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Qualitative study of an eco-toxicant model with migration

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Abstract

In this article, an ecology toxicant food chain system with Lotka-Volterra functional response for predator population includes prey protection zone has been suggested and studied. Toxins are excreted by all organisms as a form of defence. The prey follows the logistic growth law. The equilibrium points have been established. The analytic approach has been used to investigate the local stability for each acceptable equilibrium point. The global dynamics of this model was studied using the Lyapunov function. Lastly, numerical simulations and graphical illustrations were used to back up our analytic results.

Keywords: Eco-toxicant model, Food chain, Protection zone, Predatory zone.
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1. Introduction

The dynamical a link between predators and they're prey is a major topic in ecological and mathematical ecology because of it is universality and significance [3, 24]. They are used to tackle a variety of difficult and unpredictable problems. As a result, it is regarded as an alternative approach for enhancing our understanding of environmental physical and biological processes [16]. Mathematics was a major influence on the modeling and understanding of biological phenomena over the last few decades. In contrast, biologists have posed a number of difficult problems to mathematicians, which have mirrored advances in the theory of nonlinear differential equations. In the field of theoretical population dynamics, differential equations of this type have long been important. One of the

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most well-known applications of mathematics to biology is differential equation models for species interactions [18].

From the beginning, ecological science has been interested in prey–predator models. It was quickly discovered that depending on the model parameters, the prey–predator system would exhibit a variety of dynamical behaviors, including steady states, oscillations, and bifurcations. Many mathematicians and ecologists have researched the population dynamics [28]. In (1925) Lotka and Volterra in (1931) suggested the first the prey predator model on this a framework in which to describe the interaction of species, but in (1992) the first step was taken by Berryman [23]. Predation is a fundamental form of interaction that has an effect on all organisms' population dynamics. The predator–prey relationship is ubiquitous in nature, so it's no surprise that it's one of mathematical biology's most popular topics [4].

The effects of toxic contaminants on ecosystem dynamics are one of the most significant issues. Any anthropogenic toxic agent released into the atmosphere as a result of human activities is classified as a toxin. A toxin, unlike poisonous agents, is a poison that is formed naturally within an organism [19]. From an environmental standpoint, the impact of toxic materials on ecological are a significant issue [5]. Various researchers have studied effects of toxicants on biological species using mathematical models suggested by Hallam and Clark [11], Hallam [12, 13], Hallam and De Luna [14], De Luna and Hallam [6], Freedman and Shukla (1991), Huaping and Ma (1991), Shukla and Dubey [32]. However, the majority of these models focus on single-species or two-species ecological communities in general, with no particular emphasis on terrestrial or aquatic ecosystems. Eco-toxicological issues in the marine world have begun to be studied mathematically. In more recent years, eco-toxicological effects of toxicants released by marine biological organisms themselves are of particular interest to researchers [20]. The toxin produced by one species may have an effect not only on that species' growth but also on that of other species [7]. Similarly, as human needs develop, factories create massive amounts of toxic chemicals, which are then released into marine waters. Toxic compounds have a negative impact on the ocean climate [30].

A food chain is a series of links in the food web that starts with producer organisms and ends with predator, insectivorous or decomposing species at the apex, few of them are associated with living in a wild environment, unlike the vast majority of research on the aquatic environment, which will be the subject of our study [21]. Many applied mathematicians and ecologists have been studying three species food chain systems in recent years. Hang and colleagues, analyzed and founded an experimental marine food chain of three levels 'microalgae - zooplankton - fish' to study the effect of feeding selection on the convey of methyl mercury 'MeHg' through the food chain system [25].

Migration occurs in a prey-predator system with multiple patches Due to many reasons such as competition, age, sex; shortage of food, climate, and season, a species may relocate from one patch to another. Making it one of the most common phenomena in nature [29]. Many zooplankton species, for example, travel vertically every day due to light and food in aquatic environments. Some species migrate downward during the day to avoid fish predation, while others migrate upward at night to eat phytoplankton [1, 27]. The interaction between patches has important effects for population stability and lifespan., according to empirical studies, for example [9, 33, 34]. The impact of these interactions, such as migration, has received considerable attention from researchers in ecosystems. Different migration rates were considered by some researchers both predators and prey. For example, the researchers created a two patches model with predator and prey migration [2]. The predator's migration is assumed to be dependent based on the prey population in each patch, while prey migration is assumed to be constant. They determined upper and lower population bounds as well positive and border equilibrium points' stability and instability [10]. A model was created of two -patches in which The predator in the higher density patch migrates to the lower density area, whereas the prey does not migrate [17]. The authors demonstrated the existence of a purely positive equilibrium in which the predator migrates at a constant rate while the prey migration is dependent on the predator density [26]. The prey (predator) migration rate was presumed to be dependent on the predator (prey) density, and it was shown that for a broad class of density-dependent migration laws, there exists a special and stable migration equilibrium [8]. Migration is also a critical demographic event that occurs in all animals. Migration is the physical transfer from one place to another [22].

Therefore, the appropriate example for the conditions of the eco-toxicant model with migration proposed is (Blue-ringed octopus, Moray eel, Grouper) where the (Moray eel) represents the first predator feeding on the (Blue-ringed octopus) represented by prey where there are some areas that (Moray eel) cannot enter to her. Therefore, the (Blue-ringed octopus) is safe in that area, which can be considered a refuge area, and the (Grouper) feeds only on the first predator, which means that it is the top predator.

Roy and Roy in [31] studied a predator- prey system with a Holling type two functional response was analyzed for predator populations including prey shelter area. Harvesting efforts were also taken into account for predators. While in this article, an ecology toxicant food chain system is suggested to investigate the effects of toxics excreted by these organisms as a protection in the food chain system of four species under the influence of migration. The law of logistic growth governs the behavior of prey. The consequences of migration and toxins are discussed, on prey and predators.

2. Model formulation:

In this portion an eco-toxicant model with migration consists of three species (prey-first predator - top predator) and the total prey population can be divided into two regions: first in protection zone with density $\mathrm{Z}_1(\mathrm{T})$ at time T and second in predatory zone with density $\mathrm{Z}_2(\mathrm{T})$ at time T, the first and top predator populations symbol by $\mathrm{Z}_3(\mathrm{T})$ and $\mathrm{Z}_4(\mathrm{T})$ at time T respectively. Interpretation of the parameters used to study the dynamic system is as follows:

- The prey species grows logistically with carrying capacity $(\mathrm{L}_1, \mathrm{L}_2) > 0$ and the intrinsic growth rate symbol by $(\mathrm{s}_1, \mathrm{s}_2) > 0$ of a prey in the protection zone and predatory zone respectively.
- The migration and emigration must take place between two regions at time t, δ_1 is the prey unit migration, δ_2 and is the prey unit emigration of the prey population in protection zone.
- Each species secretes a toxic substance on the other as a defense (prey-first predator-top predator) according to the rate of toxins symbol by $(b_1, b_2, b_3, b_4) > 0$.
- The natural mortality rate of the prey in refuge zone and predatory zone symbol by $(\mathrm{m}_1, \mathrm{m}_2 > 0)$ and mortality rate of first and top predator in absence of it is feeding respectively symbol by $(\mathrm{m}_3, \mathrm{m}_4 > 0)$.
- The first- predator consumes the prey in predatory zone according to type Lotka -Volttra functional response with consumption rates $\gamma_1 > 0$,and the uptake rates of food from the prey by first- predator in the predatory zone $0 < \beta_1 < 1$.
- Finally, the top- predator consumes the first -predator according to type Lotka -Volttra functional response with the consumption of rates $\gamma_2 > 0$, and the uptake rates of food from the first -predator by top- predator $0 < \beta_2 < 1$.

We can now describe the dynamics of the above-mentioned assumption model by using a number of non -linear differential equations:

$$\begin{aligned}
\frac{dZ_1}{dT} &= s_1Z_1\left(1-\frac{z_1}{L_1}\right)-\delta_1Z_1+\delta_2Z_2-m_1Z_1,\\
\frac{dZ_2}{dT} &= s_2Z_2\left(1-\frac{z_2}{L_2}\right)+\delta_1Z_1-\delta_2Z_2-\gamma_1Z_2Z_3-b_1Z_2^2Z_3-m_2Z_2,\\
\frac{dZ_3}{d\ T} &= \beta_1Z_2Z_3-\gamma_2Z_3Z_4-b_2Z_3^2Z_2-b_3Z_3^2Z_4-m_3Z_3,\\
\frac{dZ_4}{d\ T} &= \beta_2Z_3Z_4-b_4Z_4^2Z_3-m_4Z_4.
\end{aligned} \tag{2.1}$$

The system (2.1) contains eighteen parameters; the system (2.1) can be dimensionless using the relationship:

$$t=s_1\ T, z_1=\frac{Z_1}{L_1}, z_2=\frac{Z_2}{L_1}, z_3=\frac{Z_3}{L_1}, z_4=\frac{Z_4}{L_1}, r_1=\frac{\delta_1}{s_1}, r_2=\frac{\delta_2}{s_1}, r_3=\frac{m_1}{s_1}, r_4=\frac{s_2}{s_1}, r_5=\frac{L_1}{L_2}, r_6=\frac{\gamma_1\ L_1}{s_1},$$
$$r_7=\frac{b_1\ {L_1}^2}{s_1}, r_8=\frac{m_2}{s_1}, r_9=\frac{\beta_1\ L_1}{s_1}, r_{10}=\frac{\gamma_2\ L_1}{s_1}, r_{11}=\frac{b_2\ {L_1}^2}{s_1}, r_{12}=\frac{b_3\ {L_1}^2}{s_1}, r_{13}=\frac{m_3}{s_1}, r_{14}=\frac{\beta_2\ L_1}{s_1},$$
$$r_{15}=\frac{b_4\ {L_1}^2}{s_1}, r_{16}=\frac{m_4}{s_1}.$$

