrium points (EP) are studied as follows

- The vanishing equilibrium point (VEP) , $\tau_0 = (0,0,0,0)$ exists.
- The axial equilibrium point (AEP), $\tau_1 = (1,0,0,0)$ exists.
- The predator free equilibrium point $(PFEP), \tau_2 = (\overline{s}, \overline{i}, 0, 0) = \left(\frac{d_1}{\alpha_1},\ \left(\frac{\alpha_1 - d_1}{\alpha_1(1+\alpha_1)}\right),\ 0, 0\right)$, exists iff the next condition holds.

$$d_1 < \alpha_1 \tag{3.1}$$

- The disease free equilibrium point $(DFEP)$,

$$\tau_3 = (\hat{s}, 0, \hat{y}_1, \hat{y}_2) = \left(\frac{d_2\left(\sigma + d_3 - \gamma_1\right)}{\theta_1\left(\sigma + d_3\right)}, 0, \frac{\left(\sigma + d_3\right)\left(\theta_1 - d_2\right) + \gamma_1}{\theta_1\left(\sigma + d_3\right)}, \frac{\gamma_1\left(\left(\sigma + d_3\right)\left(\theta_1 - d_2\right) + \gamma_1\right)}{\theta_1\left(\sigma + d_3\right)^2}\right),$$

exists iff the next condition holds

$$\frac{\gamma_1}{(\sigma + d_3)} < d_2 < \frac{\theta_1\left(\sigma + d_3\right) + \gamma_1}{(\sigma + d_3)}. \tag{3.2}$$

- The positive equilibrium point (PEP), $\tau_4 = (s^*, i^*, y_1^*, y_2^*)$, where

$$i^* = \frac{(\beta + d_1) - s^*\left(\beta + \alpha_1\right)}{\beta\left(1 + \alpha_1\right)},\ y_1^* = \frac{\alpha_1 s^* - d_1}{\beta},\ y_2^* = \frac{\gamma_1\left(\alpha_1 s^* - d_1\right)}{\beta\left(\sigma + d_3\right)}, \tag{3.3}$$

while, the following polynomial equation of second order has a positive root s^* ,

$$H_1 s^{*2} + H_2 s^* + H_3 = 0 \tag{3.4}$$

where

$$\begin{aligned}
H_1 &= \alpha_1\left(\sigma + d_3\right)\left(\theta_1\beta\left(1 + \alpha_1\right) - \theta_2\left(\beta + \alpha_1\right)\right), \\
H_2 &= \left(\theta_2\left(\sigma + d_3\right)\left(\alpha_1\left(\beta + d_1\right)\right) + \gamma_1\alpha_1\beta\left(1 + \alpha_1\right)\right) - \beta\left(1 + \alpha_1\right)\left(\sigma + d_3\right)\left(\theta_1 d_1 - d_2\alpha_1\right), \\
H_3 &= d_1 d_2\beta\left(1 + \alpha_1\right)\left(\sigma + d_3\right) - d_1\left(\theta_2\left(\beta + d_1\right)\left(\sigma + d_3\right) + \gamma_1\beta\left(1 + \alpha_1\right)\right)
\end{aligned}$$

Now, τ_4 exists iff the following condition holds

$$\overline{s} < s^* < \frac{\beta + d_1}{\beta + \alpha_1} \tag{3.5}$$

together with the following sets of conditions

$$\left.\begin{array}{c} H_1 < 0 \text{ and } H_3 > 0 \\ \text{or} \\ H_1 > 0 \text{ and } H_3 < 0 \end{array}\right\} \tag{3.6}$$

The local dynamical behaviors are carried out by calculating $J(\tau_i)$, $i = 0,1,2,3,4$ and then computing the eigenvalues, which specify the stability type of each points.

$$J(\tau_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -d_1 & 0 & 0 \\ 0 & 0 & -d_2 & 1 \\ 0 & 0 & \gamma_1 & -(\sigma + d_3) \end{bmatrix} \tag{3.7}$$

Now, $J(\tau_0)$ has a positive eigenvalue $\lambda_{01} = 1 > 0$. Then, the VEP is a saddle point.

$$J(\tau_1) = \begin{bmatrix} -1 & -(1+\alpha_1) & -1 & 0 \\ 0 & \alpha_1 - d_1 & 0 & 0 \\ 0 & 0 & \theta_1 - d_2 & 1 \\ 0 & 0 & \gamma_1 & -(\sigma + d_3) \end{bmatrix} \tag{3.8}$$

The eigenvalues of $J(\tau_1)$ are computed by

$$\lambda_{11} = -1 < 0 \tag{3.9a}$$
$$\lambda_{12} = \alpha_1 - d_1 \tag{3.9b}$$
$$\lambda_{13} + \lambda_{14} = (\theta_1 - d_2) - (\sigma + d_3) \tag{3.9c}$$
$$\lambda_{13}.\lambda_{14} = -(\theta_1(\sigma + d_3) + \gamma_1) + d_2(\sigma + d_3) \tag{3.9d}$$

Accordingly, all the eigenvalues are negative provided that

$$\alpha_1 < d_1 \tag{3.9e}$$
$$\theta_1 + \frac{\gamma_1}{(\sigma + d_3)} < d_2 \tag{3.9f}$$

Clear that, from constraint (3.9e), the AEP is locally asymptotically stable (LAS) iff the PFEP is not exist.

$$J(\tau_2) = \begin{bmatrix} 1 - 2\bar{s} - (1+\alpha_1)\bar{i} & -(1+\alpha_1)\bar{s} & -\bar{s} & 0 \\ \alpha_1\bar{i} & \alpha_1\bar{s} - d_1 & -\beta\bar{i} & 0 \\ 0 & 0 & \theta_1\bar{s} + \theta_2\bar{i} - d_2 & 1 \\ 0 & 0 & \gamma_1 & -(\sigma + d_3) \end{bmatrix} \tag{3.10}$$

The eigenvalues of $J(\tau_2)$ are computed by

$$\lambda_{21} + \lambda_{22} = -\frac{d_1}{\alpha_1} < 0 \tag{3.11a}$$
$$\lambda_{21}.\lambda_{22} = \frac{d_1}{\alpha_1}(\alpha_1 - d_1) > 0 \tag{3.11b}$$
$$\lambda_{23} + \lambda_{24} = \frac{\theta_1 d_1 \ (1+\alpha_1) + \theta_2(\alpha_1 - d_1)}{\alpha_1 \ (1+\alpha_1)} - (d_2 + (\sigma + d_3)) \tag{3.11c}$$
$$\lambda_{23}.\lambda_{24} = d_2(\sigma + d_3) - \left((\sigma + d_3)\left(\frac{\theta_1 d_1 \ (1+\alpha_1) + \theta_2(\alpha_1 - d_1)}{\alpha_1 \ (1+\alpha_1)}\right) + \gamma_1\right) \tag{3.11d}$$

So, the eigenvalues in the s - direction and i - direction, λ_{21} and λ_{22} are negative. Whilst, the eigenvalues in the y_1 - direction and y_2 - direction, λ_{23} and λ_{24} are negative. Now the PFEP is LAS provided that condition (6) in addition to the following condition hold.

