Fixed point theorems on orthogonal complete metric spaces with an application

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Abstract

In this paper, we introduce the concept of orthogonal contractive mappings and prove some fixed point theorems for such contractions. We establish our results in orthogonal bounded complete metric spaces via the notion of τ-distances. Moreover, an application to a differential equation is given.

Keywords: Fixed point, O-complete metric space, O-contractive map, O-E-weakly contractive map, τ-distance, Hausdorff topological space
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1. Introduction

Over the years many authors have improved, extend and generalized Banach's fixed point in many directions.

In 2003, Aamri and El Moutwakil [1] introduced the concept of τ-distances in general topological spaces which extend many known spaces in the literature. Moreover, they proved a version of the Banach's fixed point in this setting.

On the other hand, in 2017, Eshaghi Gordji et al [4] introduced the notion of orthogonal sets and gave an extension of Banach's fixed point. Since then, many results have appeared in the literature concerning this notion [5, 6, 7, 9].

On the other hand, Eivazi Damirchi Darsi Olia et al [3] introduced the concept of orthogonal cone metric spaces and established some versions of fixed point theorems in incomplete orthogonal cone

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metric spaces. Khalehoghli et al, in [8] defined R-topological spaces and SR-topological spaces and showed that this type of spaces is very powerful and applicable to many cases for example in the context of fixed point theory, functional analysis...

Also, Hosseini et al [6] established some fixed point theorems for Banach's contraction and Suzuki type θ-contraction in the setting of orthogonal modular metric spaces.

Very recently, in 2019, Touail et al [11] proved, some fixed point theorems for contractive mappings in bounded complete metric spaces via τ-distances without using the compactness, some related results can be found in [12, 13, 14, 15].

In this paper, we modify some concepts defined in [11] to orthogonal sets and establish some fixed point theorems via two technical lemmas in this direction. Our results generalize and improve the proven results in [11] and many known results in the literature. Furthermore, we apply our results to prove the existence and the uniqueness of a solution for a differential equation.

2. Preliminaries

The aim of this section is to present some notions and results used in the paper.

Let (X, τ) be a topological space and $p : X \times X \to [0, \infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$.

Definition 2.1. *(Definition 2.1 [1]) The function p is said to be $\tau-$distance if for each $x \in X$ and any neighborhood V of x, there exists $\varepsilon > 0$ such that $B_p(x, \varepsilon) \subset V$.*

Definition 2.2. *A sequence $\{x_n\}$ in a Hausdorff topological space X is a $p-$Cauchy if it satisfies the usual metric condition with respect to p, in other words, if $\lim_{n,m \to \infty} p(x_n, x_m) = 0$.*

Definition 2.3. *(Definition 3.1 [1])*
Let (X, τ) be a topological space with a τ-distance p.

1. *X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x, x_n) = 0$.*
2. *X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n = x$ with respect to τ.*
3. *X is said to be p-bounded if $\sup\{p(x, y)/x, y \in X\} < \infty$.*

Lemma 2.4. *(Lemma 3.1[1])*
Let (X, τ) be a Hausdorff topological space with a $\tau-$distance p, then

1. *Let (x_n) be an arbitrary sequence in X, $x \in X$ and (α_n) be a sequence in $\mathbb{R}^+$ converging to 0 such that $p(x, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$. Then (x_n) converges to x with respect to the topology τ.*
2. *$p(x, y) = 0$ implies $x = y$.*
3. *Let (x_n) be a sequence in X such that $\lim_{n \to \infty} p(x, x_n) = 0$ and $\lim_{n \to \infty} p(y, x_n) = 0$, then $x = y$.*

Definition 2.5. *([1]) Ψ is the class of all functions $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying:*
i) ψ is nondecreasing,
ii) $\lim \psi^n(t) = 0$, for all $t \in [0, \infty)$.

Theorem 2.6. *(Theorem 4.1 [1])*
Let (X, τ) be a Hausdorff topological space with a $\tau-$distance p. Suppose that X is $p-$bounded and $S-$complete. Let T be a selfmapping of X such that

$$p(Tx, Ty) \leq \psi(p(x, y)),$$

for all $x, y \in X$, where $\psi \in \Psi$. Then T has a unique fixed point.

Now, we recall the definition of an orthogonal set and some related basic notions.