Then the dimensionless system is as follows:

$$\begin{aligned}
\frac{dz_1}{dt} &= z_1\left[(1-z_1)-r_1+\frac{r_2z_2}{z_1}-r_3\right]=z_1f_1\left(z_1,z_2,z_3,z_4\right),\\
\frac{dz_2}{dt} &= z_2\left[r_4\left(1-r_5z_2\right)+\frac{r_1z_1}{z_2}-r_2-r_6z_3-r_7z_2z_3-r_8\right]=z_2f_2\left(z_1,z_2,z_3,z_4\right),\\
\frac{dz_3}{dt} &= z_3\left[r_9z_2-r_{10}z_4-r_{11}z_3z_2-r_{12}z_3z_4-r_{13}\right]=z_3f_3\left(z_1,z_2,z_3,z_4\right),\\
\frac{dz_4}{dt} &= z_4\left[r_{14}z_3-r_{15}z_4z_3-r_{16}\right]=z_4f_4\left(z_1,z_2,z_3,z_4\right).
\end{aligned} \tag{2.2}$$

It is noted that in the method, the number of parameters was reduced from eighteen in system (2.1) to sixteen in system (2.2).Obviously right-hand side interaction functions are continuous and have continuous partial derivatives in relation to the dependent variable z_1, z_2, z_3 and z_4 in the following positive four-dimensional space:

$$R_+^4=\left\{(z_1,z_2,z_3,z_4)\in R^4 : z_1(0)\geq 0, z_2(0)\geq 0, z_3(0)\geq 0, z_4(0)\geq 0\right\}.$$

As a result, these functions are Lipschitzian on R_+^4, and the system (2.2) solution exists and is unique. Furthermore, as shown in the theorem below, all solutions of the system (2.2) with non-negative initial conditions are uniformly bounded.

Theorem 2.1. *All the solutions of system (2.2) with initial condition in* R_+^4 *are uniformly bounded.*

Proof . Let $(z_1(t), z_2(t), z_3(t), z_4(t))$ be a solution of the system (2.2) with an initial non-negative condition $(z_1(0), z_2(0), z_3(0), z_4(0)) \in R_+^4$. Now define the function: $N(t) = z_1(t)+z_2(t)+z_3(t)+z_4(t)$, and then, take the function time derivative along with the solution of the system (2.2), we get:

$$\frac{\mathrm{dN}}{\mathrm{dt}} = \mathrm{z}_1 (1 - \mathrm{z}_1) + \mathrm{r}_4\mathrm{z}_2 (1 - \mathrm{r}_5\mathrm{z}_2) - \mathrm{r}_7\mathrm{z}_2^2\mathrm{z}_3 - \mathrm{r}_{11}\mathrm{z}_3^2\mathrm{z}_2 - \mathrm{r}_{12}\mathrm{z}_3^2\mathrm{z}_4 - \mathrm{r}_{15}\mathrm{z}_4^2\mathrm{z}_3$$
$$- (r_6 - r_9) z_2 z_3 - (r_{10} - r_{14}) z_3 z_4 - r_3 z_1 - r_8 z_2 - r_{13} z_3 - r_{16} z_4.$$

So, according to the biological facts always $\mathrm{r}_6 > \mathrm{r}_9, \mathrm{r}_{10} > \mathrm{r}_{14}$ and since the function $\mathrm{f}(\mathrm{z}_1) = \mathrm{z}_1 (1 - \mathrm{z}_1)$ and $\mathrm{f}(\mathrm{z}_2) = \mathrm{r}_4\mathrm{z}_2 (1 - \mathrm{r}_5\mathrm{z}_2)$, the terms represents a logistic function with respect to $\mathrm{z}_1, \mathrm{z}_2$ respectively and hence it is bounded above by the constants $\frac{1}{4}$ and $\frac{\mathrm{r}_4}{4\mathrm{r}_5}$ respectively so,

$$\frac{\mathrm{dN}}{\mathrm{dt}} \leq \frac{1}{4} + \frac{\mathrm{r}_4}{4\mathrm{r}_5} - (\mathrm{r}_3\mathrm{z}_1 + \mathrm{r}_8\mathrm{z}_2 + \mathrm{r}_{13}\mathrm{z}_3 + \mathrm{r}_{16}\mathrm{z}_4)$$

and, hence,
$\frac{\mathrm{dN}}{\mathrm{dt}} + \mathrm{nN} \leq \mu$, where $\mu = \frac{1}{4} + \frac{\mathrm{r}_4}{4\mathrm{r}_5}$ and $\mathrm{n} = \min(\mathrm{r}_3, \mathrm{r}_8, \mathrm{r}_{13}, \mathrm{r}_{16})$.
So, by using the comparison theorem [15] on the above differential initial value inequalities $\mathrm{N}(0) = \mathrm{N}_0$ we have:
$\mathrm{N}(\mathrm{t}) \leq \frac{\mu}{\mathrm{n}} + \left(\mathrm{N}_0 - \frac{\mu}{\mathrm{n}}\right) \mathrm{e}^{-\mathrm{nt}}$, Then, $\lim_{\mathrm{t}\to\infty} \mathrm{N}(\mathrm{t}) \leq \frac{\mu}{\mathrm{n}}$.
That is all solution of the system (2.2) necessary satisfies $0 \leq \mathrm{N}(\mathrm{t}) \leq \frac{\mu}{\mathrm{n}}, \forall \mathrm{t} > 0$.
Thus, all system solutions are bounded uniformly. □

3. The existence of equilibrium points:

In this portion, It is discussed whether model (2.2) has all potential equilibrium uses. As can be observed, the model (2.2) takes into account four equilibrium points, which are represented as follows:

- The trivial equilibrium point $\mathrm{E}_0 = (0, 0, 0, 0)$, which is always exists.
- The planer equilibrium point $E_1 = (\check{z}_1, \check{z}_2, 0, 0)$ where $\check{z}_1$ is the unique positive root of the following polynomial equation:

$$\mathrm{D}_1\mathrm{z}_1^3 + \mathrm{D}_2\mathrm{z}_1^2 + \mathrm{D}_3\mathrm{z}_1 + \mathrm{D}_4 = 0, \tag{3.1}$$

where:

$$\begin{aligned}
\mathrm{D}_1 &= -\mathrm{r}_4\mathrm{r}_5 < 0,\\
\mathrm{D}_2 &= 2\mathrm{r}_4\mathrm{r}_5 (1 - (\mathrm{r}_1 + \mathrm{r}_3)),\\
\mathrm{D}_3 &= \mathrm{r}_2 - \frac{\mathrm{r}_4\mathrm{r}_5 (1 - (\mathrm{r}_1 + \mathrm{r}_3))^2}{\mathrm{r}_4 - (\mathrm{r}_2 + \mathrm{r}_g)},\\
\mathrm{D}_4 &= \mathrm{r}_2 [(1 - (\mathrm{r}_1 + \mathrm{r}_3)) [-\mathrm{r}_4 + (\mathrm{r}_2 + \mathrm{r}_8)] + \mathrm{r}_1\mathrm{r}_2],
\end{aligned}$$

and $\check{z}_2 = \frac{\check{z}_1[\check{z}_1 - (1 - (r_1 + r_3))]}{r_2}$.
So, E_1 exists provided that:

$$(\mathrm{r}_1 + \mathrm{r}_3) < 1, \tag{3.2}$$
$$\mathrm{r}_4 < (\mathrm{r}_2 + \mathrm{r}_8), \tag{3.3}$$
$$\check{z}_1 > (1 - (r_1 + r_3)). \tag{3.4}$$

- The top - predator free equilibrium point exists if and only if there is a positive root to the following

equations:

$$\mathrm{z}_1-\mathrm{z}_1^2-\mathrm{r}_1\mathrm{z}_1+\mathrm{r}_2\mathrm{z}_2-\mathrm{r}_3\mathrm{z}_1=0, \tag{3.5}$$
$$\mathrm{r}_4\mathrm{z}_2-\mathrm{r}_4\mathrm{r}_5\mathrm{z}_2^2+\mathrm{r}_1\mathrm{z}_1-\mathrm{r}_2\mathrm{z}_2-\mathrm{r}_6\mathrm{z}_2\mathrm{z}_3-\mathrm{r}_7\mathrm{z}_2^2\mathrm{z}_3-\mathrm{r}_8\mathrm{z}_2=0, \tag{3.6}$$
$$r_9z_2-r_{11}z_3z_2-r_{13}=0, \tag{3.7}$$

From Eq. (3.5) we get:

$$\mathrm{z}_2=\frac{\mathrm{z}_1\left[\mathrm{z}_1-(1-(\mathrm{r}_1+\mathrm{r}_3))\right]}{\mathrm{r}_2} \tag{3.8}$$

From Eq. (3.7) we get:

$$\mathrm{z}_3=\frac{\mathrm{r}_9\left[\mathrm{z}_1\left[\mathrm{z}_1-(1-(\mathrm{r}_1+\mathrm{r}_3))\right]\right]-\mathrm{r}_2\mathrm{r}_{13}}{\mathrm{r}_{11}\left[\left[\mathrm{z}_1-(1-(\mathrm{r}_1+\mathrm{r}_3))\right]\right]} \tag{3.9}$$

