Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1855-1860
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5321

Studying the A4-graphs for elements of order three in tits group T and Mathieu group M

Zainab Hasan Msheree[a,*], Mohammed Mukheef Abed[a], Wissam Fadhel Abid[b]

[a] *Middle Technical University, Technical Instructors Training Institute, Iraq*
[b] *Middle Technical University, Institute of Technology, Baghdad, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

Assume that X is a subset of the finite group G. The A4-graph is known as a simple graph denoted by $\mathcal{A}_4(G,X)$ having X as a vertex set and two vertices $x, y \in X$, is linked by an edges if $x \neq y$ and $xy^{-1} = yx^{-1}$. In this paper, we consider $\mathcal{A}_4(G,X)$ when G is either Tits group T or Mathieu group M_{20} and X is G-conjugacy class of elements of order three. Valuable results reached, for example, disc structure, girth, clique number, and diameters of the A4-graph.

Keywords: Finite simple groups, A4-graph, connectivity, cliques.
2020 MSC: 20D05, 68R01, 05C40, 05C69.

1. Introduction

Analyzing the action of a group on a graph has become one of the current techniques for examining the structure of a group that has recently been verified to be of outstanding quality. Similarly, equivalent techniques are used to analyze the algebraic structures of rings [1]. In terms of algebraic properties of finite groups, the elements of order 3 are crucial in terms of the fundamental key roles in describing involution in finite groups. For example, many of the research acquired in this way may be seen in [4, 5]. Suppose that G is finite group with X is a class of elements of order 3 in G. In [1] Aubad introduced the A4-graph of finite group on G as follows: the A4-graph is denoted by $\mathcal{A}_4(G,X)$ with vertex set X and two vertices $x, y \in X$ is connected by an edges if x not equal to y and xy^{-1} equal to yx^{-1}. In that paper many result related to A4-graph is given. For Example, describing the relationship between the alternating group A4 and the elements of order 3 in the exceptional group

*Corresponding author
Email addresses: Zainab.hasan@mtu.edu.iq (Zainab Hasan Msheree), Mohammed.mukheef@mtu.edu.iq (Mohammed Mukheef Abed), Wissam@mtu.edu.iq (Wissam Fadhel Abid)

Received: March 2021 *Accepted:* July 2021

${}^3D_4(2)$. One thing to keep in mind regarding $\mathcal{A}_4(G,X)$ is that the alternating group of degree 4 is created by x, y. As a result, the attitude of the A_4 group may be studied inside the bigger groups that include it, as well as the manner by which it is formed there.

Now, through the body of this article, we may let G is either Tits group T or Mathieu group M_{20} with t be an element of order 3 in G. assume that $X = t^G$ class of elements of order 3 in G. The objectives of this article is to look at some of the characteristics of the A4-graph $\mathcal{A}_4(G,X)$. It was worthwhile to evaluate the discs design and measure the diameters for such graphs.
Suppose that $r \in X$ the i^{th} disc of r, typified by $\Delta_i(r)$, $i \in \mathbb{N}$ is referred to as

$$\Delta_i(r) = \{y \in X | d(r,y) = i\}$$

According to the graph $\mathcal{A}_4(G,X), d(r,y)$ is adopted as the standard metric for measuring distance. It is indeed worth noting that $\Delta_i(r)$ exactly separates into a union of certain $C_{G(t)}$–orbits.(once G acts on X through conjugation). As a result, we can calculate $C_{G(t)}$ –orbits of X in carried out to investigate the features of the $\mathcal{A}_4(G,X)$. It seems to be interesting to note that G acting on X via conjugation is included G in the group of graph automorphisms of $\mathcal{A}_4(G,X)$, also upon its vertices of $\mathcal{A}_4(G,X), G$ is transitive. Diam $\mathcal{A}_4(G,X)$ denotes the diameter of the $A4$-graph, which would also be described as

$$Diam\ \mathcal{A}_4(G,X) = \max_{r \in X}\{i | \Delta_i(r) \neq \phi\ and\ \Delta_{i+1}(r) = \phi\}$$

Finally, for the name of G-conjugacy classes, we would also use the Atlas [1].

