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Classification of singular points of perturbed quadratic systems

Asadollah Aghajani[a,*], Mohsen Mirafzal[a]

[a]*School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844-13114, Iran*

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Abstract

We consider the following two-dimensional differential system:

$$\begin{cases} \dot{x} = ax^2 + bxy + cy^2 + \Phi(x,y)\,, \\ \dot{y} = dx^2 + exy + fy^2 + \Psi(x,y)\,, \end{cases}$$

in which $\lim_{(x,y)\to(0,0)} \frac{\Phi(x,y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{x^2+y^2} = 0$ and $\Delta = (af-cd)^2 - (ae-bd)(bf-ce) \neq 0$. By calculating Poincare index and using Bendixson formula we will find all the possibilities under definite conditions for classifying the system by means of kinds of sectors around the origin which is an equilibrium point of degree two.

Keywords: Quadratic system, Classification of singular points
2010 MSC: Primary 34C05.

1. Introduction and preliminaries

The subject of classification of singular points of second order homogeneous ODE's was initiated for the first time in [4] without any explicit criteria. Another approach based on analyzing the integral curves can be found in [6]. Also some applications and related results can be find in ([1, 2, 3, 5, 7, 8]). Almost all of them are about the systems of equations without perturbations. In this paper, we present a complete characterization of singular points for a more general class of quadratic systems

*Corresponding author
Email addresses: aghajani@iust.ac.ir (Asadollah Aghajani), mirafzal@iust.ac.ir (Mohsen Mirafzal)

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with perturbations. Preliminary results, definitions and terminologies come from [9].
Consider the following second order system of equations

$$\begin{cases} \dot{x} &= ax^2 + bxy + cy^2 , \\ \dot{y} &= dx^2 + exy + fy^2 \ . \end{cases} \tag{1.1}$$

It is not hard to see that (1.1) has not a critical point $(x, y) \neq (0, 0)$, with $y \neq 0$ if and only if the following equations hold

$$\begin{cases} am^2 + bm + c = 0 , \\ dm^2 + em + f = 0 \end{cases}$$

and with $x \neq 0$ if and only if the following ones, have not a common root.

$$\begin{cases} cm^2 + bm + a = 0 , \\ fm^2 + em + d = 0 . \end{cases}$$

Each case is equivalent to

$$\Delta = (af - cd)^2 - (ae - bd)(bf - ce) \neq 0.$$

In the system

$$\begin{cases} \dot{x} = ax^2 + bxy + cy^2 + \Phi(x, y) , \\ \dot{y} = dx^2 + exy + fy^2 + \Psi(x, y) \ , \end{cases} \tag{1.2}$$

with the additional condition $\lim_{(x,y)\to(0,0)} \frac{\Phi(x,y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{x^2+y^2} = 0$ the origin is an isolated critical point. To see this, note that the following system

$$\begin{cases} ax^2 + bxy + cy^2 + \Phi(x, y) = 0 , \\ dx^2 + exy + fy^2 + \Psi(x, y) = 0 , \end{cases}$$

when $x^2 + y^2 \neq 0$ can be rewritten in the form of

$$\begin{cases} aX^2 + bXY + cY^2 + \frac{\Phi(x,y)}{x^2+y^2} = 0 , \\ dX^2 + eXY + fY^2 + \frac{\Psi(x,y)}{x^2+y^2} = 0 , \end{cases}$$

in which $X = \frac{x}{\sqrt{x^2+y^2}}$ and $Y = \frac{y}{\sqrt{x^2+y^2}}$ or

$$\overrightarrow{S}(X, Y) + \overrightarrow{F}(x, y) = 0,$$

in which $\overrightarrow{S}(X, Y) = \left(aX^2 + bXY + cY^2, dX^2 + eXY + fY^2\right)$ and $\overrightarrow{F}(x, y) = \left(\frac{\Phi(x,y)}{x^2+y^2}, \frac{\Psi(x,y)}{x^2+y^2}\right)$. When $x^2 + y^2 \neq 0$ we have $X^2 + Y^2 = 1$, therefore $\min \|S(X, Y)\| = k > 0$. Since $\lim_{(x,y)\to(0,0)} F(x, y) = 0$, implies $\|F(x, y)\| < k$ in $0 < x^2 + y^2 < \delta$, for some $\delta > 0$, then the above equation has not any solution in $0 < x^2 + y^2 < \delta$. So we have proved the following proposition.