By substitute Eq. (3.8) and (3.9) in Eq. (3.6) we have:

$$\mathrm{R}_1\mathrm{z}_1^4+\mathrm{R}_2\mathrm{z}_1^3+\mathrm{R}_3\mathrm{z}_1^2+\mathrm{R}_4\mathrm{z}_1+\mathrm{R}_5=0, \tag{3.10}$$

where:

$$\begin{aligned}
&\mathrm{R}_1=-\left(\mathrm{r}_4\mathrm{r}_5\mathrm{r}_{11}+\mathrm{r}_7\mathrm{r}_9\right)<0,\\
&\mathrm{R}_2=-2\mathrm{R}_1\left(1-(\mathrm{r}_1+\mathrm{r}_3)\right),\\
&\mathrm{R}_3=\left[\mathrm{R}_1\left(1-(\mathrm{r}_1+\mathrm{r}_3)\right)^2+\mathrm{M}\right],\\
&\mathrm{R}_4=\left[\mathrm{r}_1\mathrm{r}_2^2\mathrm{r}_{11}-\mathrm{M}\left(1-(\mathrm{r}_1+\mathrm{r}_3)\right)\right],\\
&\mathrm{R}_5=\mathrm{r}_2^2\mathrm{r}_6\mathrm{r}_{13}>0,\ \text{and}\\
&\mathrm{M}=\left[-\mathrm{r}_{11}\left(-\mathrm{r}_4+(\mathrm{r}_2+\mathrm{r}_8)\right)-\mathrm{r}_6\mathrm{r}_9+\mathrm{r}_7\mathrm{r}_{13}\right].
\end{aligned}$$

Obviously, according to the Descartes rule, eq. (3.10) either has three positive roots or no positive roots, depending upon whether following conditions are hold or not $\mathrm{R}_2>0, \mathrm{R}_3<0, \mathrm{R}_4>0$
That is, there are three equilibrium points $E_2=(\dot{z}_1,\dot{z}_2,\dot{z}_3,0)$, where $\dot{z}_2=z_2\left(\dot{z}_1\right)$ and $\dot{z}_3=\mathrm{z}_3\left(\dot{\mathrm{z}}_1\right), \mathrm{E}_3=(\overline{\mathrm{z}}_1,\overline{\mathrm{z}}_2,\overline{\mathrm{z}}_3,0)$ and $\mathrm{E}_4=\left(\overline{\overline{\mathrm{z}}}_1,\overline{\overline{\mathrm{z}}}_2,\overline{\overline{\mathrm{z}}}_3,0\right)$ under conditions (3.2),(3.3) and the following conditions :

$$\dot{z}_1>\left(1-(\mathrm{r}_1+\mathrm{r}_3)\right), \tag{3.11}$$
$$\mathrm{r}_2\mathrm{r}_{13}<\mathrm{r}_9\left[\dot{\mathrm{z}}_1\left[\dot{\mathrm{z}}_1-(1-(\mathrm{r}_1+\mathrm{r}_3))\right]\right], \tag{3.12}$$
$$\mathrm{r}_{11}\left(-\mathrm{r}_4+(\mathrm{r}_2+\mathrm{r}_8)\right)+\mathrm{r}_6\mathrm{r}_9>\mathrm{r}_7\mathrm{r}_{13}. \tag{3.13}$$

- The positive equilibrium point $E_5=(z_1^*,z_2^*,z_3^*,z_4^*)$ where z_1^* is the unique positive root of the following polynomial equation:

$$\mathrm{C}_1\mathrm{z}_1^8+\mathrm{C}_2\mathrm{z}_1^7+\mathrm{C}_3\mathrm{z}_1^6+\mathrm{C}_4\mathrm{z}_1^5+\mathrm{C}_5\mathrm{z}_1^4+\mathrm{C}_6\mathrm{z}_1^3+\mathrm{C}_7\mathrm{z}_1^2+\mathrm{C}_8\mathrm{z}_1+\mathrm{C}_9=0, \tag{3.14}$$

where:

$C_1 = -r_{15}\ A_1\ B_1, C_2 = -r_{15}\left(\ A_2\ B_1 + A_1\ B_2\right), C_3 = -B_7\left(r_{14}\ A_1 - r_{16}\ A_5\right) - r_{15}\left(\ A_3\ B_1 + A_2\ B_2 + A_1\ B_3\right),$
$C_4 = r_{14}\left(A_1B_8 + A_2B_7\right) - r_{15}\left(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4\right) - r_{16}\left(A_5B_8 + A_6B_7\right),$
$C_5 = r_{14}\left(A_1B_9 + A_2B_8 + A_3B_7\right) - r_{15}\left(A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5\right) - r_{16}\left(A_5B_9 + A_6B_8 + A_7B_7\right),$
$C_6 = r_{14}\left(\ A_1\ B_{10} + A_2\ B_9 + A_3\ B_8 + A_4\ B_7\right) - r_{15}\left(\ A_4\ B_3 + A_3\ B_4 + A_2\ B_5 + A_1\ B_6\right)$
$- r_{16}\left(\ A_5\ B_{10} + A_6\ B_9 + A_7\ B_8 + A_8\ B_7\right), C_7 = r_{14}\left(A_2B_{10} + A_3B_9 + A_4B_8\right) - r_{15}\left(A_4B_4 + A_3B_5 + A_2B_6\right)$
$- r_{16}\left(A_6B_{10} + A_7B_9 + A_8B_8\right), C_8 = r_{14}\left(\ A_3\ B_{10} + A_4\ B_9\right) - r_{15}\left(\ A_4\ B_5 + A_3\ B_6\right) - r_{16}\left(\ A_7\ B_{10} + A_8\ B_9\right),$
$C_9 = B_{10}\left(r_{14}\ A_4 - r_{16}\ A_8\right) - r_{15}\ A_4\ B_6,$

and $z_2^* = \dfrac{z_1^*\left[z_1^* - \left(1 - \left(r_1 + r_3\right)\right)\right]}{r_2}, z_3 = \dfrac{z_2^*\left[-\left(r_2 + r_8\right) + r_4\left(1 - r_5 z_2^*\right)\right] + r_1 z_1^*}{\left(r_6 + r_7 z_2^*\right) z_2^*} = \dfrac{A_1z_1^4 + A_2z_1^3 + A_3z_1^2 + A_4z_1}{A_5z_1^4 + A_6z_1^3 + A_7z_1^2 + A_8z_1},$
$z_4 = \dfrac{\left(r_9 - r_{11}z_3^*\right) z_2^* - r_{13}}{\left(r_{10} + r_{12}z_3^*\right)} = \dfrac{B_1z_1^6 + B_2z_1^5 + B_3z_1^4 + B_4z_1^3 + B_5z_1^2 + B_6z_1}{B_7z_1^4 + B_8z_1^3 + B_9z_1^2 + B_{10}z_1},$

where

$$A_1 = -r_4r_5, A_2 = 2r_4r_5\left(1 - \left(r_1 + r_3\right)\right), A_3 = r_4 - \left[r_4r_5\left(1 - \left(r_1 + r_3\right)\right)^2 + \left(r_2 + r_8\right)\right],$$
$$A_4 = \left[-r_4 + \left(r_2 + r_8\right)\right]\left(1 - \left(r_1 + r_3\right)\right) + r_1r_2,\ A_5 = r_7,\ A_6 = -2r_7\left(1 - \left(r_1 + r_3\right)\right),$$
$$A_7 = r_2r_6 + r_7\left(1 - \left(r_1 + r_3\right)\right)^2, A_8 = -\left[r_2r_6\left(1 - \left(r_1 + r_3\right)\right)\right],$$

with

$$B_1 = r_9\ A_5 - r_{11}\ A_1,\ B_2 = r_9\left(\ A_6 - A_5\left(1 - \left(r_1 + r_3\right)\right)\right) - r_{11}\left(\ A_2 - A_1\left(1 - \left(r_1 + r_3\right)\right)\right),$$
$$B_3 = r_9\left(\ A_7 - A_6\left(1 - \left(r_1 + r_3\right)\right)\right) - r_{11}\left(\ A_3 - A_2\left(1 - \left(r_1 + r_3\right)\right)\right) - r_2r_{13}\ A_5,$$
$$B_4 = r_9\left(-A_7\left(1 - \left(r_1 + r_3\right)\right) + A_8\right) + r_{11}\left(\ A_3\left(1 - \left(r_1 + r_3\right)\right) - r_4\right) - r_{12}r_{13}\ A_6,$$
$$B_5 = \left(1 - \left(r_1 + r_3\right)\right)\left(-r_9\ A_8 + r_{11}\ A_4\right) - r_2r_{13}\ A_7,\ B_6 = -r_2r_{13}\ A_8,\ B_7 = r_{10}\ A_5 + r_{12}\ A_1,$$
$$B_8 = r_{10}\ A_6 + r_{12}\ A_2,\ B_9 = r_{10}\ A_7 + r_{12}\ A_3,\ B_{10} = r_{10}\ A_8 + r_{12}\ A_4$$

Thus, E_5 exists under conditions (3.2), (3.3) and the following conditions.

$$z_1^* > (1 - (r_1 + r_3)), \tag{3.15}$$
$$r_4 < r_4 r_5 (1 - (r_1 + r_3))^2 + (r_2 + r_8), \tag{3.16}$$
$$A_2 z_1^3 + A_4 z_1 > A_1 z_1^4 + A_3 z_1^2, \tag{3.17}$$
$$A_5 z_1^4 + A_7 z_1^2 > A_6 z_1^3 + A_8 z_1, \tag{3.18}$$
$$A_3 < \min\left\{A_2 B_1, \frac{r_4}{B_1}\right\}, \tag{3.19}$$
$$r_9 (A_7 - A_6 (1 - (r_1 + r_3))) > r_{11} (A_3 - A_2 (1 - (r_1 + r_3))) - r_2 r_{13} A_5, \tag{3.20}$$
$$r_9 (-A_7 (1 - (r_1 + r_3)) + A_8) + r_{11} (A_3 (1 - (r_1 + r_3)) - r_4) < -(r_{12} r_{13} A_6), \tag{3.21}$$
$$(1 - (r_1 + r_3)) (-r_9 A_8 + r_{11} A_4) > r_2 r_{13} A_7, \tag{3.22}$$
$$r_{10} > \max\left\{r_{12}\left[\frac{A_1}{A_5}, \frac{A_3}{A_7}, \frac{A_4}{A_8}\right]\right\}, \tag{3.23}$$
$$B_1 z_1^6 + B_3 z_1^4 + B_4 z_1^3 + B_5 z_1^2 + B_6 z_1 > B_2 z_1^5, \tag{3.24}$$
$$B_7 z_1^4 + B_9 z_1^2 > B_8 z_1^3 + B_{10} z_1, \tag{3.25}$$
$$- B_7 (r_{14} A_1 - r_{16} A_5) > r_{15} (A_3 B_1 + A_2 B_2 + A_1 B_3), \tag{3.26}$$
$$A_4 B_1 + A_3 B_2 + A_2 B_3 > A_1 B_4, \tag{3.27}$$
$$A_2 B_4 > A_1 B_5 + A_3 B_3 + A_4 B_2, \tag{3.28}$$
$$A_4 B_3 + A_2 B_5 > A_1 B_6 + A_3 B_4, \tag{3.29}$$
$$A_3 < \min\left\{\frac{A_4 B_4 + A_2 B_6}{B_5}, \frac{A_4 B_5}{B_6}\right\}, \tag{3.30}$$