2. Discs Structure of $\mathcal{A}_4(G,X)$

To investigate the structure of the discs of the A4-graphs. The next table contains information about the order 3 classes in Tits group T or Mathieu group M_{20} . This information including class size, $C_G(t)$ structure, and permutation rank.

Table 1: The Classes Information of $\mathcal{A}_4(G,X)$

Group	$X = t^G$	$\|X\|$	Permutation Rank	$C_G(t)$ structures
T	3A	166400	1564	$\big((C_3 \times C_3) : C_3\big) : C_4$
M_{20}	3A	320	108	C_3

Dealing computationally using 1600-potints representation for the groups T and 20-points representation for the groups M_{20}, as describes in details with the OnLine Atlas [3]. Computational style was used in association with Gap [6] and the OnLine Atlas to obtain the findings of A4-graphs for the above finite groups

2.1. Discs Structure of $C(T, 3A)$

Check out the following set where C is an arbitrary G-Conjugacy class:

$$X_C = \{x \in X |\ tx \in X\}.$$

We note that if $X_C \neq \phi$, so it is a union of $C_G(t)$-orbits of X We can see that if $X_C \neq \phi$, then it would be a union of $C_G(t)$-orbits for the class X. (where $C_G(t)$ has a conjugation action on the class

X). As the path of X_C divided into $C_G(t)$-orbits, it might be useful to determine which discs of t possess the vertices in X_C. It is indeed beneficial to understand how big X_C is since it helps us figure out what class structure constants to utilize.

The lengths of the following set are indicated as class structure constants:

$$\{(r_1, r_2) \in C_1 \times C_2 | \; r_1 r_2 = r\}$$

Where r is an arbitrary element in the class G-conjugacy class C_3 and C_2,C_3 are also G-conjugacy classes. The complex character table of the group G, which is contained in the Atlas and accessible digitally in the computer algebra packages of the standard libraries of GAP, is now used to precisely measure the aforementioned constants.

If we assume C_1 = C, C_2 = X = C_3 and $r = t$, then, in this situation,

$$|X_C| = \frac{|G|}{|C_G\,(t)|\,|C_G(h)|} \sum_{i=1}^{n} \frac{\chi_i(h)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(1)}$$

Where $h \in C$ and $\chi_1, \chi_2, \ldots, \chi_n$ are the complex irreducible characters of the group G.

The data we gathered to look at the disc structure of $A_{\mathbf{4}}$ (T,3A), as well as the size of the $C_G(t)$(t)-orbits as they acts on X_C by conjugation , for the non-empty set X_C and G-conjugacy classes C of tx such that $x \in \Delta_i(t)$; $i \in \mathbb{N}$.

Exponential terminology is being used to represent the multiplicity of a size in the tables that follow. In this part, we go over how to retrieve the tables in more depth. We utilize the class names from the Atlas, as mentioned earlier. We compress the letter portion of the class name even further since we intend to combine these classes, and their characters are in alphabetical order to make the work easier. For example, in Table 2, when $G \cong T$ and $X = 3A$, 12AB is short-hand for $12A \cup 12B$. We also note that in Table 2 that the entry $((108)^{10}, (108)^{100}, (108)^{10})$ in the class name of 12AB means X_{12AB} is a union of 120 $C_G(t)$-orbits with size 108 such that 10 of them in $\Delta_i(t)$ with $i \in \{2, 4\}$ and 100 of them in $\Delta_3(t)$.

Table 2: Discs structure of $A_4(T, 3A)$

Class Name	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$
1A			1	
2A			27	
2B		54^2		
3A	$27, 54^2$		$12^2, 36^6, 108^8$	
4AB			108^8	
5A		108^6	108^{34}	108^2
6A		108^{14}	$27^2, 54^2, 108^{114}$	$27^2, 54^4$
8AB		108^6	108^{54}	
8CD		108^8	108^{76}	108^{12}
10A		108^8	108^{104}	108^8
12AB		108^{10}	108^{100}	108^{10}
13AB		108^{15}	108^{100}	108^8
16ABCD		108^4	108^{82}	108^{10}

A very remarkable result can be seen for the above table that the distance between t and $x \in A_4(G, X)$ is assorted by using the G-conjugacy class C such that $tx \in C$ Furthermore, the class C is one of the following G-conjugacy classes $\{1A, 2AB\}$.