(3.11e)

$$\left(\frac{\theta_1 d_1 \ (1+\alpha_1) + \theta_2(\alpha_1 - d_1)}{\alpha_1 \ (1+\alpha_1)}\right) + \frac{\gamma_1}{(\sigma + d_3)} < d_2 \tag{3.11f}$$

$$J(\tau_3) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix} \tag{3.12a}$$

where

$$\begin{aligned} & b_{11} = 1 - 2\hat{s} - \hat{y}_1, \quad b_{12} = -(1+\alpha_1)\hat{s}, \quad b_{13} = -\hat{s}, \quad b_{14} = 0. \\ & b_{21} = 0, \quad b_{22} = \alpha_1 \hat{s} - \beta \hat{y}_1 - d_1, \quad b_{23} = 0, \quad b_{24} = 0. \\ & b_{31} = \theta_1 \hat{y}_1, \quad b_{32} = \theta_2 \hat{y}_1, \quad b_{33} = \theta_1 \hat{s} - d_2, \quad b_{34} = 1. \\ & b_{41} = 0, \quad b_{42} = 0, \quad b_{43} = \gamma_1, \quad b_{44} = -(\sigma + d_3). \end{aligned}$$

Clearly, one of the eigenvalues of $J(\tau_3)$ is $\lambda_{32} = \alpha_1 \hat{s} - \beta \hat{y}_1 - d_1$, will be negative if the following condition holds:

$$\alpha_1 \hat{s} < \beta \hat{y}_1 + d_1 \tag{3.12b}$$

However, the other eigenvalues of $J(\tau_3)$ are roots of following equation:

$$\left(\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3\right) = 0, \tag{3.12c}$$

where

$$\begin{aligned} & A_1 = -(b_{11} + b_{33} + b_{44}). \\ & A_2 = (b_{11}b_{33} - b_{13}b_{31}) + (b_{33}b_{44} - b_{34}b_{43}) + b_{11}b_{44}. \\ & A_3 = -(b_{11}(b_{33}b_{44} - b_{34}b_{43}) - b_{13}b_{31}b_{44}). \end{aligned}$$

while

$$\begin{aligned} \Delta = A_1 \, A_2 - A_3 = & -(b_{11}b_{33} - b_{13}b_{31})(b_{11} + b_{33} + b_{44}) - b_{11}b_{44}(b_{11} + b_{44}) \\ & - (b_{33}b_{44} - b_{34}b_{43})(b_{33} + b_{44}) - b_{44}(b_{11}b_{33} + b_{13}b_{31}) \end{aligned}$$

By Routh-Hawirtiz Criterion [12], $J(\tau_3)$ have negative real parts provided that condition (3.12b) with $A_1 > 0, \quad A_3 > 0$ and $\Delta > 0$, as follow

$$1 < 2\hat{s} + \hat{y}_1 \tag{3.12d}$$

$$\theta_1 \hat{s} < d_2 \tag{3.12e}$$

$$\frac{\theta_1 \hat{s}(\sigma + d_3) + \gamma_1}{(\sigma + d_3)} < d_2 < \frac{\theta_1 \hat{s}(1 - 2\hat{s} - 2\hat{y}_1)}{1 - 2\hat{s} - 2\hat{y}_1} \tag{3.12f}$$

Moreover, the Jacobian matrix at PEP is

$$J(\tau_4) = [a_{\mathrm{ij}}]_{4\times 4}. \tag{3.13}$$

Here

$$a_{11} = 1 - 2s^* - (1+\alpha_1)\, i^* - y_1^*, \quad a_{12} = -(1+\alpha_1)\, s^*, \quad a_{13} = -s^*, \quad a_{14} = 0,$$
$$a_{21} = \alpha_1 i^*, \quad a_{22} = \alpha_1 s^* - \beta y_1^* - d_1, \quad a_{23} = -\beta i^*, \quad a_{24} = 0,$$
$$a_{31} = \theta_1 y_1^*, \quad a_{32} = \ \theta_2 y_1^*, \quad a_{33} = \theta_1 s^* + \theta_2 i^* - d_2, \quad a_{34} = 1,$$
$$a_{41} = 0\ , \quad a_{42} = 0\ , \quad a_{43} = \gamma_1\ , \quad a_{44} = -(\sigma + d_3)\, .$$

By Gershgorin theorem [8], if the next conditions hold

$$1 < 2s^* + (1+\alpha_1)\, i^* + y_1^* \tag{3.14a}$$
$$\alpha_1 s^* < \beta y_1^* + d_1 \tag{3.14b}$$
$$\theta_1 s^* + \theta_2 i^* < d_2 \tag{3.14c}$$
$$\sigma + d_3 > 1 \tag{3.14d}$$
$$\frac{1 - ((1-\theta_1)\, y_1^* + i^*)}{2} < s^* < min\left\{\frac{d_1 + (\beta - \theta_2)\, y_1^*}{1 + 2\alpha_1}, \frac{d_2 - (\gamma_1 + (\beta + \theta_2)\, i^*)}{1 + \theta_1}\right\} \tag{3.14e}$$

Then, every the eigenvalues of $J(\tau_4)$ exists in the left half plane. Then, the (PEP) is LAS in Int.$\mathbb{R}_+^4$.

Theorem 3.1. *Presume that AEP is LAS, then it's a globally asymptotically stable* (GAS) *in the Int.* $\mathbb{R}_+^4$ *provided the following conditions hold.*

$$(1+\alpha_1) < d_1, \tag{3.15a}$$
$$1 + \theta_1 + \frac{\gamma_1}{\sigma} < d_2, \tag{3.15b}$$
$$1 < \sigma + d_3. \tag{3.15c}$$
$$\theta_2 < \beta. \tag{3.15d}$$

Proof . *Let* $w_1(s, i, y_1, y_2) = \int_k^s \frac{u-1}{u} du \ + i + y_1 + y_2$.
Now, straightforward calculations give that:

$$\frac{dw_1}{dt} = -(s-1)^2 - (1+\alpha_1)\, si + (1+\alpha_1)\, i - sy_1 + y_1 + \alpha_1 si - \beta i y_1 - d_1 i + \theta_1 s y_1 + \theta_2 i y_1$$
$$+ y_2 - d_2 y_1 + \gamma_1 y_1 - \sigma y_2 - d_3 y_2$$

we obtain that

$$\frac{dw_1}{dt} = -(s-1)^2 - si - (\beta - \theta_2)\, i y_1 - s y_1 - (d_1 - (1+\alpha_1))\, i - (d_2 - (1 + \theta_1 s + \gamma_1))\, y_1$$
$$- ((\sigma + d_3) - 1)\, y_2$$
$$\frac{dw_1}{dt} \le -(s-1)^2 - (d_1 - (1+\alpha_1))\, i - (d_2 - (1 + \theta_1 + \gamma_1))\, y_1 - ((\sigma + d_3) - 1)\, y_2.$$

when conditions (3.15a)-(3.15d) *are hold,* $\frac{dw_1}{dt}$ *is negative definite. Then* w_1 *is Lyapunov function* *(L.F.) and AEP is GAS.* □

Theorem 3.2. *Presume that PFEP is LAS, then it's GAS in the Int. $\mathbb{R}_{+}^{4}$ provided that the following sufficient conditions hold.*

$$\overline{s}+\beta\overline{i}+\gamma_1<d_2 \tag{3.16a}$$
$$1<(\sigma+d_3) \tag{3.16b}$$
$$\theta_2<\beta \tag{3.16c}$$
$$\theta_1<1 \tag{3.16d}$$
$$\left(i-\overline{i}\right)^2<\left((s-\overline{s})+\left(i-\overline{i}\right)\right)^2 \tag{3.16e}$$

Proof . *Let*

$$w_2\left(s,i,y_1,y_2\right)=\int_{\overline{s}}^{s}\frac{u-\overline{s}}{u}du+\int_{\overline{i}}^{i}\frac{v-\overline{i}}{v}dv+y_1+y_2.$$