Proposition 1.1. *The following set of conditions*

$$\begin{array}{l} \lim_{(x,y)\to(0,0)} \frac{\Phi(x,y)}{x^2+y^2} = 0 , \\ \lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{x^2+y^2} = 0, \\ \Delta = (af - cd)^2 - (ae - bd)(bf - ce) \neq 0 . \end{array}$$

implies that origin is an isolated equilibrium point for the system

$$\begin{cases} \dot{x} = ax^2 + bxy + cy^2 + \Phi(x, y) , \\ \dot{y} = dx^2 + exy + fy^2 + \Psi(x, y) . \end{cases}$$

Lemma 1.2. *The Poincare index of (1.2) is*

$$J = \begin{cases} -2 & ae - bd < 0 \ \textit{and} \ \Delta < 0 \\ 0 & \Delta > 0 \\ 2 & ae - bd > 0 \ \textit{and} \ \Delta < 0 \end{cases}$$

Proof . First note that the Poincare index of following system

$$\begin{cases} \dot{x} = ax^2 + bxy + cy^2, \\ \dot{y} = dx^2 + exy + fy^2. \end{cases}$$

is calculated by

$$\begin{aligned} J &= \tfrac{1}{2\pi} \oint_c d[\arctan(\tfrac{ax^2+bxy+cy^2}{dx^2+exy+fy^2})] \\ &= \tfrac{1}{\pi} \int_{-\infty}^{+\infty} d[\arctan(\tfrac{a\xi^2+b\xi+c}{d\xi^2+e\xi+f})] \ , \end{aligned}$$

which has only the values $2, -2, 0$ when $\Delta \neq 0$. On the other hand

$$J = J(a,b,c,d,e,f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(ae-bd)\xi^2 - 2(af-cd)\xi + (bf-ce)}{(a\xi^2+b\xi+c)^2 + (d\xi^2+e\xi+f)^2} d\xi$$

is continuous on $\mathbb{R}^6 - \{\Delta = 0\}$. This set has three disjoint open connected subsets:

$$\mathbb{R}^6 - \{\Delta = 0\} = \{\Delta > 0\} \cup \{\Delta < 0, ae - bd < 0\} \cup \{\Delta < 0, ae - bd > 0\}.$$

Therefore the proof is complete for system (1.1). By the Roche theorem we have the same conclusion for (1.2). □

2. Main results

The orbits of system (1.2) in polar coordinates satisfy the following equation

$$r\frac{d\theta}{dr} = \frac{G(\theta) + o(1)}{H(\theta) + o(1)} \quad \text{as} \quad r \to 0$$

in which

$$G(\theta) = cos\theta(d\cos^2\theta + e\sin\theta\cos\theta + f\sin^2\theta) - \sin\theta(a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta)$$

and

$$H(\theta) = \sin\theta(d\cos^2\theta + e\sin\theta\cos\theta + f\sin^2\theta) + \cos\theta(a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta).$$

It is clear that the equations $G(\theta) = 0$ and $H(\theta) = 0$ have not a common root. Therefore the necessary condition for $\theta = \theta_0$ to be a critical direction is $G(\theta_0) = 0$. By some elementary calculations, we can find that there are two, four, or six roots for $G(\theta) = 0$, which at least two of them have odd multiplicity, then we have at least two orbit which tends to zero along these directions. So the origin is not a focus or center. Let $\theta = \theta_k$ be a real root of the characteristic equation $G(\theta) = 0$ and consider the sectors

$$\triangle\widehat{OA_kB_k}: \quad |\theta - \theta_k| \leqslant \varepsilon, \quad r \leqslant r_0,$$