$$\begin{aligned} r_{15} > \max\Bigg\{ & \frac{r_{14} (A_1 B_8 + A_2 B_7) - r_{16} (A_5 B_8 + A_6 B_7)}{(A_4 B_1 + A_3 B_2 + A_2 B_3 + A_1 B_4)}, \\ & \frac{r_{14} (A_1 B_{10} + A_2 B_9 + A_3 B_8 + A_4 B_7) - r_{16} (A_5 B_{10} + A_6 B_9 + A_7 B_8 + A_8 B_7)}{(A_4 B_3 + A_3 B_4 + A_2 B_5 + A_1 B_6)}, \\ & \frac{r_{14} (A_3 B_{10} + A_4 B_9) - r_{16} (A_7 B_{10} + A_8 B_9)}{(A_4 B_5 + A_3 B_6)} \Bigg\}. \end{aligned} \tag{3.31}$$

4. Local stability analysis:

In this portion, investigates the local stability analysis of the system (2.2) for each of equilibrium points by computing the Jacobean matrix $J(z_1, z_2, z_3, z_4)$ of the system (2.2) as bellow: $J = [y_{ij}]_{4\times 4}$ where:

$$y_{11} = 1 - 2z_1 - (r_1 + r_3), y_{12} = r_2, y_{13} = 0, y_{14} = 0,$$
$$y_{21} = r_1, y_{22} = r_4 (1 - 2r_5 z_2) - (r_2 + r_8) - (r_6 + 2r_7 z_2) z_3, y_{23} = -(r_6 + r_7 z_2) z_2, y_{24} = 0,$$
$$y_{31} = 0, y_{32} = (r_9 - r_{11} z_3) z_3, y_{33} = (r_9 - 2r_{11} z_3) z_2 - (r_{10} + 2r_{12} z_3) z_4 - r_{13},$$
$$y_{34} = -(r_{10} + r_{12} z_3) z_3, y_{41} = 0, y_{42} = 0, y_{43} = (r_{14} - r_{15} z_4) z_4, y_{44} = (r_{14} - 2r_{15} z_4) z_3 - r_{16}.$$

4.1. Local stability analysis of E_0 :

The Jacobean matrix at E_0 is given by:
$J_0 = J(\mathrm{E}_0) = [\mathrm{y_{ij}}]_{4\times4}$ and can be written as:

$$J_0 = J(E_0) = \begin{bmatrix} 1-(r_1+r_3) & r_2 & 0 & 0 \\ r_1 & r_4-(r_2+r_8) & 0 & 0 \\ 0 & 0 & -r_{13} & 0 \\ 0 & 0 & 0 & -r_{16} \end{bmatrix},$$

Then the characteristic equation of J_0 :

$$\left[\lambda^2 - A_1\lambda + A_2\right](-r_{13}-\lambda)(-r_{16}-\lambda) = 0, \tag{4.1}$$

So, either $[\lambda^2 - \mathrm{A}_1\lambda + \mathrm{A}_2] = 0$, where
$\mathrm{A}_1 = \lambda_{0\mathrm{z}_1} + \lambda_{0\mathrm{z}_2} = 1-(\mathrm{r}_1+\mathrm{r}_3)+\mathrm{r}_4-(\mathrm{r}_2+\mathrm{r}_8)$ and,
$\mathrm{A}_2 = \lambda_{0\mathrm{z}_1}\lambda_{0\mathrm{z}_2} = [1-(\mathrm{r}_1+\mathrm{r}_3)][\mathrm{r}_4-(\mathrm{r}_2+\mathrm{r}_8)] - \mathrm{r}_1\mathrm{r}_2$,
which gives the first two eigenvalues of J_0 with negative real parts due to the reverse condition (3.2) and (3.3) with the following condition:

$$[1-(r_1+r_3)][r_4-(r_2+r_8)] > r_1 r_2. \tag{4.2}$$

Or
$(-r_{13}-\lambda)(-r_{16}-\lambda) = 0$, which gives:

$$\lambda_{0_{z_3}} = -r_{13} < 0, \lambda_{0_{z4}} = -r_{16} < 0.$$

So, E_0 is stable. It's unstable point otherwise.

4.2. Local stability analysis of $\mathbf{E}_1$ *:*

The Jacobean matrix at $\mathrm{E}_1 = (\check{\mathrm{z}}_1, \check{\mathrm{z}}_2, 0, 0)$ is given by:
$\check{J}_1 = \check{J}(\mathrm{E}_1) = [\check{\mathrm{y}}_{\mathrm{ij}}]_{4\times4}$ and can be written as:

$$\check{\mathrm{J}}_1 = \check{\mathrm{J}}(\mathrm{E}_1) = \begin{bmatrix} \check{\mathrm{y}}_{11} & \check{\mathrm{y}}_{12} & 0 & 0 \\ \check{\mathrm{y}}_{21} & \check{\mathrm{y}}_{22} & \check{\mathrm{y}}_{23} & 0 \\ 0 & 0 & \check{\mathrm{y}}_{33} & 0 \\ 0 & 0 & 0 & \check{\mathrm{y}}_{44} \end{bmatrix}$$

where: $\check{\mathrm{y}}_{11} = 1-2\check{z}_2-(r_1+r_3), \check{\mathrm{y}}_{12} = r_2 > 0, \check{\mathrm{y}}_{13} = 0, \check{\mathrm{y}}_{14} = 0,$
$\check{\mathrm{y}}_{21} = r_1 > 0, \check{\mathrm{y}}_{22} = r_4(1-2r_5\check{z}_2)-(r_2+r_8), \check{\mathrm{y}}_{23} = -(r_6+r_7\check{z}_2)\check{z}_2 < 0, \check{\mathrm{y}}_{24} = 0,$
$\check{\mathrm{y}}_{31} = 0, \check{\mathrm{y}}_{32} = 0, \check{\mathrm{y}}_{33} = r_9\check{z}_2 - r_{13}, \check{\mathrm{y}}_{34} = 0, \check{\mathrm{y}}_{41} = 0, \check{\mathrm{y}}_{42} = 0, \check{\mathrm{y}}_{43} = 0, \check{\mathrm{y}}_{44} = -r_{16} < 0.$
Then the characteristic equation of $\check{\mathrm{J}}_1$:

$$\left[\lambda^2 - \widetilde{\mathrm{A}}_1\lambda + \widetilde{\mathrm{A}}_2\right]((\mathrm{r}_9\check{\mathrm{z}}_2 - \mathrm{r}_{13}) - \lambda)(-\mathrm{r}_{16}-\lambda) = 0, \tag{4.3}$$

So, either $\left[\lambda^2 - \check{\mathrm{A}}_1\lambda + \check{\mathrm{A}}_2\right] = 0$, where

$\breve{A}_1 = \lambda_{1z_1} + \lambda_{1z_2} = 1 - 2\breve{z}_2 - (r_1 + r_3) + r_4(1 - 2r_5\breve{z}_2) - (r_2 + r_8)$ and,
$\breve{A}_2 = \lambda_{1z_1}\lambda_{1z_2} = [1 - 2\breve{z}_2 - (r_1 + r_3)][r_4(1 - 2r_5\breve{z}_2) - (r_2 + r_8)] - r_1 r_2$.
which gives the first two eigenvalues of $\breve{J}_1$ with negative real parts due to the conditions:

$$1 < 2\breve{z}_1 + (\mathrm{r}_1 + \mathrm{r}_3), \tag{4.4}$$

$$\breve{\mathrm{z}}_2 > \frac{1}{2\mathrm{r}_5}, \tag{4.5}$$

$$[1 - 2\breve{z}_2 - (r_1 + r_3)][r_4(1 - 2r_5\breve{z}_2) - (r_2 + r_8)] > r_1 r_2. \tag{4.6}$$

Or
$((r_9\breve{z}_2 - r_{13}) - \lambda)(-r_{16} - \lambda) = 0$, which gives:

$$\lambda_{1z_3} = \mathrm{r}_9\breve{z}_2 - \mathrm{r}_{13},$$
$$\lambda_{1_{z4}} = -\mathrm{r}_{16} < 0.$$

So, E_1 is stable if the above-mentioned conditions are hold with condition:

$$\breve{\mathrm{z}}_2 < \frac{r_{13}}{r_9}. \tag{4.7}$$

It's unstable point otherwise.