Now, straightforward calculations give that

$$\begin{aligned}\frac{dw_2}{dt}=&-(s-\overline{s})^2-((1+\alpha_1)-\alpha_1)(s-\overline{s})\left(i-\overline{i}\right)-(\beta-\theta_2)iy_1-(1-\theta_1)sy_1\\&-\left(d_2-\left(\overline{s}+\beta\overline{i}+\gamma_1\right)\right)y_1-((\sigma+d_3)-1)y_2\end{aligned}$$

We obtain that

$$\frac{dw_2}{dt}\leq-(s-\overline{s})^2-(s-\overline{s})\left(i-\overline{i}\right)-\left(d_2-\left(\overline{s}+\beta\overline{i}+\gamma_1\right)\right)y_1-((\sigma+d_3)-1)y_2.$$
$$\frac{dw_2}{dt}<-(s-\overline{s})^2-(s-\overline{s})\left(i-\overline{i}\right)-\left(i-\overline{i}\right)^2-\left(d_2-\left(\overline{s}+\beta\overline{i}+\gamma_1\right)\right)y_1-((\sigma+d_3)-1)y_2+\left(i-\overline{i}\right)^2.$$
$$\frac{dw_2}{dt}<-\left((s-\overline{s})+\left(i-\overline{i}\right)\right)^2-\left(d_2-\left(\overline{s}+\beta\overline{i}+\gamma_1\right)\right)y_1-((\sigma+d_3)-1)y_2+\left(i-\overline{i}\right)^2.$$

when conditions (3.16a)-(3.16e) *are hold,* $\frac{dw_2}{dt}$ *is negative definite. Then,* w_2 *is L.F. and PFEP is GAS.*□

Theorem 3.3. *Presume that DFEP is LAS, then it's GAS in the Int. $\mathbb{R}_{+}^{4}$ provided the following conditions hold.*

$${q_{12}}^2<2q_{11}q_{22} \tag{3.17a}$$
$${q_{23}}^2<2q_{22}q_{33} \tag{3.17b}$$
$$\theta_2<\beta \tag{3.17c}$$
$$(1+\alpha_1)\hat{s}<\theta_2\hat{y}_1+d_1 \tag{3.17d}$$

Proof . *Let*

$$w_3\left(s,i,y_1,y_2\right)=\int_{\hat{s}}^{s}\frac{u-\hat{s}}{u}du+i+\int_{\hat{y}_1}^{y_1}\frac{v-\hat{y}_1}{v}dv+\int_{\hat{y}_2}^{y_2}\frac{w-\hat{y}_2}{w}dw.$$

Now, straightforward calculations give that

$$\begin{aligned}\frac{dw_3}{dt}=&-(s-\hat{s})^2-(1+\alpha_1)(s-\hat{s})i-(s-\hat{s})(y_1-\hat{y}_1)+\alpha_1si-\beta iy_1-d_1i+\theta_1(s-\hat{s})(y_1-\hat{y}_1)\\&+\theta_2i(y_1-\hat{y}_1)(y_1-\hat{y}_1)\left(\frac{y_2}{y_1}-\frac{\hat{y}_2}{\hat{y}_1}\right)+\gamma_1(y_2-\hat{y}_2)\left(\frac{y_1}{y_2}-\frac{\hat{y}_1}{\hat{y}_2}\right).\end{aligned}$$

we obtain that

$$\frac{dw_3}{dt} \leq -(s-\hat{s})^2 + (1+\alpha_1)\hat{s}i + (\theta_1 - 1)(s-\hat{s})(y_1-\hat{y}_1) - (\beta-\theta_2)iy_1 - d_1 i - \theta_2 i\hat{y}_1 + \frac{1}{y_1}(y_1-\hat{y}_1)(y_2-\hat{y}_2) - \frac{\hat{y}_2}{y_1\hat{y}_1}(y_1-\hat{y}_1)^2 + \frac{\gamma_1}{y_2}(y_1-\hat{y}_1)(y_2-\hat{y}_2) - \frac{\gamma_1\hat{y}_1}{y_2\hat{y}_2}(y_2-\hat{y}_2)^2$$

$$\frac{dw_3}{dt} < -q_{11}(s-\hat{s})^2 + q_{12}(s-\hat{s})(y_1-\hat{y}_1) - q_{22}(y_1-\hat{y}_1)^2 + q_{23}(y_1-\hat{y}_1)(y_2-\hat{y}_2) - q_{33}(y_2-\hat{y}_2)^2 - (d_1 + \theta_2\hat{y}_1 - (1+\alpha_1)\hat{s})i.$$

here $q_{11} = 1$, $q_{12} = \theta_1 - 1$, $q_{22} = \frac{\hat{y}_2}{y_1\hat{y}_1}, q_{23} = \frac{1}{y_1} + \frac{\gamma_1}{y_2}, q_{33} = \frac{\gamma_1\hat{y}_1}{y_2\hat{y}_2}$.

$$\frac{dw_3}{dt} < -\left(\sqrt{q_{11}}(s-\hat{s}) - \sqrt{\frac{1}{2}q_{22}}(y_1-\hat{y}_1)\right)^2 - \left(\sqrt{\frac{1}{2}q_{22}}(y_1-\hat{y}_1) - \sqrt{q_{33}}(y_2-\hat{y}_2)\right)^2 - (d_1 + \theta_2\hat{y}_1 - (1+\alpha_1)\hat{s})i.$$

when conditions (3.17a)-(3.17d) *are hold,* $\frac{dw_3}{dt}$ *is negative definite.*
Then, w_3 *is L.F. and DFEP is GAS.* □

Theorem 3.4. *Presume that, PEP is LAS, then it's GAS in the Int.* $\mathbb{R}^4_+$ *provided the following conditions hold.*

$$q_{12}{}^2 < q_{11}\ q_{22} \tag{3.18a}$$

$$q_{13}{}^2 < \frac{2}{3}\ q_{11}\ q_{33} \tag{3.18b}$$

$$q_{23}{}^2 < \frac{2}{3}\ q_{22}\ q_{33} \tag{3.18c}$$

$$q_{34}{}^2 < \frac{4}{3}\ q_{33}\ q_{44} \tag{3.18d}$$

$$(i-i^*)^2 < \left(\sqrt{\frac{1}{2}q_{11}}\ (s-s^*) + \sqrt{\frac{1}{2}q_{22}}\ (i-i^*)\right)^2 + \left(\sqrt{\frac{1}{2}q_{22}}\ (i-i^*) + \sqrt{\frac{1}{3}q_{33}}\ (y_1-y_1{}^*)\right)^2 \tag{3.18e}$$

Proof . *Let*

$$w_4(s,i,y_1,y_2) = \int_{s^*}^{s}\frac{u-s^*}{u}du + \int_{i^*}^{i}\frac{v-i^*}{v}dv + \int_{y_1^*}^{y_1}\frac{w-y_1^*}{w}dw + \int_{y_2^*}^{y_2}\frac{z-y_2^*}{z}dz.$$

Now, straightforward calculations give that

$$\frac{dw_4}{dt} = -(s-s^*)^2 - (s-s^*)(i-i^*) \pm (i-i^*)^2 - (1-\theta_1)(s-s^*)(y_1-y_1^*) - (\beta-\theta_2)(i-i^*)\star (y_1-y_1^*) + \left(\frac{1}{y_1}+\frac{\gamma_1}{y_2}\right)(y_1-y_1^*)(y_2-y_2{}^*) - \frac{y_2{}^*}{R_1}(y_1-y_1^*)^2 - \frac{\gamma_1 y_1{}^*}{R_2}(y_2-y_2{}^*)^2$$

where $R_1 = y_1y_1^*$, $R_2 = y_2y_2{}^*$.