such that they have no points in common except the origin. In the outside of these sectors, all orbits move from one radial side to another radial side and cannot tend to the critical points in there. When $\theta = \theta_k$ the real roots of the characteristic equation $G(\theta) = 0$, have odd multiplicity l and if $G^{(l)}(\theta_k)H(\theta_k) > 0$ then all orbits in the sector tend to the critical point along the direction $\theta = \theta_k$. In this case, we call the sector, normal region of the first type. Also, if $G^{(l)}(\theta_k)H(\theta_k) < 0$, then there exist a point or closed subarc in $\widehat{A_kB_k}$ such that any orbit starting from there will tend to the critical point along the direction $\theta = \theta_k$, we call the sector in this case a normal region of the second type. The problem of determining whether there is one or infinitely many orbits tending to the critical point O along $\theta = \theta_k$, will be called the first classification problem. When l is even, there is no orbit in $\widehat{A_kB_k}$ tending to the critical point O or there exists a $P \in OB_k$ or $\widehat{A_kB_k}$, such that for any $R \in OP$ or $R \in OB_k \cup \widehat{B_kP}$, hence the orbit stating from R must tend to the critical point O along the direction $\theta = \theta_k$, then in this case, we call the sector a normal region of the third type. The problem that there may be infinitely many orbits or no orbit tending to the critical point O along $\theta = \theta_k$, will be called the second classification problem.

Theorem 2.1. *Suppose* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) < 0$*,* $ae - bd < 0$ *and*

$$\lim_{(x,y)\to(0,0)} \frac{\Phi(x,y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{x^2+y^2} = 0.$$

Then the origin is the isolated equilibrium point of (1.2) with six hyperbolic sectors.

Proof . By Lemma (1.2), we have $J = -2$, then the Poincare-Bendixson formula

$$J = 1 + \frac{e-h}{2}$$

gives that $e = 0$ and $h = 6$. Thus (1.1) has six straight line orbits and the characteristic equation $G(\theta) = 0$ has exactly six simple roots in which around each of them there is a normal region of the second type. Therefore, we have six hyperbolic sectors for (1.2). □

Theorem 2.2. *Suppose* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) > 0$ *and*

$$\lim_{(x,y)\to(0,0)} \frac{\Phi(x,y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{x^2+y^2} = 0.$$

Then the origin is the isolated equilibrium point of (1.2) with two hyperbolic sectors, no elliptic sector and the others are parabolic.

Proof . By Lemma (1.2), we have $J = 0$, then Poincare-Bendixson formula

$$J = 1 + \frac{e-h}{2}$$

gives that $(e,h) = (0,2)$, $(1,3)$ or $(2,4)$. In (1.1) because of symmetry, e and h are even numbers, therefore the second case is impossible. Also if we have $e = 2$ and $h = 4$ we must have six sectors and six straight line orbits which normal regions about two of them are of the third type. Therefore the related directions must have even multiplicity in characteristic equation. This contradicts that the characteristic equation has at most six roots, with calculation of multiplicities, therefore in (1.1) we must have $e = 0$ and $h = 2$, and the proof is completed for (1.1). Now we will prove for (1.2). When the characteristic equation $G(\theta) = 0$ has two roots, then in (1.1) there are two straight line orbits and two hyperbolic sectors therefore the sectors about the roots are normal regions of the second type and there are two hyperbolic sectors for (1.2) and no elliptic and parabolic sector. When we have four roots, because two of them are of the multiplicity two we have two normal regions of the third type and because we have two hyperbolic sectors in (1.1), the other two roots have normal regions of the second type therefore we have in (1.2) two hyperbolic sectors, no elliptic sector and by the second classification problem may or may not have prabolic sectors. When we have six roots, since all of them are simple, the normal regions about each of them are of the first and second type, because we have two hyperbolic sectors in (1.1), there are consecutive roots θ_1, θ_2 and $\theta_1 + \pi, \theta_2 + \pi$ such that the normal regions about them are of the second type. Therefore, the other two directions have normal region of the first type about themselves. Then we have two hyperbolic sectors, four parabolic sectors and no elliptic sector in (1.2). □

Theorem 2.3. *Suppose* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) < 0$ *and* $ae - bd > 0$ *and*

$$\lim_{(x,y)\to(0,0)} \frac{\Phi(x,y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\Psi(x,y)}{x^2+y^2} = 0.$$

Then the origin is the isolated equilibrium point of (1.2) with two elliptic sectors, no hyperbolic sector and the others are parabolic.