4.3. Local stability analysis of $\mathbf{E}_2$ *:*

The Jacobean matrix of system (2.2) at the top − predator free equilibrium point $E_2 = (\dot{z}_1, \dot{z}_2, \dot{z}_3, 0)$ similarly for $\mathrm{E}_3 = (\overline{\mathrm{z}}_1, \overline{\mathrm{z}}_2, \overline{\mathrm{z}}_3, 0)$ and $\mathrm{E}_4 = \left(\overline{\overline{\mathrm{z}}}_1, \overline{\overline{\mathrm{z}}}_2, \overline{\overline{\mathrm{z}}}_3, 0\right)$ is given by:
$\dot{\mathrm{J}}_2 = \dot{\mathrm{J}}(\mathrm{E}_2) = [\dot{\mathrm{y}}_{\mathrm{ij}}]_{4\times 4}$ and can be written as:

$$\dot{\mathrm{J}}_2 = \dot{\mathrm{J}}(\mathrm{E}_2) = \begin{bmatrix} \dot{\mathrm{y}}_{11} & \dot{\mathrm{y}}_{12} & 0 & 0 \\ \dot{\mathrm{y}}_{21} & \dot{\mathrm{y}}_{22} & \dot{\mathrm{y}}_{23} & 0 \\ 0 & \dot{\mathrm{y}}_{32} & \dot{\mathrm{y}}_{33} & \dot{\mathrm{y}}_{34} \\ 0 & 0 & 0 & \dot{\mathrm{y}}_{44} \end{bmatrix},$$

where: $\dot{y}_{11} = 1 - 2\dot{z}_1 - (r_1 + r_3), \dot{y}_{12} = r_2 > 0, \dot{y}_{13} = 0, \dot{y}_{14} = 0$,
$\dot{y}_{21} = \mathrm{r}_1 > 0, \dot{y}_{22} = \mathrm{r}_4(1 - 2r_5\dot{z}_2) - (r_2 + r_8) - (r_6 + 2r_7\dot{z}_2)\dot{z}_3, \dot{y}_{23} = -(r_6 + r_7\dot{z}_2)\dot{z}_2 < 0, \dot{y}_{24} = 0$,
$\dot{y}_{31} = 0, \dot{y}_{32} = (r_9 - r_{11}\dot{z}_3)\dot{z}_3, \dot{y}_{33} = (r_9 - 2r_{11}\dot{z}_3)\dot{z}_2 - r_{13}, \dot{y}_{34} = -(r_{10} + r_{12}\dot{z}_3)\dot{z}_3 < 0$,
$\dot{y}_{41} = 0, \quad \dot{y}_{42} = 0, \dot{y}_{43} = 0, \dot{y}_{44} = r_{14}\dot{z}_3 - r_{16}$.
Then the characteristic equation of $\dot{\mathrm{J}}_2$:

$$\left[\lambda^3 + \dot{\mathrm{A}}_1\lambda^2 + \dot{\mathrm{A}}_2\lambda + \dot{\mathrm{A}}_3\right](\dot{\mathrm{y}}_{44} - \lambda) = 0, \tag{4.8}$$

So, either $\lambda_{2z_4} = r_{14}\dot{z}_3 - r_{16}$ which is negative under the condition

$$\dot{z}_3 < \frac{r_{16}}{r_{14}}. \tag{4.9}$$

Or

$$\left[\lambda^3 + \dot{\mathrm{A}}_1\lambda^2 + \dot{\mathrm{A}}_2\lambda + \dot{\mathrm{A}}_3\right] = 0$$

Where:

$$\dot{A}_1 = -\left[\dot{y}_{11} + \dot{y}_{22} + \dot{y}_{33}\right],$$
$$\dot{A}_2 = \dot{y}_{11}\left(\dot{y}_{22} + \dot{y}_{33}\right) + \dot{y}_{22}\dot{y}_{33} - \dot{y}_{23}\dot{y}_{32} - \dot{y}_{12}\dot{y}_{21},$$
$$\dot{A}_3 = \dot{y}_{11}\dot{y}_{23}\dot{y}_{32} - \dot{y}_{33}\left(\dot{y}_{11}\dot{y}_{22} - \dot{y}_{12}\dot{y}_{21}\right).$$

Now, by utilizing of the Routh- Hurwitz criterion, eq. (4.8) has negative real parts roots, if and only if $\dot{A}_1 > 0, \dot{A}_3 > 0$ and $\Delta = \left(\dot{A}_1\dot{A}_2 - \dot{A}_3\right)\dot{A}_3 > 0$.
Obviously, $\dot{A}_i > 0, i = 1$ and 3 if the following conditions are hold:

$$1 < 2\dot{z}_1 + (r_1 + r_3), \tag{4.10}$$
$$\dot{z}_2 > \frac{1}{2r_5}, \tag{4.11}$$
$$\frac{r_9}{2r_{11}} < \dot{z}_3 < \frac{r_9}{r_{11}}, \tag{4.12}$$
$$\left[1 - 2\dot{z}_1 - (r_1 + r_3)\right]\left[r_4\left(1 - 2r_5\dot{z}_2\right) - (r_2 + r_8) - \left(r_6 + 2r_7\dot{z}_2\right)\dot{z}_3\right] > r_1 r_2. \tag{4.13}$$

Straightforward computation shows that:

$$\dot{\Delta} = \dot{A}_1\dot{A}_2 - \dot{A}_3 = -\dot{y}_{11}^2\left(\dot{y}_{22} + \dot{y}_{33}\right) - \dot{y}_{11}\left[\left(y_{22} + \dot{y}_{33}\right)^2 - \dot{y}_{12}\dot{y}_{21}\right] + \dot{y}_{23}\dot{y}_{32}\left(\dot{y}_{22} + \dot{y}_{33}\right)$$
$$- \dot{y}_{22}\left[\dot{y}_{33}\left(\dot{y}_{22} + \dot{y}_{33}\right) - \dot{y}_{12}\dot{y}_{21}\right]$$

Consequently, $\Delta = \left(\dot{A}_1\dot{A}_2 - \dot{A}_3\right)\dot{A}_3 > 0$ if in addition of the conditions (4.10-4.13) the following condition hold:

$$\dot{y}_{12}\dot{y}_{21} < \min\left\{\left(\dot{y}_{22} + \dot{y}_{33}\right)^2, \dot{y}_{33}\left(\dot{y}_{22} + \dot{y}_{33}\right)\right\} \tag{4.14}$$

So, E_2 is stable. It's unstable point otherwise.

4.4. Local stability of $\mathbf{E}_5$ *:*

The Jacobean matrix at $E_5 = (z_1^*, z_2^*, z_3^*, z_4^*)$ is given by:
$J_5^* = J^*(E_5) = \left[y_{ij}^*\right]_{4\times 4}$ and can be written as:

$$J_5^* = J^*(E_5) = \begin{bmatrix} y_{11}^* & y_{12}^* & 0 & 0 \\ y_{21}^* & y_{22}^* & y_{23}^* & 0 \\ 0 & y_{32}^* & y_{33}^* & y_{34}^* \\ 0 & 0 & y_{43}^* & y_{44}^* \end{bmatrix},$$

where: $y_{11}^* = 1 - 2z_1^* - (r_1 + r_3), y_{12}^* = r_2 > 0, y_{13}^* = 0, y_{14}^* = 0, y_{21}^* = r_1 > 0,$
$y_{22}^* = r_4\left(1 - 2r_5 z_2^*\right) - \left[(r_2 + r_8) + \left(r_6 + 2r_7 z_2^*\right)z_3^*\right], y_{23}^* = -\left(r_6 + r_7 z_2^*\right)z_2^* < 0, y_{24}^* = 0,$
$y_{31}^* = 0, y_{32}^* = \left(r_9 - r_{11}z_3^*\right)z_3^*, y_{33}^* = r_9 z_2^* - r_{10}z_4^* - 2z_3^*\left(r_{11}z_2^* + r_{12}z_4^*\right) - r_{13},$
$y_{34}^* = -\left(r_{10} + r_{12}z_3^*\right)z_3^* < 0, y_{41}^* = 0, y_{42}^* = 0, = \left(r_{14} - r_{15}z_4^*\right)z_4^*, y_{43}^* = \left(r_{14} - 2r_{15}z_4^*\right)z_3^* - r_{16}.$
Then the characteristic equation of J_5^* :

$$\left[\lambda^4 + A_1^*\lambda^3 + A_2^*\lambda^2 + A_3^*\lambda + A_4^*\right] = 0, \tag{4.15}$$

where:

$$\begin{aligned}
A_1^* &= -\left[k_0 + k_1\right], \\
A_2^* &= k_2 + k_3 + k_4 + k_5 + k_6 + k_7 - \left(k_8 + k_9 + k_{10}\right), \\
A_3^* &= k_1\left(-k_2 + k_{10}\right) - k_0\left(k_7 - k_9\right) + k_8 k_{11}, \\
A_4^* &= \left(k_2 - k_{10}\right)\left(k_7 - k_9\right) - k_4 k_8.
\end{aligned}$$

with

$$\begin{aligned}
&k_0 = y_{11}^* + y_{22}^*, k_1 = y_{33}^* + y_{44}^*, k_2 = y_{11}^* y_{22}^*, k_3 = y_{11}^* y_{33}^*, k_4 = y_{11}^* y_{44}^*, k_5 = y_{22}^* y_{33}^*, k_6 = y_{22}^* y_{44}^*, \\
&k_7 = y_{33}^* y_{44}^*, k_8 = y_{23}^* y_{32}^*, k_9 = y_{34}^* y_{43}^*, k_{10} = y_{12}^* y_{21}^*, k_{11} = y_{11}^* + y_{44}^*.
\end{aligned}$$

Now, by utilizing the Routh-Hurwitz criterion, eq. (4.15) has negative real parts roots, if and only if $A_1^* > 0,\ A_3^* > 0,\ A_4^* > 0$ and $\Delta = \left(A_1^*\ A_2^* - A_3^*\right) A_3^* - {A_1^*}^2\ A_4^* > 0$.
Obviously, $A_i^* > 0, i = 1, 3$ and 4 if the following conditions are hold:

$$1 < 2z_2^* + \left(r_1 + r_3\right), \tag{4.16}$$

$$z_2^* > \frac{1}{2r_5}, \tag{4.17}$$

$$\frac{r_9}{2r_{11}} < z_3^* < \frac{r_9}{r_{11}}, \tag{4.18}$$

$$\frac{r_{14}}{2r_{15}} < z_4^* < \frac{r_{14}}{r_{15}}, \tag{4.19}$$

$$\left[1 - \breve{z}_2 - \left(r_1 + r_3\right)\right]\left[r_4\left(1 - 2r_5\breve{z}_2\right) - \left[\left(r_2 + r_8\right) + \left(r_6 + 2r_7\breve{z}_2\right)\breve{z}_3\right] > r_1 r_2\ . \tag{4.20}$$