$$\frac{dw_4}{dt} = -q_{11}(s-s^*)^2 - q_{12}(s-s^*)(i-i^*) - q_{22}(i-i^*)^2 + (i-i^*)^2 - q_{13}(s-s^*)(y_1-y_1^*) - q_{23}(i-i^*)(y_1-y_1^*) + q_{34}(y_1-y_1^*)(y_2-y_2{}^*) - q_{33}(y_1-y_1^*)^2 - q_{44}(y_2-y_2{}^*)^2$$

here $q_{11}=1,\ q_{12}=1,\ q_{22}=1,\ q_{13}=1-\theta_1,\ q_{23}=\beta-\theta_2,\ q_{34}=\frac{1}{y_1}+\frac{\gamma_1}{y_2}\ ,\ q_{33}=\frac{{y_2}^*}{R_1},\ q_{44}=\frac{\gamma_1 {y_1}^*}{R_2}.$

$$\frac{dw_4}{dt}\le-\left(\sqrt{\frac{1}{2}q_{11}}\ (s-s^*)+\sqrt{\frac{1}{2}q_{22}}\ (i-i^*)\right)^2-\left(\sqrt{\frac{1}{2}q_{22}}\ (s-s^*)+\sqrt{\frac{1}{3}q_{33}}\ (y_1-{y_1}^*)\right)^2$$
$$-\left(\sqrt{\frac{1}{2}q_{22}}\ (i-i^*)+\sqrt{\frac{1}{3}q_{33}}\ (y_1-{y_1}^*)\right)^2-\left(\sqrt{\frac{1}{3}q_{33}}\ (y_1-{y_1}^*)-\sqrt{q_{44}}\,(y_2-{y_2}^*)\right)^2+(i-i^*)^2.$$

Hence, under condition (3.18a)-(3.18e), $\frac{dw_4}{dt}$ *is negative definite. Then,* w_4 *is L.F.*
Therefore, PEP is GAS. □

Now, the persistence of system (2.3) is discussed in the next theorem.

Theorem 3.5. *Presume that condition* (3.1) *along with the following condition holds:*

$$\left(\frac{\theta_1 d_1\ (1+\alpha_1)+\theta_2\,(\alpha_1-d_1)}{\alpha_1\ (1+\alpha_1)}\right)+\frac{\gamma_1}{(\sigma+d_3)}>d_2 \tag{3.19a}$$
$$\alpha_1\hat{s}>\beta\hat{y}_1+d_1 \tag{3.19b}$$

Then, system (2.3) *uniformly persists.*

Proof . *Presume that the point* $\mathcal{P}$ *is in the Int.* $\mathbb{R}^4_+$ *and the orbit through* $\mathcal{P}$ *is denoted by* $o(\mathcal{P})$.
Let $\Omega(\mathcal{P})$ *be omega limit set of* $o(\mathcal{P})$. *Note that* $\Omega(\mathcal{P})$ *is bounded, due to theorem (1).*
Now to show that $\tau_0\notin\Omega(\mathcal{P})$, *presume the contrary.*
τ_0 *is saddle point, by Butler-McGhee lemma [5], there exist at least one another point* $\mathcal{Q}_1$ *such that* $\mathcal{Q}_1\in\omega^s(\tau_0)\cap\Omega(\mathcal{P})$.
Moreover, since $\omega^s(\tau_0)$ *is the* $\mathbb{R}^3_+(iy_1y_2)$ *space and* $o(\mathcal{Q}_1)$ *is the entire orbit through* $\mathcal{Q}_1$ *contain in* $\Omega(\mathcal{P})$.

Now, if $\mathcal{Q}_1$*on ether boundary axes of* $\mathbb{R}^3_+(iy_1y_2)$, *then the positive specific axis is contained in* $\Omega(\mathcal{P})$ *and this is contradicting to it's boundedness.*
Else, $\mathcal{Q}_1\in$ *Int.* $\mathbb{R}^3_+(iy_1y_2)$ *and there is no equilibrium point in the Int.* $\mathbb{R}^3_+(iy_1y_2)$, *the* $o(\mathcal{Q}_1)$ *must be unbounded and this leads to contradiction. We get that* $\tau_0\notin\Omega(\mathcal{P})$.
Presently to proof $\tau_1\notin\Omega(\mathcal{P})$, *presume the contrary.*
τ_1 *is a saddle point provided condition* (3.1), *by Butler-McGhee lemma* $\mathcal{Q}_2\in\omega^s(\tau_1)\cap\Omega(\mathcal{P})$. *Moreover, since* $\omega^s(\tau_1)$ *is* $\mathbb{R}^3_+(sy_1y_2)$ *space.*

Now, if $\mathcal{Q}_2$ *on boundary axes of* $\mathbb{R}^3_+(sy_1y_2)$, *we obtain the contradiction in above part of proof. In case of* $\mathcal{Q}_2\in$ *Int.* $\mathbb{R}^3_+(sy_1y_2)$ *there is no equilibrium point in Int.* $\mathbb{R}^3_+(sy_1y_2)$ *we get* $o(\mathcal{Q}_2)\subset\Omega(\mathcal{P})$ *is undounded and this leads to contradiction. Then, we get* $\tau_1\notin\Omega(\mathcal{P})$.
Presently to proof $\tau_2\notin\Omega\,(\mathcal{P})$, *presume the contrary.* τ_2 *is a saddle point provided condition (23a), by Butler-McGhee lemma* $\mathcal{Q}_3\in\omega^s(\tau_1)\cap\Omega(\mathcal{P})$. *Moreover, since* $\omega^s(\tau_2)$ *is* $\mathbb{R}^3_+(siy_2)$ *space.*

Now, if $\mathcal{Q}_3$ *on boundary axes of* $\mathbb{R}^3_+(siy_2)$, *we obtain contradiction in above part of proof. In case of* $\mathcal{Q}_3\in$ *Int.* $\mathbb{R}^3_+(siy_2)$ *there is no equilibrium point in Int.* $\mathbb{R}^3_+(siy_2)$ *we get* $o(\mathcal{Q}_3)\subset\Omega(\mathcal{P})$ *is undounded and this leads to contradiction. Therefore, we get* $\tau_2\notin\Omega(\mathcal{P})$.
Finally, τ_3 *is a saddle point provided condition* (3.19b). *Similarity, by using the argument we obtain* $\tau_3\notin\Omega(\mathcal{P})$. *Then* $\Omega(\mathcal{P})$ *must be in the Int.* $\mathbb{R}^4_+$. □

4. Bifurcations Analyses

Rewrite system (2.3) as the follow:

$$\frac{\mathrm{dX}}{\mathrm{dt}} = f(X)$$

where $X = (s, i, y_1, y_2)^T$ and $f = (f_1, f_2, f_3, f_4)^T$ with $f_i; i = 1, 2, 3, 4$. Then by J of system (2.3), Let $V = (v_1, v_2, v_3, v_4)^T$ be any nonzero vector and the second directional derivative write as follow

$$D^2 f(s, i, y_1, y_2)(V, V) = \begin{pmatrix} -2{v_1}^2 - 2(1+\alpha_1) v_1 v_2 - 2v_1 v_3 \\ 2v_1 v_3 - 2\beta v_2 v_3 \\ 2\theta_1 v_1 v_3 + 2\theta_2 v_2 v_3 \\ 0 \end{pmatrix} \tag{4.1}$$

Moreover, the third directional derivative given by

$$D^3 f(s, i, y_1, y_2)(V, V, V) = (0, 0, 0, 0)^T.$$

Then, system (2.3) has no pitchfork bifurcation.