Proof . By Lemma (1.2), we have $J = 2$, then Poincare-Bendixson formula

$$J = 1 + \frac{e-h}{2}$$

gives that $(e, h) = (2, 0)$, $(3, 1)$ or $(4, 2)$. In system (1.1), because of the symmetry, e and h are even numbers, therefore the second case is impossible. Also if we have $e = 4$ and $h = 2$ we must have six sectors and six straight line orbits which normal regions around two of them are of the third type. Hence, the related directions must have even multiplicity in characteristic equation. This contradicts with the fact that the characteristic equation has at most six roots, with algebraic multiplicity, therefore in (1.1) we must have $e = 2$ and $h = 0$, and the proof is completed for (1.1). Now we will prove the result for the system (1.2). When the characteristic equation $G(\theta) = 0$ has two roots, then in (1.1) there are two straight line orbits and two elliptic sectors, thus the sectors around the roots are normal regions of the first type and there are two elliptic sectors for (1.2) and no hyperbolic or parabolic sector. When we have four roots, since two of them are of the multiplicity two we have two normal regions of the third type, and since we have two elliptic sectors in (1.1), the other two roots have normal regions of the first type. Therefore, we have in (1.2), two elliptic sectors, no hyperbolic sector, and by the second classification problem may or may not have parabolic sectors. When we have six roots, since all of them are simple, the normal regions around each of them are of the first and second type, because we have two elliptic sectors in (1.1), there are consecutive roots θ_1, θ_2 and $\theta_1 + \pi, \theta_2 + \pi$ such that the normal regions around them are of the first type. Therefore, the other two directions have normal region of the second type around themselves. Then we have two elliptic sectors, four parabolic sectors and no hyperbolic sector in (1.2) □

3. Applications and examples

The method that we employed for the proofs of main results in the previous section also classify the system by means of behavior of orbits in the vicinity of the origin, as we shall see in the following examples.

Example 3.1. *Consider the system*

$$\begin{cases} \dot{x} = -x^2 + y^2\,, \\ \dot{y} = 2xy\,, \end{cases}$$

which is a special case of (1.1) with $a = -1, b = 0, c = 1, d = 0, e = 2, f = 0$. *Therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) < 0$ *and* $ae - bd < 0$, *then by Theorem (2.1) there are six hyperbolic sectors. Also* $G(\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta$, *then there are six critical directions and along each direction we have one straight line orbit which tends to the origin, correspond to* $\theta_1 = 0$, $\theta_2 = \frac{\pi}{3}$, $\theta_3 = \frac{2\pi}{3}$ $\theta_4 = \pi$, $\theta_5 = \frac{4\pi}{3}$ *and* $\theta_6 = \frac{5\pi}{3}$. *Moreover, we have* $H(\theta) = 3\sin^2\theta\cos\theta - \cos^3\theta$ *and* $G'(\theta) = 3\cos^3\theta - 9\sin^2\theta\cos\theta$. *Then* $G'(\theta_i)H(\theta_i) < 0$ *for* $i = 1, 2, ..., 6$, *therefore we have six isolated rays or six normal regions of the second type. (Figure 1)*

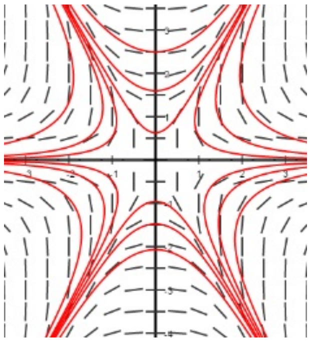

Figure 1:

Example 3.2. *Now consider the system*

$$\begin{cases} \dot{x} = y^2, \\ \dot{y} = x^2. \end{cases}$$

Here we have $a = 0, b = 0, c = 1, d = 1, e = 0, f = 0$. *Therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) > 0$, *then by Theorem (2.2) there are two hyperbolic sectors. Also* $G(\theta) = \cos^3\theta - \sin^3\theta$, *so there are two critical directions and along each direction we have one straight line orbit which tends to the origin, correspond to* $\theta_1 = \frac{\pi}{4}$ *and* $\theta_2 = \frac{5\pi}{4}$. *Moreover, we have* $H(\theta) = \sin\theta\cos\theta(\cos\theta + \sin\theta)$ *and* $G'(\theta) = -3\sin\theta\cos^2\theta - 3\cos\theta\sin^2\theta$. *Then* $G'(\frac{\pi}{4})H(\frac{\pi}{4}) < 0$ *and* $G'(\frac{5\pi}{4})H(\frac{5\pi}{4}) < 0$, *thus we have two isolated rays or two normal regions of the second type. (Figure 2)*