Definition 2.7. *[4]. Let $X \neq \emptyset$ and let $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following hypothesis:*

$$\exists x_0 : (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y), \tag{2.1}$$

then it called an orthogonal set (briefly $O-set$). we denote this $O-set$ by $(X, \perp)$.

Note that in the above Definition, x_0 is said to be an orthogonal element.

Remark 2.8. *In general, x_0 is not unique, otherwise, $(X, \perp)$ is called unique orthogonal set and the element x_0 is said to be a unique orthogonal element.*

Definition 2.9. *[4]. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, O-sequence) if*

$$(\forall n, x_n \perp x_{n+1}) \text{ or } (\forall n, x_{n+1} \perp x_n).$$

Definition 2.10. *[4] The triplet $(X, \perp, d)$ is called an orthogonal metric space if (X, d) is a metric space and $(X, \perp)$ is an O-set.*

Definition 2.11. *[4] Let $(X, \perp, d)$ be an Orthogonal metric space. Then, a mapping $T : X \to X$ is said to be orthogonally continuous (briefly $\perp$-continuous) in $x \in X$, if for each O-sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to \infty$, we obtain $Tx_n \to Tx$ as $n \to \infty$. Also, T is said to be $\perp -continuous$ on X if T is $\perp -continuous$ in each $x \in X$.*

Definition 2.12. *[4] Let $(X, \perp, d)$ be an Orthogonal metric space. Then, X is said to be orthogonally complete (or $\perp -complete$) if every Cauchy $O-sequence$ is convergent.*

Definition 2.13. *[4] Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp -preserving$ if $Tx \perp Ty$ whenever $x \perp y$.*

Remark 2.14. *[4] Every complete metric space (continuous mapping) is O-complete metric space ($\perp -continuous$ mapping) and the converse is not true.*

Theorem 2.15. *[4]. Let $(X, \perp, d)$ O-complete metric space and T a selfmapping on X which is $\perp -preserving$ and $\perp -continuous$. If there exists $k \in [0, 1)$ such that for all $x, y \in X$*

$$x \perp y \text{ implies } d(Tx, Ty) \leq kd(x, y).$$

Then T has a unique fixed point.

Now, we give some examples of orthogonal spaces

Example 2.16. *[4] Let $X = Z$. Define the binary relation $\perp$ on X by $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence, $(X, \perp)$ is an O-set.*

Example 2.17. *[4] Let X be an inner product space with the inner product $(., .)$. Define the binary relation $\perp$ on X by $x \perp y$ if $(x, y) = 0$. It is easy to see that $0 \perp x$ for all $x \in X$. Hence, $(X, \perp)$ is an O-set.*

For more details, we refer the reader to see [4].

At the end of this section, we recall the proven results in [11]

Theorem 2.18. *(Theorem 3 [11]) Let $T : X \to X$ be a mapping of a bounded complete metric space (X, d) such that*

$$\inf_{x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0. \tag{2.2}$$

Then T has a unique fixed point.

Definition 2.19. *Definition 8 [11]) Let $T : X \to X$ be a mapping of a metric space (X, d). T will be said an E-weakly contractive map if for all $x, y \in X$*

$$d(Tx, Ty) \leq d(x, y) - \phi[1 + d(x, y)],$$

where $\phi : [1, \infty) \to [0, \infty)$ is a function satisfying
  i) $\phi(1) = 0$,
  ii) $\inf_{t > 1} \phi(t) > 0$.

Theorem 2.20. *(Theorem 9 [11]) Let $T : X \to X$ be an E-weakly contractive map of a bounded complete metric space (X, d). Then T has a unique fixed point.*

3. Main results

In this section, we start the following definitions and lemmas.