Theorem 4.1. *Presume that condition (13f) holds, system* (2.3) *do not undergoes any types of local bifurcation at AEP when d_1 passes through $d_1^* = \alpha_1$.*

Proof . *From $J(\tau_1)$, system* (2.3) *at AEP and $d_1 = d_1^*$ has $J(\tau_1, d_1^*) = J_1$, which has zero eigenvalue, say $\lambda_i^* = 0$.*

$$J_1 = \begin{bmatrix} -1 & -(1+\alpha_1) & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 - d_2 & 1 \\ 0 & 0 & \gamma_1 & -(\sigma + d_3) \end{bmatrix}$$

Now, let $U^{[1]} = \left(u_1^{[1]}, u_2^{[1]}, u_3^{[1]}, u_4^{[1]}\right)^T$ is the eigenvector corresponding to $\lambda_i^* = 0$.

Now, $J_1 U^{[1]} = \mathbf{0}$ leads to $U^{[1]} = \left(\delta u_2^{[1]}, u_2^{[1]}, 0, 0\right)^T$, where $u_2^{[1]}$ is nonzero real numbers and $\delta = -(1+\alpha_1) < 0$. Let $\psi^{[1]} = \left(\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}, \psi_4^{[1]}\right)^T$ is the eigenvector corresponding to $\lambda_i^* = 0$ of ${J_1}^T$. Hence, due to condition (13f), ${J_1}^T \psi^{[1]} = \mathbf{0}$ gives that $\psi^{[1]} = \left(0, \psi_2^{[1]}, 0, 0\right)^T$, where $\psi_2^{[1]}$ is any nonzero real numbers.

Now,

$$\frac{\partial f}{\partial d_1} = f_{d_1}(X, d_1) = \left(\frac{\partial f_1}{\partial d_1}, \frac{\partial f_2}{\partial d_1}, \frac{\partial f_3}{\partial d_1}, \frac{\partial f_4}{\partial d_1}\right)^T = (0, -i, 0, 0)^T.$$

Thus $f_{d_1}(\tau_1, d_1^*) = (0, 0, 0, 0)^T$, which gives $\left(\psi^{[1]}\right)^T f_{d_1}(\tau_1, d_1^*) = 0$. By Sotomayor's theorem system (2.3) *has no saddle $-$ node bifurcation at $d_1 = d_1^*$. Furthermore*

$$Df_{d_1}(\tau_1, d_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can show,

$$\left(\psi^{[1]}\right)^T \left(Df_{d_1}(\tau_1, d_1^*) U^{[1]}\right) = \left(0, \psi_2^{[1]}, 0, 0\right) \left(0, -u_2^{[1]}, 0, 0\right)^T = -\psi_2^{[1]} u_2^{[1]} \neq 0$$

Moreover using Eq.(4.1) with τ_1, d_1^* and $U^{[1]}$ gives

$$D^2 f\left(\tau_1, d_1^*\right)\left(U^{[1]}, U^{[1]}\right) = -2\delta\left(u_2^{[1]}\right)^2\left(\delta + (1+\alpha_1), 0, 0,\ 0\right)^T .$$

Hence it is obtained that

$$\left(\psi^{[1]}\right)^T D^2 f\left(\tau_1, d_1^*\right)\left(U^{[1]}, U^{[1]}\right) = 0.$$

Now, a transcritical bifurcation does not occurs as d_1 passes through the value d_1^*. Therefore, the AEP has no any types of local bifurcation. □

Theorem 4.2. *system (2.3) undergoes a transcritical bifurcation at PFEP when d_2 passs through $d_2^* = \frac{\gamma_1}{(\sigma+d_3)} + \theta_1\overline{s} + \theta_2\overline{i}$ provided that*

$$\left[\theta_1\mu_1 + \theta_2\mu_2\right] \neq 0 \tag{4.2}$$

here μ_1 and μ_2 are given in the proof.

Proof . *From $J(\tau_2)$, system (2.3) at PFEP and $d_2 = d_2^*$ has $J(\tau_2, d_2^*) = J_2$, which has zero eigenvalue, say $\lambda_{y_1}^* = 0$.*

$$J_2 = \begin{bmatrix} 1-2\overline{s}-(1+\alpha_1)\,\overline{i} & -(1+\alpha_1)\,\overline{s} & \overline{s} & 0 \\ \alpha_1\overline{i} & \alpha_1\overline{s}-d_1 & -\beta\overline{i} & 0 \\ 0 & 0 & -\frac{\gamma_1}{(\sigma+d_3)} & 1 \\ 0 & 0 & \gamma_1 & -(\sigma+d_3) \end{bmatrix}$$

Now, let $U^{[2]} = \left(u_1^{[2]}, u_2^{[2]}, u_3^{[2]}, u_4^{[2]}\right)^T$ is the eigenvector corresponding to $\lambda_{y_1}^* = 0$.

Now, $J_2U^{[2]} = \mathbf{0}$ leads to $U^{[2]} = \left(\mu_1 u_4^{[2]}, \mu_2 u_4^{[2]}, \mu_3 u_4^{[2]}, u_4^{[2]}\right)^T$, where $u_4^{[2]}$ is nonzero real numbers, $\mu_1 = \frac{\beta(\sigma+d_3)}{\alpha_1\gamma_1}$, $\mu_2 = -\frac{(\beta+\alpha_1)(\sigma+d_3)}{\alpha_1\gamma_1(1+\alpha_1)}$ and $\mu_3 = \frac{(\sigma+d_3)}{\gamma_1}$.

Let $\psi^{[2]} = \left(\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]}\right)^T$ is the eigenvector corresponding to $\lambda_{y_1}^* = 0$ of ${J_2}^T$.

${J_2}^T\psi^{[2]} = \mathbf{0}$ leads to $\psi^{[2]} = \left(0, 0, \eta\psi_4^{[2]}, \psi_4^{[2]}\right)^T$, where $\psi_4^{[2]}$ is nonzero real numbers and $\eta = (\sigma + d_3)$.

Now,

$$\frac{\partial f}{\partial d_2} = f_{d_2}\left(X, d_2\right) = \left(\frac{\partial f_1}{\partial d_2}, \frac{\partial f_2}{\partial d_2}, \frac{\partial f_3}{\partial d_2}, \frac{\partial f_4}{\partial d_2}\right)^T = \left(0, 0, -y_1, 0\right)^T .$$

Thus $f_{d_2}\left(\tau_2, d_2^*\right) = (0,0,0,0)^T$, which gives $\left(\psi^{[2]}\right)^T f_{d_2}\left(\tau_2, d_2^*\right) = 0$.