Example 3.3. *Consider the system*

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = xy + y^2. \end{cases}$$

Here we have $a = 1, b = 0, c = 0, d = 0, e = 1, f = 1$ *therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) > 0$, *then by Theorem (2.2) there are two hyperbolic sectors. Also* $G(\theta) = sin^2\theta cos\theta$, *then there are four critical directions and along each direction we have one straight line orbit which tends to the origin, correspond to* $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, $\theta_3 = \pi$ *and* $\theta_4 = \frac{3\pi}{2}$. *Here, the roots* $\theta_1 = 0$, $\theta_3 = \pi$ *are of multiplicity two. Then the normal reign around them are of the third type. Moreover, we have* $H(\theta) = \sin^2\theta\cos\theta + \sin^3\theta + \cos^3\theta$ *and* $G'(\theta) = 2\sin\theta\cos^2\theta - \sin^3\theta$. *Then* $G'(\frac{\pi}{2})H(\frac{\pi}{2}) < 0$ *and* $G'(\frac{3\pi}{2})H(\frac{3\pi}{2}) < 0$, *imply that we have two isolated rays or two normal regions of the second type. (Figure 3)*

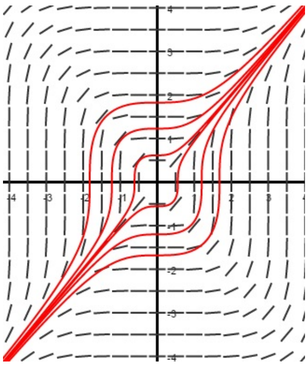

Figure 2:

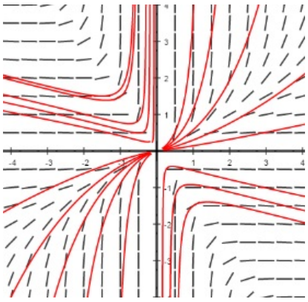

Figure 3:

Example 3.4. *Now consider the system*

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = -xy + 2y^2. \end{cases}$$

We have $a = 1, b = 0, c = 0, d = 0, e = -1, f = 2$. *Therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) > 0$, *then by the Theorem (2.2) there are two hyperbolic sectors. Also* $G(\theta) = sin\theta cos\theta(2sin\theta - 2cos\theta)$, *then there are six critical directions and along each direction we have one straight line orbit which tends to the origin (correspond to* $\theta_1 = 0$, $\theta_2 = \frac{\pi}{4}$, $\theta_3 = \frac{\pi}{2}$ $\theta_4 = \pi$, $\theta_5 = \frac{5\pi}{4}$, $\theta_6 = \frac{3\pi}{2}$*). Moreover, we have* $H(\theta) = -\sin^2\theta\cos\theta + 2\sin^3\theta + \cos^3\theta$ *and* $G'(\theta) = -2\cos^3\theta - 2\sin^3\theta + 4\sin^2\theta\cos\theta + 4\sin\theta\cos^2\theta$. *Then* $G'(\theta_i)H(\theta_i) < 0$ *for* $i = 1, 3, 4, 6$, *therefore we have four isolated ray or four normal region of second type and* $G'(\theta_i)H(\theta_i) > 0$ *for* $i = 2, 5$, *therefore we have two nodal rays or two normal regions of the first type. (Figure 4)*

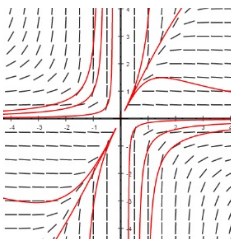

Figure 4:

Example 3.5. *Consider the system*

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = 2xy. \end{cases}$$

We have $a = 1, b = 0, c = -1, d = 0, e = 2, f = 0$. *Therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) < 0$ *and* $ae - bd > 0$, *then by the Theorem (2.3) there are two elliptic sectors.*
Also $G(\theta) = \sin\theta\cos^2\theta + \sin^3\theta$, *then there are two critical directions and along each direction we have one straight line orbit which tends to the origin (correspond to* $\theta_1 = 0$, $\theta_2 = \pi$*). Moreover, we have* $H(\theta) = \cos^3\theta + \cos\theta\sin^2\theta$ *and* $G'(\theta) = \cos^3\theta + \cos\theta\sin^2\theta$. *Then* $G'(0)H(0) > 0$ *and* $G'(\pi)H(\pi) > 0$, *therefore we have two nodal rays or two normal regions of the first type. (Figure 5)*