Definition 3.1. *The triplet $(X, \tau, \perp)$ is called an orthogonal Hausdorff topological space with a $\tau-$distance p if (X, τ) is a Hausdorff topological space with a $\tau-$distance p and $(x, \perp)$ is an orthogonal set.*

Definition 3.2. *Let $(X, \tau, \perp)$ be an orthogonal Hausdorff topological space with a $\tau-$distance p.*

- *Let $\{x_n\}$ an O-sequence in X. If $\{x_n\}$ is p-Cauchy then it is called orthogonal p-Cauchy sequence (O-p-Cauchy sequence).*

- *X is said to be orthogonal S-complete if for every O-p-Cauchy sequence $\{x_n\}$ there exists $u \in X$ such that $\lim p(u, x_n) = 0$.*

Lemma 3.3. *Let $(X, \tau, \perp)$ be an orthogonal Hausdorff topological space with a $\tau-$distance p. Suppose that X is p-bounded and orthogonal S-complete. Let T be a $\perp$-continuous and $\perp$-preserving selfmapping of X such that, for all $x, y \in X$*

$$x \perp y \quad implies \quad p(Tx, Ty) \leq \psi(p(x, y)), \tag{3.1}$$

where $\psi \in \Psi$. Then T has a unique fixed point.

Proof . Since X is an orthogonal set, there exists at least $x_0 \in X$ such that

$$\forall y \in X \; x_0 \perp y \text{ or } \forall y \in X y \perp x_0 \tag{3.2}$$

This implies that $x_0 \perp T x_0$ or $T x_0 \perp x_0$. Consider the iterated sequence $\{x_n\}$ such that $x_n = T^n x_0$ for all $n \in \mathbb{N}$. As T is a $\perp$−perserving, we obtain either $T^n x_0 \perp T^{n+1} x_0$ or $T^{n+1} x_0 \perp T^n x_0$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a O-sequence. Now, let $n, m \in \mathbb{N}$, we obtain from (3.2) $x_0 \perp x_m$ or $x_m \perp x_0$, using the fact that T is $\perp$-preserving, we get $x_n \perp x_{n+m}$ or $x_{n+m} \perp x_n$, which implies by (3.1) that

$$p(x_n, x_{n+m}) \leq \psi[p(x_{n-1}, x_{n+m-1})]$$
$$\vdots \tag{3.3}$$
$$\leq \psi^n[p(x_0, x_m)]$$
$$\leq \psi^n[M]$$

or

$$p(x_{n+m}, x_n) \leq \psi[p(x_{n+m-1}, x_{n-1})]$$
$$\vdots \tag{3.4}$$
$$\leq \psi[p(x_m, x_0)]$$
$$\leq \psi^n[M],$$

where $M = \sup\{p(x, y)/x, y \in X\}$.

It can be seen that $\lim p(x_n, x_{n+m}) = 0$, then (x_n) is an O-p-Cauchy sequence. Since X is orhtogonal S-complete, we deduce that there exists $u \in X$ such that $\lim(u, x_n) = 0$, and hence by Lemma 2.4 we obtain that (x_n) converges to u with respect to u. On the other hand, T is O-continuous, hence ,$(T x_n)$ converges to Tu. By the uniqueness of the limit, we obtain $Tu = u$.

For uniqueness, let $v \in X$ a fixed point of T, hence we have either $x_0 \perp v$ or $v \perp x_0$. From the orthogonality preserving, we get $x_n \perp v$ or $v \perp x_n$ for all $n \in \mathbb{N}$. So,

$$p(v, x_n) \leq \psi(p(v, x_{n-1}))$$
$$\leq \psi^n(p(v, x_0)). \tag{3.5}$$

Using Lemma 2.4 and letting $n \to \infty$ in (3.5), we obtain $u = v$. $\square$

Corollary 3.4. *(Theorem 4.1 [1])*
Let (X, τ) be a Hausdorff topological space with a $\tau-$distance p. Suppose that X is $p-$bounded and $S-$complete. Let T be a selfmapping of X such that

$$p(Tx, Ty) \leq \psi(p(x, y)),$$

for all $x, y \in X$, where $\psi \in \Psi$. Then T has a unique fixed point.

Lemma 3.5. *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x, y) = e^{d(x,y)} - 1. \tag{3.6}$$

Then p is a τ_d-distance on X where τ_d is the metric topology.

Proof . Let (X, τ_d) be the topological space with the metric topology τ_d, let $x \in X$ and V an arbitrary neighborhood of x, then there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset V$, where $B_d(x, \varepsilon) = \{y \in X, d(x, y) < \varepsilon\}$ is the open ball.
It easy to see that $B_p(x, e^\varepsilon - 1) \subset B_d(x, \varepsilon)$, indeed:
Let $y \in B_p(x, e^\varepsilon - 1)$, then $p(x, y) < e^\varepsilon - 1$, which implies that $e^{d(x,y)} < e^\varepsilon$, and hence $d(x, y) < \varepsilon$. $\square$

Definition 3.6. *Let $(X, \perp, d)$ be an orthogonal metric space and $T : X \longrightarrow X$ be a mapping. Suppose that T is $\perp$-preserving and $\perp$-continuous such that*

$$\inf_{x \perp y, x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0. \tag{3.7}$$

Then T is called orthogonal contractive mapping.