By Sotomayor's theorem, system (2.3) has no saddle $-$ node bifurcation at $d_2 = d_2^*$.

$$Df_{d_2}\left(\tau_2, d_2^*\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then its obtain that

$$\left(\psi^{[2]}\right)^T\left(Df_{d_2}\left(\tau_2, d_2^*\right)U^{[2]}\right) = \left(0, 0, \eta\ \psi_4^{[2]}, \psi_4^{[2]}\right)\left(0, 0, -\mu_3 u_4^{[2]}, 0\right)^T = -\eta\ \mu_3\psi_4^{[2]}u_4^{[2]} \neq 0$$

Again by using Eq.(4.1) with τ_2, d_2^* and $U^{[2]}$ gives that

$$D^2 f\left(\tau_2, d_2^*\right)\left(U^{[2]}, U^{[2]}\right) = 2\left(u_4^{[2]}\right)^2 \begin{pmatrix} -\mu_1\left[\mu_1 + (1+\alpha_1)\,\mu_2 + \mu_3\right] \\ \mu_3\left[\mu_1 - \beta\mu_2\right] \\ \mu_3\left[\theta_1\mu_1 + \theta_2\mu_2\right] \\ 0 \end{pmatrix}$$

Hence it is obtain that:

$$\left(\psi^{[2]}\right)^T D^2 f\left(\tau_2, d_2^*\right)\left(U^{[2]}, U^{[2]}\right) = 2\ \eta\ \mu_3 \psi_4^{[2]}\left[\theta_1\mu_1 + \theta_2\mu_2\right]\left(u_4^{[2]}\right)^2\ \left[\theta_1\mu_1 + \theta_2\mu_2\right] \neq 0$$

Therefore, if the condition (4.2) satisfies, system (2.3) has a transcritical bifurcation at PFEP as d_2 passes through value d_2^*. □

Theorem 4.3. *system (2.3) undergoes a transcritical bifurcation at DFEP when α_1 passs through $\alpha_1^* = \frac{\beta\hat{y}_1 + d_1}{\hat{s}}$ provided that*

$$\xi_2\left(\xi_1 - \beta\right) \neq 0. \tag{4.3}$$

here ξ_1 and ξ_2 are given in the proof.

Proof . *From $J(\tau_3)$, system (2.3) at DFEP and $\alpha_1 = \alpha_1^*$ has $J(\tau_3, \alpha_1^*) = J_3$, which has zero eigenvalue, say $\lambda_i^* = 0$.*

$$J_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix}$$

where, $b_{11} = 1 - 2\hat{s} - \hat{y}_1$, $b_{12} = -\left(1 + \alpha_1^*\right)\hat{s}$, $b_{13} = \hat{s}$, $b_{31} = \theta_1\hat{y}_1$, $b_{32} = \theta_2\hat{y}_1$, $b_{33} = \theta_1\hat{s} - d_2$, $b_{34} = 1$, $b_{43} = \gamma_1$, $b_{44} = -\left(\sigma + d_3\right)$.

Now, let $U^{[3]} = \left(u_1^{[3]}, u_2^{[3]}, u_3^{[3]}, u_4^{[3]}\right)^T$ is the eigenvector corresponding to $\lambda_i^* = 0$.

Now, $J_3 U^{[3]} = \mathbf{0}$ leads to $U^{[3]} = \left(\xi_1 u_2^{[3]}, u_2^{[3]}, \xi_2 u_2^{[3]}, \xi_3 u_2^{[3]}\right)^T$, where $u_2^{[3]}$ is nonzero real numbers with $\xi_1 = -\left(\frac{b_{12}(b_{44}(b_{11}b_{33} - b_{13}b_{31}) - b_{11}b_{34}b_{43}) + b_{13}b_{44}(b_{12}b_{31} - b_{11}b_{32})}{b_{44}(b_{11}b_{33} - b_{13}b_{31}) - b_{11}b_{34}b_{43}}\right)$, $\xi_2 = \frac{b_{44}(b_{12}b_{31} - b_{11}b_{32})}{b_{44}(b_{11}b_{33} - b_{13}b_{31}) - b_{11}b_{34}b_{43}}$ *and* $\xi_3 = -\frac{b_{43}(b_{12}b_{31} - b_{11}b_{32})}{b_{44}(b_{11}b_{33} - b_{13}b_{31}) - b_{11}b_{34}b_{43}}$.

Let $\psi^{[3]} = \left(\psi_1^{[3]}, \psi_2^{[3]}, \psi_3^{[3]}, \psi_4^{[3]}\right)^T$ is the eigenvector corresponding to $\lambda_i^* = 0$ of ${J_3}^T$.

${J_3}^T\psi^{[3]} = \mathbf{0}$ *leads to* $\psi^{[3]} = \left(0, \psi_2^{[3]}, 0, 0\right)^T$, *where $\psi_2^{[3]}$ is nonzero real numbers.*

Now,

$$\frac{\partial f}{\partial\alpha_1} = f_{\alpha_1}\left(X, \alpha_1\right) = \left(\frac{\partial f_1}{\partial\alpha_1}, \frac{\partial f_2}{\partial\alpha_1}, \frac{\partial f_3}{\partial\alpha_1}, \frac{\partial f_4}{\partial\alpha_1}\right)^T = \left(-si, si, 0, 0\right)^T.$$

Thus $f_{\alpha_1}\left(\tau_3, \alpha_1^*\right) = (0,0,0,0)^T$, which gives $\left(\psi^{[3]}\right)^T f_{\alpha_1}\left(\tau_3, \alpha_1^*\right) = 0$.

By Sotomayor's theorem, system (2.3) has no saddle $-$ node bifurcation at $\alpha_1 = {\alpha_1}^*$.

$$Df_{\alpha_1}\left(\tau_3, \alpha_1^*\right) = \begin{bmatrix} 0 & -\hat{s} & 0 & 0 \\ 0 & \hat{s} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then its obtain that

$$\left(\psi^{[3]}\right)^T\left(Df_{\alpha_1}\left(\tau_3, \alpha_1^*\right)U^{[3]}\right) = \hat{s}\psi_2^{[3]}u_2^{[3]} \neq 0.$$

Again by using Eq.(4.1) with τ_3, α_1^* and $U^{[3]}$ gives that

$$\left(\psi^{[3]}\right)^T D^2 f\left(\tau_3, \alpha_1^*\right)\left(U^{[3]}, U^{[3]}\right) = \xi_2\left(\xi_1 - \beta\right)\psi_2^{[3]}\left(u_2^{[3]}\right)^2 \neq 0.$$

Therefore, if the condition (4.3) satisfies, system (2.3) has a transcritical bifurcation at DFEP as α_1 passes through value α_1^*. □

5. Numerical Analysis

The aim is the study the impact of changing the value of all parameters on the dynamical behavior of system (2.3). It is spotted that, for the next set of presumptive parameters that satisfies stability restrictions of the PEP, system (2.3) has a GAS as shown in Figure.1.

$$\begin{aligned} &\alpha_1 = 0.6, \quad \beta = 0.1 \quad d_3 = 0.05, \quad \theta_1 = 0.1, \quad \theta_2 = 0.06 \\ &d_2 = 0.1, \quad \gamma_1 = 0.05, \quad \sigma = 0.6, \quad d_3 = 0.05 \end{aligned} \tag{5.1}$$

Clearly, Figure.1 shows the system has a GAS as the solution of the system which approaches asymptotically to the $PEP = (0.20, 0.04, 0.72, 0.05)$, starting from three various initial points.