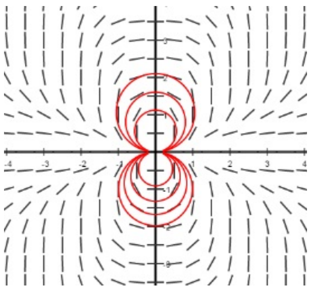

Figure 5:

Example 3.6. *Consider the system*

$$\begin{cases} \dot{x} = x^2 + xy, \\ \dot{y} = xy + \frac{1}{2}y^2. \end{cases}$$

We have $a = 1, b = 1, c = 0, d = 0, e = 1, f = \frac{1}{2}$. *Therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) < 0$ *and* $ae - bd > 0$, *then by the Theorem (2.3) there are two elliptic sectors. Also,* $G(\theta) = -\frac{1}{2}sin^2\theta cos\theta$ *then there are four critical directions and along each direction we have one straight line orbit which tends to the origin,* $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, $\theta_3 = \pi$, $\theta_4 = \frac{3\pi}{2}$. *Here, the roots* $\theta_1 = 0$, $\theta_3 = \pi$ *are of multiplicity two. Then the normal reign about them are of the third type. Moreover, we have* $H(\theta) = \sin^2\theta\cos\theta + \sin\theta\cos^2\theta + \cos^3\theta + \frac{1}{2}\sin^3\theta$ *and* $G'(\theta) = -\sin\theta\cos^2\theta + \frac{1}{2}\sin^3\theta$. *Then* $G'(\frac{\pi}{2})H(\frac{\pi}{2}) > 0$ *and* $G'(\frac{3\pi}{2})H(\frac{3\pi}{2}) > 0$, *therefore we have two nodal ray or two normal region of the first type. (Figure 6)*

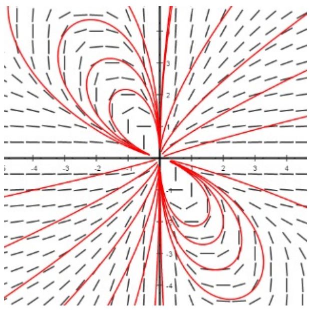

Figure 6:

Example 3.7. *Consider the system*

$$\begin{cases} \dot{x} = 2x^2 - 2y^2, \\ \dot{y} = xy, \end{cases}$$

we have $a = 2, b = 0, c = -2, d = 0, e = 1, f = 0$. *Therefore* $\Delta = (af - cd)^2 - (ae - bd)(bf - ce) < 0$ *and* $ae - bd > 0$, *then by the Theorem (2.3) there are two elliptic sectors. Also* $G(\theta) = -sin\theta cos^2\theta + 2sin^3\theta$

then there are six critical directions and along each direction we have one straight line orbit which tends to the origin, $\theta_1 = 0$, $\theta_2 = arctg(\frac{1}{\sqrt{2}})$, $\theta_3 = \pi - arctg(\frac{1}{\sqrt{2}})$ $\theta_4 = \pi$, $\theta_5 = \pi + arctg(\frac{1}{\sqrt{2}})$, $\theta_6 = 2\pi - arctg(\frac{1}{\sqrt{2}})$*. Moreover, we have* $H(\theta) = -\sin^2\theta\cos\theta + 2\cos^3\theta$ *and* $G'(\theta) = -\cos^3\theta + 8\sin^2\theta\cos\theta$*. Then* $G'(\theta_i)H(\theta_i) < 0$ *for* $i = 1, 4$*, therefore we have two isolated ray or two normal region of the second type, also* $G'(\theta_i)H(\theta_i) > 0$ *for* $i = 2, 3, 5, 6$*, therefore we have four nodal rays or four normal regions of the first type. (Figure 7)*

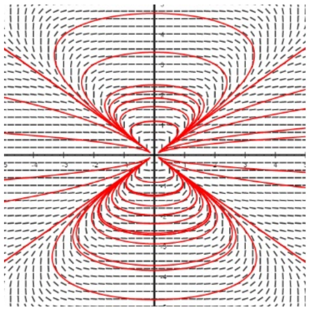

Figure 7:

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