Now, we give the first fixed point theorem.

Theorem 3.7. *Let $(X, \perp, d)$ be an O-complete orthogonal bounded metric space and $T : X \longrightarrow X$ be an orthogonal contractive mapping. Then T has a unique fixed point.*

Proof . Let $\alpha = \inf_{x \perp y, x \neq y \in X} \{d(x, y) - d(Tx, Ty)\}$ which implies that for all $x \neq y \in X$, with $x \perp y$

$$d(Tx, Ty) \leq d(x, y) - \alpha.$$

Then

$$e^{d(Tx,Ty)} \leq k e^{d(x,y)},$$

where $k = e^{-\alpha} < 1$. Also,

$$p(Tx, Ty) \leq k p(x, y), \tag{3.8}$$

for all $x, y \in X$ such that $x \perp y$, with $p(x, y) = e^{d(x,y)} - 1$ is the function mentioned in Lemma 3.5. Now , if we take $\psi(t) = kt$ for all $t \in [0, \infty)$ in Lemma 3.3, we deduce that T has a unique fixed point. $\square$

Corollary 3.8. *(Theorem 3 [11]) Let $T : X \to X$ be a mapping of a bounded complete metric space (X, d) such that*

$$\inf_{x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0. \tag{3.9}$$

Then T has a unique fixed point.

Example 3.9. *Let $X = \{0, 1, 2\}$ endowed with the usual metric $d(x, y) = |x - y|$. Consider the mapping $T : X \to X$ defined as*

$$T0 = 0, \ T1 = 0 \ and \ T2 = 1.$$

Define a relation $\perp$ on X by

$$x \perp y \ if \ and \ only \ if \ xy \in \{0, 1\}.$$

Let $x \neq y \in X$, then $xy \in \{0\}$, and hence, it easy to see that T is $\perp$-continuous and T is $\perp$-preserving. So we have the following cases:
Case 1: *$d(0, 1) - d(T0, T1) = 1$.*
Case 2: *$d(0, 2) - d(T0, T2) = 1$.*
Therefore, all conditions of Theorem 3.7 are satisfied and so T has the unique fixed point 0. On the other hand, since $d(1, 2) - d(T1, T2) = 0$, so Corollary 3.8 does not ensure the existence of the fixed point.

Now, motivated by [2, 10, 11], we define a new class of weakly contractive maps and we use Theorem 3.7 to prove a fixed point theorem for this type of maps.

Definition 3.10. *Let $T : X \to X$ be a mapping of an orthogonal metric space $(X, \perp, d)$. Suppose that T is $\perp$-preserving and $\perp$-continuous, then T will be said an orthogonal E-weakly contractive map ($\perp$-E-weakly contractive map) if for all $x, y \in X$*

$$x \perp y \ \ implies \ \ d(Tx, Ty) \leq d(x, y) - \phi[1 + d(x, y)],$$

where $\phi : [1, \infty) \to [0, \infty)$ is a function satisfying
   i) $\phi(1) = 0$,
   ii) $\inf\limits_{t>1} \phi(t) > 0$.

Theorem 3.11. *Let $T : X \to X$ be an $\perp$-E-weakly contractive map of a bounded $\perp$-complete metric space $(X, \perp, d)$. Then T has a unique fixed point.*

Proof . Let $x \neq y \in X$ such that $x \perp y$, using Definition 3.10, we obtain

$$
\begin{aligned}
0 &< \inf_{t>1} \phi(t) \\
&\leq \phi[d(x, y) + 1] \\
&\leq d(x, y) - d(Tx, Ty),
\end{aligned}
\tag{3.10}
$$

and hence, $\inf\limits_{x \perp y, x \neq y \in X} \{d(x, y) - d(Tx, Ty)\}$. Now, from Theorem 3.7, we conclude that T has a unique fixed point. $\square$