Straightforward computation shows that:
$\Delta = G_1 - G_2$, where

$$\begin{aligned}
G_1 &= k_0^2\left(k_7 - k_9\right) - k_1^2\left(-k_2 + k_{10}\right) - k_1\left[-k_0\left(k_7 - k_9\right) + k_8 k_{11}\right] - k_0\left[k_1\left(-k_2 + k_{10}\right) + k_8 k_{11}\right] \\
&\quad \left(k_2 + k_3 + k_4 + k_5 + k_6 + k_7 - \left(k_8 + k_9 + k_{10}\right)\right), \\
G_2 &= k_1^2\left(-k_2 + k_{10}\right)^2 + k_0^2\left(k_7 - k_9\right)^2 + k_8^2 k_{11}^2 + 2k_1\left(-k_2 + k_{10}\right)\left[-k_0\left(k_7 - k_9\right) + k_8 k_{11}\right] \\
&\quad + \left(k_7 - k_9\right)\left[-2k_0 k_8 k_{11} + \left(k_0 + k_1\right)^2\left(k_2 - k_{10}\right)\right] - \left(k_0 + k_1\right)^2 k_8 k_{11}.
\end{aligned}$$

Consequently, $\Delta = \left(A_1^* A_2^* - A_3^*\right) A_3^* - A_1^{*2} A_4^* > 0$ if in addition of the above conditions $(4.16 - 4.20)$ the following conditions hold:

$$G_1 > G_2 \tag{4.21}$$

So, E_5 is stable. It's unstable point otherwise.

5. Global stability analysis:

In this portion, the Lyapunov function method is used to scientifically concentrate the global stability analysis for the equilibrium points of the system (2.2) that are locally asymptotically stable, as shown below.

Theorem 5.1. *The equilibrium point E_0 is globally asymptotically stable on a subregion $\omega_0 \subset R^4_+$, where $\omega_0 = \left\{(\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3, \mathrm{z}_4) \in \mathrm{R}^4_+\right\}$, that satisfies the following conditions:*

$$\mathrm{z}_1 > 1, \tag{5.1}$$

$$\mathrm{z}_2 > \frac{1}{\mathrm{r}_5}, \tag{5.2}$$

Proof . Consider the following function: $F_0(z_1, z_2, z_3, z_4) = z_1 + z_2 + z_3 + z_4$.
Obviously $\mathrm{F}_0 : \mathrm{R}^4_+ \to \mathrm{R}$ is a C^1 positive definite function.
So, by differentiating F_0 with respect to time t and performing various algebraic operations, the following result is obtained:

$$\frac{\mathrm{dF}_0}{\mathrm{dt}} < \mathrm{z}_1(1-\mathrm{z}_1) + \mathrm{r}_4\mathrm{z}_2(1-\mathrm{r}_5\mathrm{z}_2) - (\mathrm{r}_6-\mathrm{r}_9)\mathrm{z}_2\mathrm{z}_3 - (\mathrm{r}_{10}-\mathrm{r}_{14})\mathrm{z}_3\mathrm{z}_4 - \mathrm{r}_3\mathrm{z}_1 - \mathrm{r}_8\mathrm{z}_2 - \mathrm{r}_{13}\mathrm{z}_3 - \mathrm{r}_{16}\mathrm{z}_4.$$

Now, by the biological facts $\mathrm{r}_6 > \mathrm{r}_9$ and $\mathrm{r}_{10} > \mathrm{r}_{14}$, we get:

$$\frac{\mathrm{dF}_0}{\mathrm{dt}} < \mathrm{z}_1(1-\mathrm{z}_1) + \mathrm{r}_4\mathrm{z}_2(1-\mathrm{r}_5\mathrm{z}_2) - (\mathrm{r}_3\mathrm{z}_1 + \mathrm{r}_8\mathrm{z}_2 + \mathrm{r}_{13}\mathrm{z}_3 + \mathrm{r}_{16}\mathrm{z}_4)$$

Consequently, F_0 is negative definite under conditions (5.1) and (5.2) and then F_0 is Lyapunov function. As a result, we can conclude that F_0 is globally asymptotically stable in the subregion ω_0, and our demonstration is complete. □

Theorem 5.2. *The equilibrium point E_1 is a globally asymptotically stable on a subregion $\omega_1 \subset R^4_+$ that satisfies the following conditions:*

$$\left(\frac{\mathrm{r}_2}{\check{\mathrm{z}}_1} + \frac{\mathrm{r}_1}{\check{\mathrm{z}}_2}\right) \le 2\sqrt{\left(1 + \frac{\mathrm{r}_2\mathrm{z}_2}{\mathrm{z}_1\check{\mathrm{z}}_1}\right)\left(\mathrm{r}_4\mathrm{r}_5 + \frac{\mathrm{r}_1\mathrm{z}_1}{\mathrm{z}_2\check{\mathrm{z}}_2}\right)}, \tag{5.3}$$

$$\check{\mathrm{u}}_1 > \check{\mathrm{u}}_2, \tag{5.4}$$

where:

$$\check{\mathrm{u}}_1 = \left[\sqrt{\left(1 + \frac{\mathrm{r}_2\mathrm{z}_2}{\mathrm{z}_1\check{z}_1}\right)}(\mathrm{z}_1 - \check{\mathrm{z}}_1) - \sqrt{\left(\mathrm{r}_4\mathrm{r}_5 + \frac{\mathrm{r}_1\mathrm{z}_1}{\mathrm{z}_2\check{z}_2}\right)}(\mathrm{z}_2 - \check{\mathrm{z}}_2)\right]^2 + \mathrm{r}_{13}\mathrm{z}_3 + \mathrm{r}_{16}\mathrm{z}_4,$$

$$\check{\mathrm{u}}_2 = (\mathrm{r}_6 + \mathrm{r}_7\mathrm{z}_2)\check{z}_2\mathrm{z}_3.$$

Proof . Consider the following function: $\mathrm{F}_1(\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3, \mathrm{z}_4) = \left(\mathrm{z}_1 - \check{\mathrm{z}}_1 - \check{\mathrm{z}}_1 \ln \frac{\mathrm{z}_1}{\check{\mathrm{z}}_1}\right) + \left(\mathrm{z}_2 - \check{\mathrm{z}}_2 - \check{z}_2 \ln \frac{z_2}{\check{z}_2}\right) + z_3 + z_4$, obviously $F_1 : R^4_+ \to R$ is a C^1 positive definite function.
So, by differentiating F_1 with respect to time t and performing various algebraic operations, the following result is obtained:

$$\frac{\mathrm{dF}_1}{\mathrm{dt}} = -\left(1 + \frac{\mathrm{r}_2\mathrm{z}_2}{\mathrm{z}_1\check{\mathrm{z}}_1}\right)(\mathrm{z}_1 - \check{\mathrm{z}}_1)^2 + \left(\frac{\mathrm{r}_2}{\check{\mathrm{z}}_1} + \frac{\mathrm{r}_1}{\check{\mathrm{z}}_2}\right)(\mathrm{z}_1 - \check{\mathrm{z}}_1)(\mathrm{z}_2 - \check{\mathrm{z}}_2) - \left(\mathrm{r}_4\mathrm{r}_5 + \frac{\mathrm{r}_1\mathrm{z}_1}{\mathrm{z}_2\check{\mathrm{z}}_2}\right)(\mathrm{z}_2 - \check{\mathrm{z}}_2)^2 - \mathrm{r}_{13}\mathrm{z}_3$$
$$- r_{16}\mathrm{z}_4 - (\mathrm{r}_6 - \mathrm{r}_9)\mathrm{z}_2\mathrm{z}_3 - (\mathrm{r}_{10} - \mathrm{r}_{14})\mathrm{z}_3\mathrm{z}_4 + (\mathrm{r}_6 + \mathrm{r}_7\mathrm{z}_2)\check{\mathrm{z}}_2\mathrm{z}_3$$

Now, when we combine the condition (5.3) with the biological facts $r_6 > r_9$ and $r_{10} > r_{14}$, we get:

$$\frac{dF_1}{dt} < -\left[\sqrt{\left(1+\frac{r_2 z_2}{z_1 \check{z}_1}\right)}(z_1-\check{z}_1)-\sqrt{\left(r_4 r_5+\frac{r_1 z_1}{z_2 \check{z}_2}\right)}(z_2-\check{z}_2)\right]^2 - r_{13}z_3 - r_{16}z_4 + (r_6 + r_7 z_2)\check{z}_2 z_3$$
$$= -\check{u}_1 + \check{u}_2$$

So, according to condition (5.4) F_1 is negative definite under above the condition and then F_1 is Lyapunov function. As a result, we can conclude that F_1 is globally asymptotically stable in the sub-region ω_1, and our demonstration is complete. □

Moreover since there are three top − predator free equilibrium points $E_2 = (\dot{z}_1, \dot{z}_2, \dot{z}_3, 0)$, $E_3 = (\bar{z}_1, \bar{z}_2, \bar{z}_3, 0)$ and $E_4 = \left(\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3, 0\right)$ in R^4_+ having the same local stability conditions but with different neighborhood of starting points then it is not possible to studying the global stability of them using Lyapunove function. Hence we'll study it numerically instead of analytically as shown in last part.