2.2. Discs Structure of $C(M_{20}, 3A)$

From Table 2.1 we note that the size of the class 3A is 320 with permutation rank 108 $C_G(t)$(t)-orbits such that $C_G(t) \equiv C_3$. Since the size of 3A is adequate to deal with it directly to generated the $A_4(M_{20}, 3A)$ then we may use the gap package YAGS [7] (especially **GraphByRelation**) to create the A4-graph. Then information we obtain about the discs structure of the $A_4(M_{20}, 3A)$ describe is following:
For a fixed representative $t \in 3A \textit{ in } M_{20}$, the $\mathcal{A}_4(M_{20}, 3A)$ spite into three discs where $|t = \Delta_0(t)| = 1$, $|\Delta_1(t)| = 39$, $|\Delta_2(t)| = 264$ *and* $|\Delta_3(t)| = 16$. Furthermore, for a fixed $x_1 \in \Delta_1(t)$ there are 14, 24 vertices in $\Delta_1(t)$ *and* $\Delta_2(t)$ linked with x_1 respectively. Also, for $x_2 \in \Delta_2(t)$ there are 34,1 and 4 vertices in $\Delta_1(t), \Delta_2(t)$ *and* $\Delta_3(t)$ linked with x_2 respectively. Also, for $x_3 \in \Delta_3(t)$ there are 24,15 vertices in $\Delta_2(t)$ *and* $\Delta_3(t)$ linked with x_3 respectively. This can be seen in the following figure:

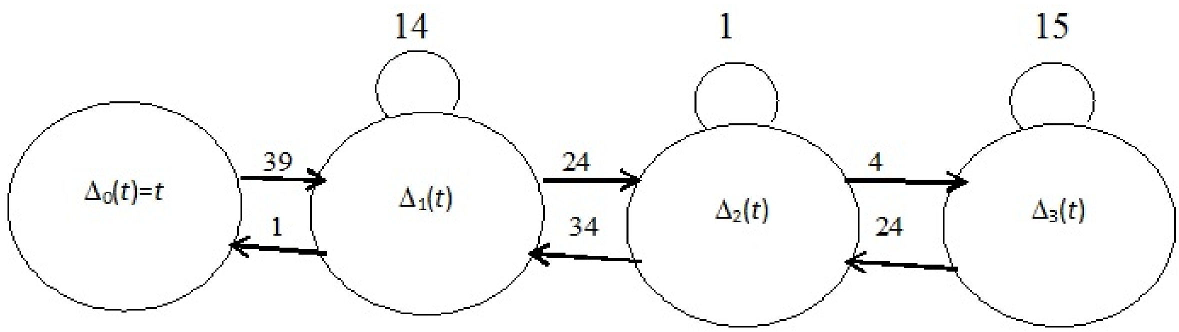

Figure 1: Discs structure of $A_4(M_{20}, 3A)$

3. Girth and Clique Number $A_4(G, X)$

The girth and clique number of $\mathcal{A}_4(G, X)$ graph is equivalent to the girth and clique number of the graph $\mathcal{A}_4(\Delta_1(t) \cup \{t\}, X)$ for X a G-conjugacy class of elements of order 3 in T or M_{20} [?]. Then by using this result there are 136 and 40 vertices when G isomorphic to T or M_{20} respectively in these A4-graphs. We only draw $\mathcal{A}_4(\Delta_1(t) \cup \{t\}, 3A)$ when $G \cong M_{20}$, as the number of vertices is small enough.