Corollary 3.12. *(Theorem 9 [11]) Let $T : X \to X$ be an E-weakly contractive map of a bounded complete metric space (X, d). Then T has a unique fixed point.*

Example 3.13. *Let $X = \{0\} \cup [1, 4]$. Suppose that $x \perp y$ if and only if $xy \leq 1$, it is easy to see that $(X, \perp)$ is an O-set. Define $T : X \to X$ by*

$$
Tx = \begin{cases} 0, & if \ x \in \{0, 1\} \\ 2x & if \ x \in (1, 2] \\ \frac{x}{2} & if \ x \in (2, 4] \end{cases}
$$

Let $x, y \in X$, $x \perp y$ implies $x, y \in \{0, 1\}$, so for all $x, y \in X$

$$x \perp y \ \ implies \ \ d(Tx, Ty) \leq d(x, y) - \phi[1 + d(x, y)],$$

where $\phi : [1, \infty) \to [0, \infty)$ is a function defined by

$$
\phi(t) = \begin{cases} 0 & if \ t = 1 \\ 1 & if \ t > 1 \end{cases} .
$$

Then T satisfies all conditions of Theorem 3.11 and 0 is the unique fixed point. Note that T does not satisfy all conditions of Corollary 3.12, indeed, $d(T1, T2) = 4 > 0 = d(1, 2) - \phi[1 + d(1, 2)]$.

4. Application

In this section, we prove the existence and uniqueness of a solution of the following differential equation:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in I := [1, \tau], \tau \in [1, \infty); \\ x(0) = a, & a \geq 2, \end{cases} \tag{4.1}$$

where $x \in X = \mathcal{C}(I)$ the space of all continuous functions from I into $\mathbb{R}$ and $f : I \times \mathbb{R} \to \mathbb{R}+$ is a continuous function satisfying the following assumptions:
There exists $M > 0$, for any $x, y \in X$ with
$x(s)y(s) \geq y(s)$ or $x(s)y(s) \geq x(s)$, we have

$$|f(s, x(s)) - f(s, y(s))| \leq \frac{1}{\tau}[|x(s) - y(s)| - M], \tag{4.2}$$

where $x(s) \neq y(s)$ for all $s \in I$.

Theorem 4.1. *Under the above assumptions the differential equation* (4.1) *has a unique solution.*

Proof . We define an orthogonal relation $\perp$ on X by

$$x \perp y \iff x(s)y(s) \geq y(s) \text{ or } x(s)y(s) \geq x(s) \quad \text{for all } s \in I. \tag{4.3}$$

From (4.3), it is easy to see that $\perp$ is an orthogonal relation on X. We endow X with the metric $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|$$

for all $x, y \in X$. Therefore, $(X, \perp, d)$ is an O-complete orthogonal metric space. Define a mapping $T : X \to X$ as follows:

$$Tx(t) := a + \int_0^t f(s, x(s))ds. \tag{4.4}$$

Now, we will show the other conditions of Theorem (3.7):
T **is generalized $\perp$ preserving.** Let $x, y \in X$ such that $x \perp y$ and $t \in I$, we obtain

$$Tx(t) := a + \int_0^t f(s, x(s))ds \geq 2,$$

and hence $Tx(t)Ty(t) \geq Ty(t)$, then

$$Tx \perp Ty.$$

T **is $\perp$ continuous.** It is clear to see From the fact $Tx(t) := a + \int_0^t f(s, x(s))ds.$ that T is an orthogonal continuous mapping.
Note that (4.1) has a unique solution if only if T has a unique fixed point.
Now, let $x, y \in X$ such that $x \perp y$ and $x \neq y$, it follows from (4.2) that for any $t \in I$

$$\begin{aligned} |T(x)(t) - T(y)(t)| &= \left| \int_0^t f(s, x(s))ds - \int_0^t f(s, y(s))ds \right| \\ &= \left| \int_0^t [f(s, x(s)) - f(s, y(s))]ds \right| \\ &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds \\ &\leq d(x, y) - M \end{aligned}$$

hence

$$d(Tx, Ty) \leq d(x, y) - M$$

for all $x, y \in X$. Then $\inf\limits_{x \perp y, x \neq y} \{d(x, y) - d(Tx, Ty)\} \geq M > 0$, which implies by Theorem 3.7 that there exists a unique solution of (4.1). $\square$

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