Theorem 5.3. *The equilibrium point E_5 is a globally asymptotically stable on a subregion $\omega_2 \subset R^4_+$ that satisfie the following conditions:*

$$\left(\frac{r_2}{z_1^*}+\frac{r_1}{z_2^*}\right) \le 2\sqrt{\left(1+\frac{r_2 z_2}{z_1 z_1^*}\right)\left(r_4 r_5+\frac{r_1 z_1}{z_2 z_2^*}\right)}, \tag{5.5}$$
$$u_1^* > u_2^*, \tag{5.6}$$

where:

$$u_1^* = \left[\sqrt{\left(1+\frac{r_2 z_2}{z_1 z_1^*}\right)}(z_1-z_1^*)-\sqrt{\left(r_4 r_5+\frac{r_1 z_1}{z_2 z_2^*}\right)}(z_2-z_2^*)\right]^2 + (r_7 z_2 + r_{11} z_3) z_2 z_3$$
$$+ (r_7 z_2^* + r_{11} z_3^*) z_2^* z_3^* + (r_{12} z_3 + r_{15} z_4) z_3 z_4 + (r_{12} z_3^* + r_{15} z_4^*) z_3^* z_4^*,$$
$$u_2^* = (r_6 - r_9)(z_2 z_3^* + z_2^* z_3) + (r_7 z_2 + r_{11} z_3) z_3^* z_2^* + (r_7 z_2^* + r_{11} z_3^*) z_3 z_2 + (r_{10} - r_{14})(z_3 z_4^* + z_3^* z_4)$$
$$+ (r_{12} z_3^* + r_{15} z_4^*) z_4 z_3 + (r_{12} z_3 + r_{15} z_4) z_3^* z_4^*.$$

Proof . Consider the following function: $F_5(z_1, z_2, z_3, z_4) = \left(z_1 - z_1^* - z_1^* \ln \frac{z_2}{z_1^*}\right) + \left(z_2 - z_2^* - z_2^* \ln \frac{z_2}{z_2^*}\right) + \left(z_3 - z_3^* - z_3^* \ln \frac{z_3}{z_3^*}\right) + \left(z_4 - z_4^* - z_4^* \ln \frac{z_4}{z_4^*}\right)$, Obviously $F_5 : R^4_+ \to R$ is a C^1 positive definite function.

So, by differentiating F_5 with respect to time t and performing various algebraic operations, the following result is obtained:

$$\frac{dF_5}{dt} = -\left(1+\frac{r_2 z_2}{z_1 z_1^*}\right)(z_1-z_1^*)^2 + \left(\frac{r_2}{z_1^*}+\frac{r_1}{z_2^*}\right)(z_1-z_1^*)(z_2-z_2^*) - \left(r_4 r_5+\frac{r_1 z_1}{z_2 z_2^*}\right)(z_2-z_2^*)^2$$
$$-(r_6-r_9) z_2 z_3 + (r_6-r_9) z_2 z_3^* + (r_6-r_9) z_2^* z_3 - (r_6-r_9) z_2^* z_3^* - (r_{10}-r_{14}) z_3 z_4$$
$$+(r_{10}-r_{14}) z_3 z_4^* + (r_{10}-r_{14}) z_3^* z_4 - (r_{10}-r_{14}) z_3^* z_4^* - (r_7 z_2 + r_{11} z_3) z_2 z_3 - (r_7 z_2^* + r_{11} z_3^*) z_2^* z_3^*$$
$$-(r_{12} z_3 + r_{15} z_4) z_3 z_4 - (r_{12} z_3^* + r_{15} z_4^*) z_3^* z_4^* + (r_7 z_2 + r_{11} z_3) z_3^* z_2^* + (r_7 z_2^* + r_{11} z_3^*) z_3 z_2$$
$$+(r_{12} z_3^* + r_{15} z_4^*) z_4 z_3 + (r_{12} z_3 + r_{15} z_4) z_3^* z_4^*$$

Now, when we combine the condition (5.5) with the biological facts $r_6 > r_9$ and $r_{10} > r_{14}$, we get:

$$\frac{dF_5}{dt} < -\left[\sqrt{\left(1+\frac{r_2 z_2}{z_1 z_1^*}\right)}(z_1 - z_1^*) - \sqrt{\left(r_4 r_5 + \frac{r_{r_1 z_1}}{z_2 z_2^*}\right)}(z_2 - z_2^*)\right]^2 - (r_6 - r_9) z_2^* z_3^* - (r_7 z_2 + r_{11} z_3) z_2 z_3 - (r_7 z_2^* + r_{11} z_3^*) z_2^* z_3^* - (r_{10} - r_{14}) z_3^* z_4^* - (r_{12} z_3 + r_{15} z_4) z_3 z_4 - (r_{12} z_3^* + r_{15} z_4^*) z_3^* z_4^* + (r_6 - r_9)(z_2 z_3^* + z_2^* z_3) + (r_7 z_2 + r_{11} z_3) z_3^* z_2^* + (r_7 z_2^* + r_{11} z_3^*) z_3 z_2 + (r_{10} - r_{14})(z_3 z_4^* + z_3^* z_4) + (r_{12} z_3^* + r_{15} z_4^*) z_4 z_3 + (r_{12} z_3 + r_{15} z_4) z_3^* z_4^* = -u_1^* + u_2^*$$

So, again by the biological facts $r_6 > r_9$ and $r_{10} > r_{14}$ with condition (5.6) F_5 is negative definite, and then F_5 is Lyapunov function. As a result, we can conclude that F_5 is globally asymptotically stable in the subregion ω_2, and our demonstration is complete. □

6. Numerical simulation:

In this portion, the dynamic behavior of the system (2.2) is numerically explored. For different parameter values and different combinations of initial conditions, large-scale numerical simulations are run. The aim is to explore different options for (biologically plausible) parametric space in order to uncover the possibilities of dynamic behavior of the food chain, Fig. (1)(a-e) illustrates that the system (2.2) has global asymptomatically stable positive equilibrium point.

$$r_1 = 0.5, r_2 = 0.5, r_3 = 0.01, r_4 = 0.3, r_5 = 0.2, r_6 = 0.5, r_7 = 0.1, r_8 = 0.01, r_9 = 0.4, r_{10} = 0.5, r_{11} = 0.1, r_{12} = 0.1, r_{13} = 0.01, r_{14} = 0.4, r_{15} = 0.1, r_{16} = 0.01. \quad (6.1)$$

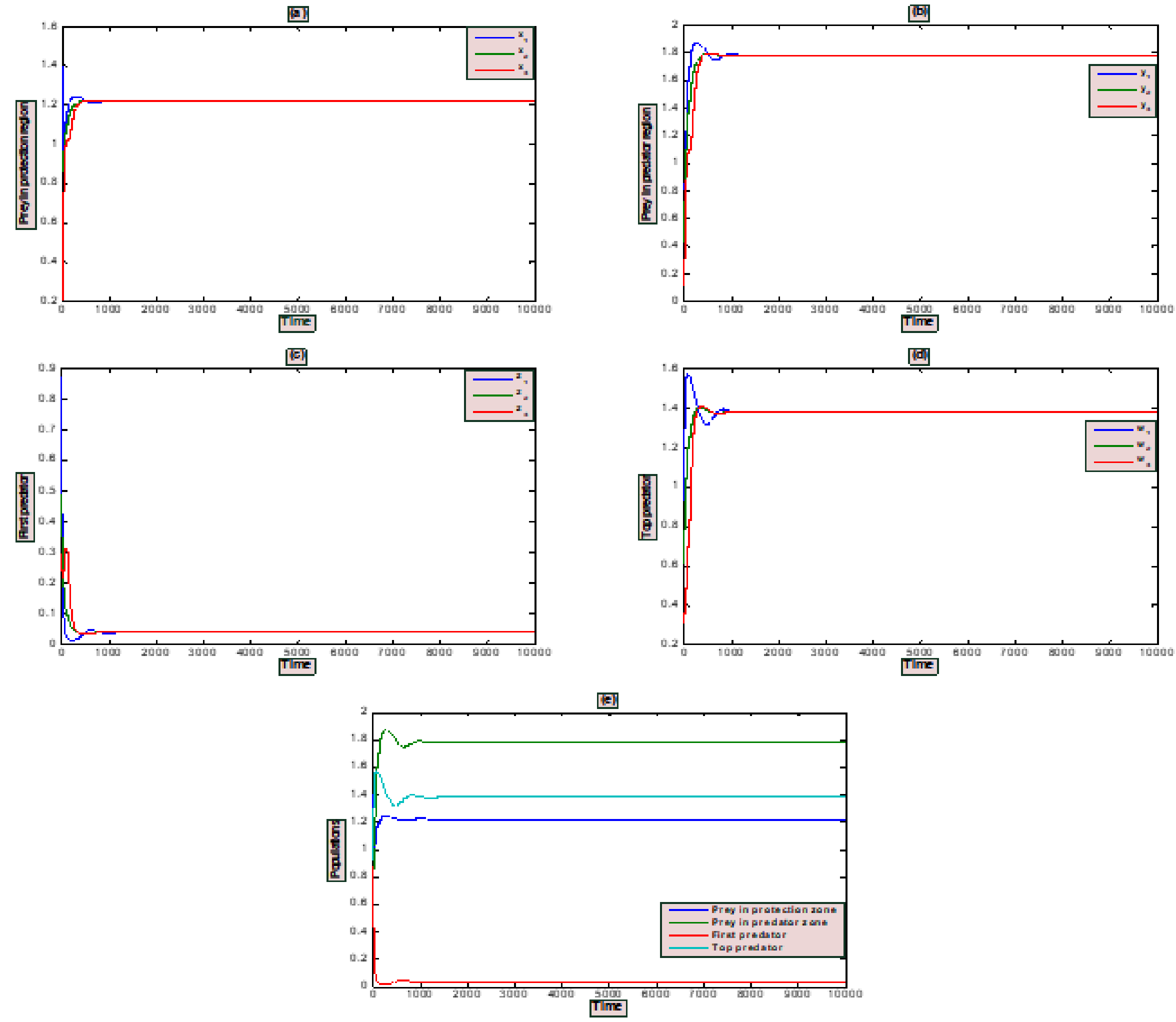

Figure 1: $(a-e)$: Time series of the solution of system (2.2) start with different initial point $(1.5, 0.8, 0.9, 0.9), (1, 0.4, 0.5, 0.6)$, and $(0.2, 0.1, 0.3, 0.3)$, for the data provided in Eq. (6.1) . (a)trajectories variable of x dependent on time (b) trajectories variable of y dependent on time, (c) trajectories variable of z dependent on time, (d) trajectoriesvariable of w dependent on time, (e) time series of the solution of system (2.2).