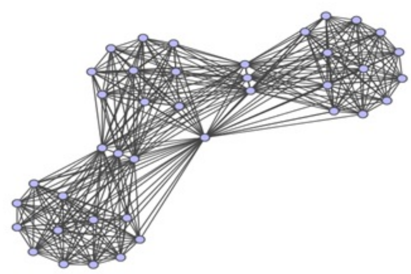

Figure 2: The graph $\mathcal{A}_4(\Delta_1(t) \cup \{t\}, X)$ when $X = 3A \in M_{20}$

Then we conclude the size of the girth and clique number of the $\mathcal{A}_4(G,X)$ in the next table:

Table 3: Girth and cliques number of $\mathcal{A}_4(G,X)$

Graph	Girth	Clique Number
$\mathcal{A}_4(T,3A)$	3	16
$\mathcal{A}_4(M_{20},3A)$	3	16

Thus in both A4-graphs the girth and clique number are equal.

4. Main Theorem

In the following theorem we describe the discs structure and diameters of the$\mathcal{A}_4(G,X)$ for our interesting groups

Theorem 4.1. *Let G be isomorphic to one of T or M_{20} groups. Then $\mathcal{A}_4(G,X)$ is connected with the following properties:*

1. *$\Delta_i(t)$ sizes of $\mathcal{A}_4(G,X)$ are presented in Table 4.*
2. *In case $(G,X)=(T,3A)$ then Dim $\mathcal{A}_4(G,X)=4$.*
3. *In case $(G,X)=(M_{20},3A)$ then Dim $\mathcal{A}_4(G,X)=3$.*

Table 4: The Discs for $\mathcal{A}_4(G,X), G\cong T$ *and* $G\cong M_{20}$

G	$X=t^G$	$\lvert\Delta_1(t)\rvert$	$\lvert\Delta_2(t)\rvert$	$\lvert\Delta_3(t)\rvert$	$\lvert\Delta_4(t)\rvert$
T	*3A*	*135*	*13284*	*139912*	*13068*
M_{20}	*3A*	*39*	*264*	*16*	

Proof . *By using the full information about discs structure of the $A_4(G,X)$ which are provided in section 2. Then applied these information to give the proof the aforementioned theorem.* □

Corollary 4.2. *Let $G\cong M_{20}$ or Tits group T, and let t be a random elements of order 3 in G. Then there are 135 and 39 subgroups isomorphic to A4 of the groups M_{20} or T respectively, generated by t and particular element in t^G.*

5. Conclusion

The altering group structure inside the groups T and M_{20} is analyzed. For that purpose we employed the A4-graph on the G-conjugacy class of elements of order 3 in these group. an interesting results have obtained. For example, discs structure, girth, clique number and the diameters A4-graphs.

For an open problem we may consider the $A4$-graph for the symmetric group S_n or Alternating group A_n. Also one can study the $A4$-graph for the linear groups or the classical groups. Studying the alternating group $A4$ inside the dihedral group can be also consider by analyzing the structure $A4$-graph .

References

[1] A. Aubad, S. kadem, and A. H. Majeed,*The non-zero divisor graph of a ring*, IJPAM. 43 (2020) 975–983.
[2] C. Cedillo, R. MacKinney-Romero, M.A. Pizaa, I.A. Robles and R. Villarroel-Flores, *Yet Another Graph System, Yags*, Version 0.0.5., 2020.
[3] H. Conway, R. T. Curtis, S. P. Norton and R. A. Parker, *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups*, Oxford Clarendon Press, 1985.
[4] A. Maksimenko and A. Mamontov, *The local finiteness of some groups generated by a conjugacy class of order 3 elements*, Siberian Math. J. 3 (2007) 508–518.
[5] S. M. Kasim and A. Nawawi, *On diameter of subgraphs of commuting graph in symplectic group for elements of order three*, Sains Malays. 2 (2021) 549–557.
[6] J. Tripp, I. Suleiman, S. Rogers R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, A. Wilson and P. Walsh, *A World Wide Web Atlas of Group Representations*, 2021.
[7] The GAP Group, *GAP Groups, Algorithms, and Programming*, Version 4.11.1, 2021.