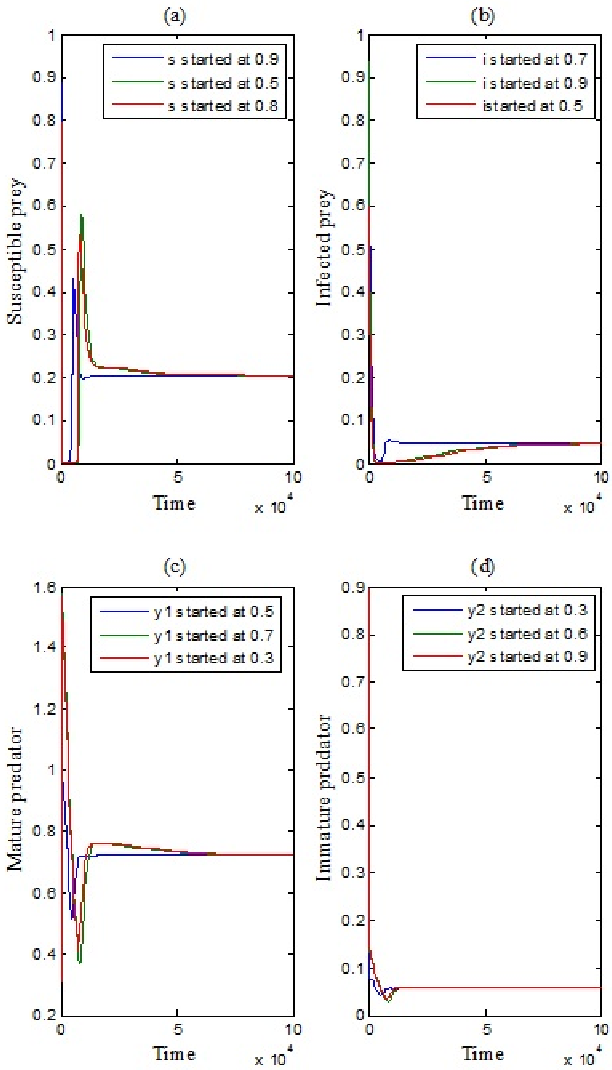

Figure 1: The trajectories of system (2.3) using data given by Eq.(5.1) with different initial points approach asymptotically to PEP, represented by $\tau_4 =$ (0.20, 0.04, 0.72, 0.05). (a) Time series of trajectories of the susceptible prey. (b) Time series of the trajectories of infected prey. (c) Time series of the trajectories of mature predator. (d) Time series of the trajectories of immature predator.

Now, the influence of varying α_1 on the dynamical behavior in the ranges $0.01 \leq \alpha_1 < 0.6$, and $0.6 \leq \alpha_1 < 1$, is investigated. Note that, the trajectory approaches asymptotically to DFEP and PEP in the Int.$\mathbb{R}_+^4$ respectively, as shown in Figure.2.

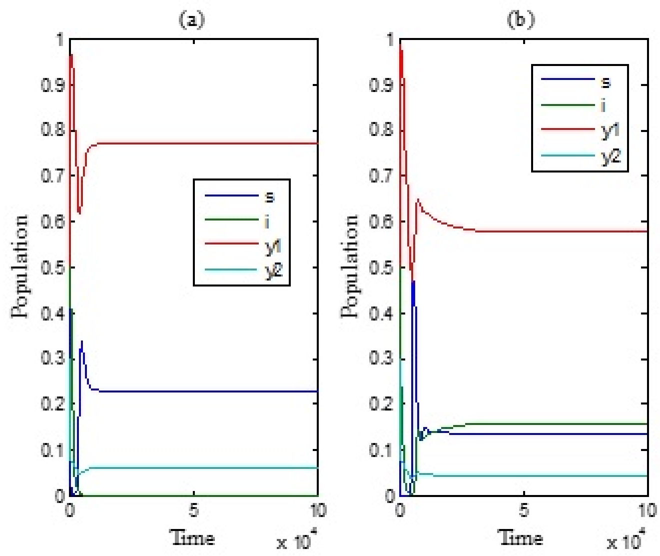

Figure 2: The trajectories of system (2.3) using data given by Eq. (5.1) with values of α_1. (a) Time series of the trajectory with $\alpha_1 = 0.2$. (b) Time series of the trajectory with $\alpha_1 = 0.8$.

The impact of changing of β on the dynamical behavior in the ranges $0 < \beta < 0.2$ and $0.2 \leq \beta < 1$ is studied. The trajectory approaches asymptotically to PEP in the Int.$\mathbb{R}^4_+$ and DFEP, respectively as illustrated in the Figure.3.

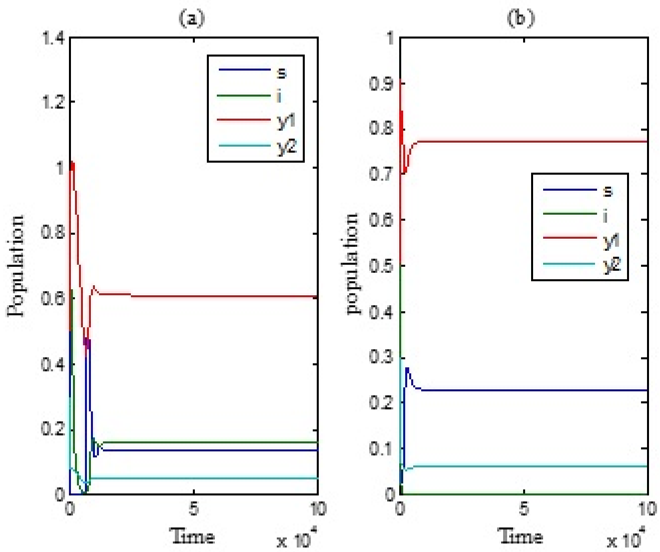

Figure 3: The trajectories of system (2.3) using data given by Eq.(5.1) with values of β. (a) Time series of the trajectory with $\beta = 0.05$. (b) Time series of the trajectory with $\beta = 0.4$.

Now, the influence of varying the parameters d_1 and θ_1 in the ranges $0 < d_1 < 0.07$, $0.07 \leq d_1 < 1$, and $0.01 \leq \theta_1 < 0.2$, $0.2 \leq \theta_1 < 1$, are studied. The trajectory approaches asymptotically to PEP and DFEP, respectively as shown above in the Figure.3.
The influence of varying θ_2 in the ranges $0.01 \leq \theta_2 < 0.03$ and $0.03 \leq \theta_2 < 1$ is studied. the trajectory approaches asymptotically to PFEP in the $si-$plane and PEP in the Int.$\mathbb{R}^4_+$ respectively as shown in Figure.4.

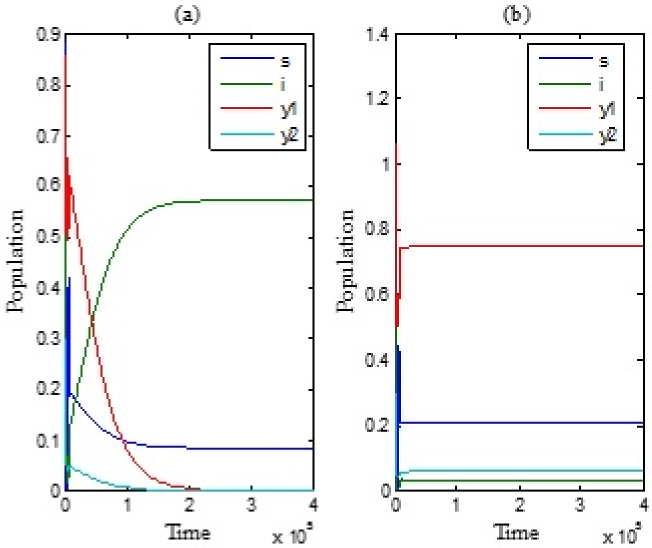

Figure 4: The trajectories of system (2.3) using data given by Eq.(5.1) with values of β (a) Time series of the trajectory with $\beta = 0.05$ (b) Time series of the trajectory with $\beta = 0.4$.