Obviously, Fig(1)(a-e) shows that system (2.2) has globally asymptotically stable as the solutions of the system (2.2) approaches to the positive point $E_5 = (1.220, 1.780, 0.038, 1.380)$ start from three various initial points and this's verifying the analytical findings we discovered.

Now, in order to discuss the outcome of the parameters values of the system (2.2) on the dynamical behavior of the system, the system is numerically resolved for the data its provided Eq.(6.1) with a different parameter each time.

It's remarked that for $0.01 \le r_1 < 0.043$, the solutions of the system (2.2) approaches to E_1, while for $0.043 \le r_1 < 1$, the solutions of the system (2) approaches to E_5,as shown in Fig. (2)$(a-b)$ for exemplary values (a)$r_{\ 1} = 0.01$ and (b)$r_1 = 0.043$

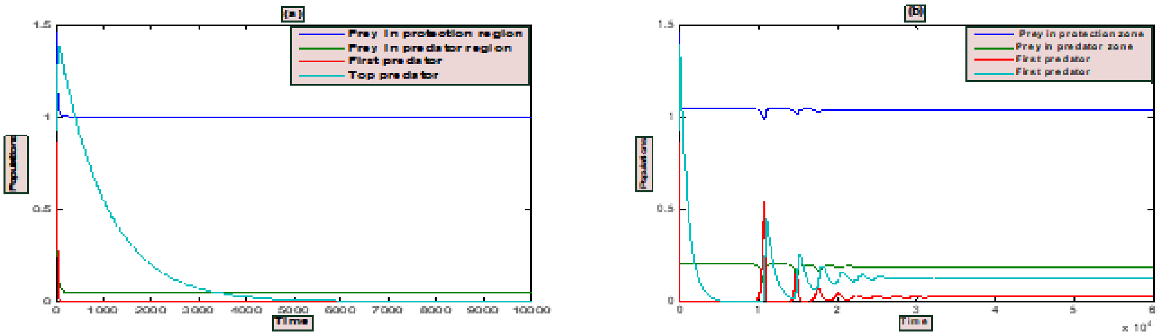

Figure 2: (a−b) : Time series of the solution of system (2.2) approaches to (a)$E_1 = (1.003, 0.047, 0, 0)$ for $r_1 = 0.01$ and (b) $E_5 = (1.039, 0.190, 0.026, 0.130)$ for $r_1 = 0.043$.

the changing of parameters $r_i, i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ and 16, for the data provided in Eq. (6.1) don't play a vital role on the dynamics of system (2.2) and the results have been summarization in table1.

Table 1: Numerical behavior of system (2.2) for the data gives in eq. (6.1) when one factor is changed at a time.

Range of parameter	The stable point	Range of parameter	The stable point
$0.01 \leq r_2 < 1$	E_5	$0.5 < r_{10} \leq 2$	E_5
$0.01 < r_3 < 1$	E_5	$0.01 < r_{11} < 1$	E_5
$0.1 < r_4 < 2$	E_5	$0.01 < r_{12} < 1$	E_5
$0.1 < r_5 < 2$	E_5	$0.01 < r_{13} < 1$	E_5
$0.5 < r_6 \leq 2$	E_5	$0.01 \leq r_{14} < 0.5$	E_5
$0.01 < r_7 < 1$	E_5	$0.01 < r_{15} < 1$	E_5
$0.01 < r_8 < 1$	E_5	$0.01 < r_{16} < 1$	E_5

Furthermore, for with $0.01 \leq r_9 < 0.012$, the solution of the system (2.2) approaches to E_1, while for $0.012 \leq r_9 < 0.015$, the solution of the system (2.2) approaches to E_2, but for $0.015 \leq r_9 < 0.5$, the solution of the system (2.2) approaches to E_5,as shown in Fig. (3)(a-c) for exemplary values (a) $r_9 = 0.01$, (b)$r_9 = 0.013$, (c)$r_9 = 0.2$

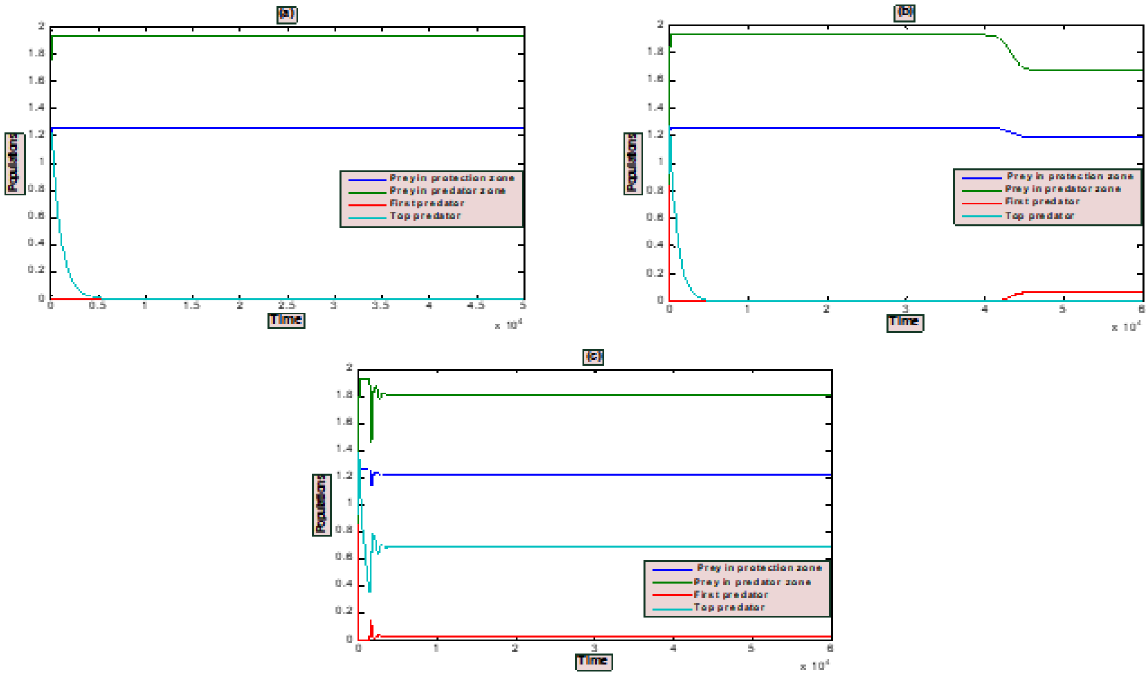

Figure 3: (a-c): Time series of the solution of system (2.2) approcches to (a) $E_1 = (1.257, 1.930, 0, 0)$ for $r_9 = 0.01 (b) E_2 = (1.191, 1.668, 0.070, 0)$ for $r_9 = 0.013$, and (c) $E_5 = (1.227, 1.809, 0.030, 0.689)$ for $r_9 = 0.2$

Finally, change the parameters $0.6 \leq \mathrm{r}_1 < 1, 0.83 < \mathrm{r}_3 < 1$ and $0.5 < \mathrm{r}_8 < 1$ at the same time, with given in Eq. (6.1), it is remarked that the solutions of the system (2.2) approaches to $\mathrm{E}_0 = (0, 0, 0, 0)$ as shown in Fig. (4) for exemplary values $\mathrm{r}_1 = 0.6, \mathrm{r}_3 = 0.83$ and $\mathrm{r}_8 = 0.5$.

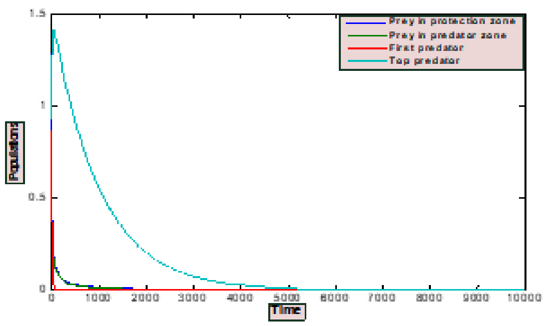

Figure 4: Time series of the solution of system (2.2) for the data provided Eq. (6.1) with $\mathrm{r}_1 = 0.6, \mathrm{r}_3 = 0.83$ and $\mathrm{r}_8 = 0.5$ which approaches to $\mathrm{E}_0 = (0, 0, 0, 0)$

Conclusions and discussion:

In this article, an eco-toxicant model made up of four autonomous non-linear differential equations that represent distinct populations, such as prey in the protective zone $\mathrm{Z}_1(\mathrm{T})$, prey in the predatory zone $\mathrm{Z}_2(\mathrm{T})$ first predator $\mathrm{Z}_3(\mathrm{T})$, and top predator $\mathrm{Z}_4(\mathrm{T})$ has been suggested and studied. Toxins are excreted by all organisms as a form of defense. The prey grows according to the logistic growth law. In order to make the calculation more visible, between the ages of eighteen and sixteen the system's boundedness has been investigated. We've shown that there is equilibrium in the system and investigated its stability. Our findings show that coexistence is possible. Lastly, numerical computations have been used to verify the theoretical stability results which are summarized as follows:

(i) The periodic solution is not existent in the system due to a set of parameters that have been imposed.

(ii) Each of the parameters $\mathrm{r}_1, \mathrm{r}_9$ played an important role in changing the behavior of the solution of system (2.2).

(iii) The parameters $r_i, i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ and 16 have no effect on dynamical behavior of the system (2.2) and the solution of system (2.2) is still approaching to the point $E_5 = (z_1^*, z_2^*, z_3^*, z_4^*)$.

(iv) For the set of parameters given in Eq. (6.1) with $r_1 = 0.6, r_3 = 0.83$ and $r_8 = 0.5$ which approaches to E_0.

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