The influence of varying d_2 in the ranges $0.08 \leq d_2 < 0.1$, $0.1 \leq d_2 < 0.2$ and $0.2 \leq d_2 < 1$ is studied. The trajectory approaches asymptotically to the DFEP, PEP in the Int.$\mathbb{R}_+^4$ and PFEP in the $si-$plane respectively, as shown in Figure.5.

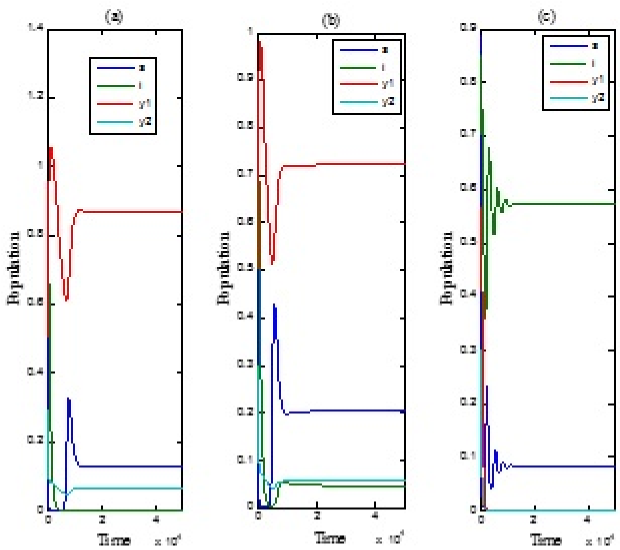

Figure 5: The trajectories of system (2.3) using data given by Eq.(5.1) with values of β (a) Time series of the trajectory with $\beta = 0.05$ (b) Time series of the trajectory with $\beta = 0.4$.

The impact of varying γ_1 in the ranges $0.01 \leq \gamma_1 < 0.04$, $0.04 \leq \gamma_1 < 0.06$ and $0.06 \leq \gamma_1 < 0.07$ is studied. The trajectory approaches asymptotically to PFEP in the si$-$plane, PEP in the Int.$\mathbb{R}_+^4$, and DFEP respectively, the trajectory of system (2.3) as explained in Figure.6.

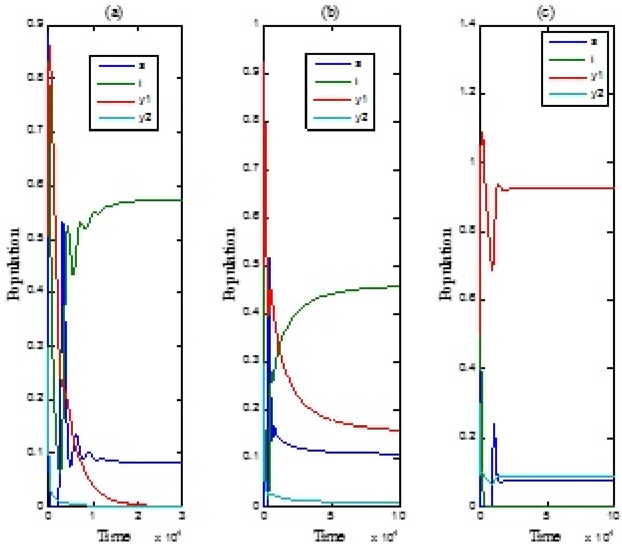

Figure 6: The trajectories of system (2.3) using data given by Eq.(5.1) with values of γ_1, (a) Time series of the trajectory with $\gamma_1 = 0.02$ (b) Time series of the trajectory with $\gamma_1 = 0.04$ (c) Time series of the trajectory with $\gamma_1 = 0.06$.

Now, the effect of varying σ in the ranges $0.5 \leq \sigma < 0.6$, $0.6 \leq \sigma < 0.9$, and $0.9 \leq \sigma < 1$ is studied. The trajectory approaches asymptotically to DFEP, PEP in the Int.$\mathbb{R}^4_+$, PFEP in the $\text{si}-$plane, as shown in the Figure.7.

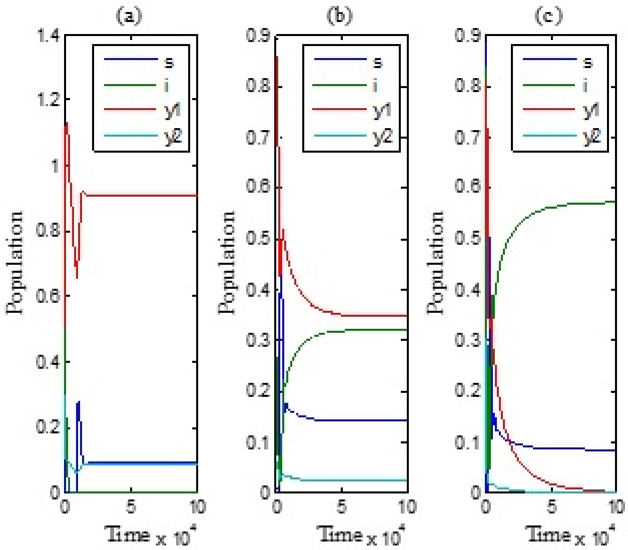

Figure 7: The trajectories of system (2.3) using data given by Eq.(5.1) with values of σ, (a) Time series of the trajectory with $\sigma = 0.5$ (b) Time series of the trajectory with $\sigma = 0.7$ (c) Time series of the trajectory with $\sigma = 0.9$.

Finally, the effect of varying d_3 in the ranges $0.01 \leq d_3 < 0.4$, $0.4 \leq d_3 < 0.3$, and $0.3 \leq d_3 < 1$ is investigated. Note that, varying d_3 has similar effects as shown with varying σ.

6. Discussion and Conclusion

In this article, a prey−predator model comprising infectious disease in prey species and stage–structure in predator species is suggested and studied. The local and global dynamics of the suggested model are investigated. The conditions of persistence and the local bifurcation are investigated. Finally, the global dynamics of the model is investigated numerically and confirmed the obtained outcomes.

Now, the summary of the numerical simulation outcomes are obtained by using data (5.1).

1. The trajectory approaches asymptotically to PEP starting from various initial points, which refers to existence of GAS.
2. when α_1 decreases below a particular value, we observed the trajectory approaches asymptotically to DFEP. While, increasing α_1 above a particular in the Int.$\mathbb{R}^4_+$.
3. If the parameter β increases above a particular value leading to approaches asymptotically to DFEP. Else, the system still persists at a PEP.
4. When d_1 and θ_1 decreases below a particular value leads to approaches asymptotically to PEP. However, increasing these parameters above a particular value leads to approaches asymptotically to DFEP.
5. When θ_2 decreases below a particular value, the trajectory approaches asymptotically to PFEP in the $si-$plane. Else, the system still persists at a PEP.
6. Decreasing d_2 below a particular value leading to approaches asymptotically to DFEP. Increasing d_2 above a particular value leads to approaches asymptotically to PEP in the Int.$\mathbb{R}^4_+$. Further increasing leads to PFEP in the $si-$plane.
7. If γ_1 decreases below a particular value leads to approaches asymptotically to PFEP in the $si-$plane. Increasing $\gamma_1$above a particular value leads to approaches asymptotically to PEP in the Int.$\mathbb{R}^4_+$. Further increasing leads to DFEP.
8. Decreasing σ and d_3 below a particular value leads to approaches asymptotically to DFEP. However, increasing these parameters above a particular values leads to approaches asymptotically to PEP in the Int.$\mathbb{R}^4_+$. Further increasing leads to approaches asymptotically to PFEP in the $si-$plane.

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