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Study of the generalized P-contractions on Banach spaces and uniqueness for stability fixed points

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Abstract

This article aims to introduce some concepts such as P-contraction and $G-P$-contraction which are defined on Banach spaces as stronger forms of the concepts of P-contraction and Ćiric type generalized P-contraction defined on complete metric spaces, respectively. Furthermore, the relationship between these concepts have been discussed. Finally, the uniqueness of fixed point for the presented concepts has been investigated.

Keywords: fixed point, metric space, Banach space, contraction mapping, F-contractions.
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1. Introduction

Fixed point theory becomes the cornerstone of many areas of mathematics such as nonlinear functional analysis, mathematical analysis, operator theory, and general topology. The Banach contraction principle [4] is the beginning in fixed point theory on metric spaces. This theorem has a lot of applications in many fields such as chemistry, physics, biology, computer science, and other branches of mathematics. The fixed point theory is split into three major types. The first is the fixed point theory on contraction mappings on complete metric spaces. The second is the fixed point theory on continuous operators on compact and convex sets of a normed space. The third one is fixed point theory on lattices. The beginning of fixed point theory on complete metric spaces is studied by Banach [4] in 1922, where (X, d) be a metric space and $T : X \to X$ be a mapping, then T is called a contraction mapping if there exists a constant $\theta \in [0, 1)$ such that $d(T(x), T(y)) \leq \theta d(x, y)$

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for all $x, y \in X$. Many other authors are interested in studying fixed point theory on metric spaces, for example see [7], [8], [1], [10], and [3]. The concept of fixed point was studied by Gulick [6], where x_0 is in the domain of T. Then x_0 is a fixed point of T, if $T(x_0) = x_0$. Many authors are interested in studying the concept of fixed point (see, [5] and [12]). Minak in [9] introduced the concept of Ćiric type generalized P-contractions, where H is the set of all function $P : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:
1− If $s < t$ in $\mathbb{R}^+$, then $P(s) < P(t)$.
2− Let s_n be a sequence in $\mathbb{R}^+$. Then $\lim_{n\to\infty} s_n = 0$ if $\lim_{n\to\infty} P(s_n) = -\infty$.
3− $\lim_{s\to 0^+} s^m P(s) = 0$, for some m, $0 < m < 1$.
and (N, d) is a metric space with $T : N \to N$ be a map. Then T is called Ćiric type generalized P-contractions if $P \in H$ and there exists $u > 0$ such that if for each $n_1, n_2 \in N$ with $d(T_{n_1}, T_{n_2}) > 0$, then $u + P(d(T_{n_1}, T_{n_2})) \le P(M(n_1, n_2))$, where $M(n_1, n_2) = \max(d(n_1, n_2), d(n_1, T_{n_2}), \frac{1}{2}[d(n_1, T_{n_2}) + d(T_{n_1}, n_2)])$. In [2] Wardowski introduced the concept of F-contractions, where (X, d) is a metric space and $T : X \to X$ is a map. Then T is called F-contraction if there exists $\beta > 0$ such that $\beta + F[d(T(x), T(y))] \le F[d(x, y)]$ for all $x, y \in X$ and $d(T(x), T(y)) > 0$. Popescu in [11] generalized some results of Wardowski.

This paper presents new fixed point theorems namely G-P- contractions on Banach spaces as stronger form of the presented concepts of Amini-Harandi [2] and Popescu [11].

2. The Main Results

Definition 2.1. *Let N be a normed space and $T : N \to N$ be a map. Then T is called P-contractions if $P \in H$ and there is $u > 0$ such that if for each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$, then $u + P(\|T_{n_1} - T_{n_2}\|) \le P(\|n_1 - n_2\|)$.*

Example 2.2. *Let $N = \{\frac{1}{a^2} : a \in \mathbb{N}\} \cup \{0\}$ and $\|n_1 - n_2\| = |n_1 - n_2|$. Then N is a normed space. Define $T : N \to N$ by*

$$T_n = \begin{cases} \frac{1}{(a+1)^2} & if \quad n = \frac{1}{a^2} \\ 0 & if \quad n = 0 \end{cases}$$

Assume that

$$P(x) = \begin{cases} \frac{\ln x}{\sqrt{x}} & if \quad 0 < x < e^2 \\ x - e^2 + \frac{2}{e} & if \quad x \ge e^2 \end{cases}$$

Then the function $P : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:
1− If $s < t$ in $\mathbb{R}^+$, then $P(s) < P(T)$.
2− Let $\{s_n\}$ be a sequence in $\mathbb{R}^+$. Then $\lim_{n\to\infty} s_n = 0$ if $\lim_{n\to\infty} P(s_n) = -\infty$.
3− $\lim_{s\to 0^+} s^m P(s) = 0$, for some m, $0 < m < 1$ (when $m = \frac{2}{3}$),
and hence $P \in H$. Put $u = \ln 2 > 0$. For each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$, then

$$u + P(\|T_{n_1} - T_{n_2}\|) \le P(\|n_1 - n_2\|).$$

Therefore T is P-contraction.

Proposition 2.3. *Every P-contractions on normed spaces is a P-contractions on metric space.*

Proof . Let N be a normed space and $T : N \to N$ be a P-contractions and $P \in H$. Then there exists $u > 0$ such that if for each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$, then

$$u + P(\|T_{n_1} - T_{n_2}\|) \le P(\|n_1 - n_2\|).$$

Define a function $d : N \times N \to R$ by $d(n_1, n_2) = \|n_1 - n_2\|$ for each $n_1, n_2 \in N$. Since N is a normed space, $\|n_1 - n_2\| \geq 0$, hence $d(n_1, n_2) \geq 0$. Now, $d(n_1, n_2) = \|n_1 - n_2\| = 0$ if and only if $n_1 - n_2 = 0$ if and only if $n_1 = n_2$. Then

$$\begin{aligned} d(n_1, n_2) =&\|n_1 - n_2\| = \|n_2 - n_1\| \\ =&d(n_2, n_1) \end{aligned}$$

and

$$\begin{aligned} d(n_1, n_2) =&|n_1 - n_2\| == \|n_1 - n_3 + n_3 - n_2\| \\ \leq&\|n_1 - n_3\| + \|n_3 - n_2\| \\ =&d(n_1, n_3) + d(n_3, n_2). \end{aligned}$$

Then (N, d) is a metric space. Consequentially, $d(T_{n_1}, T_{n_2}) > 0$ and thus $u + P(d(T_{n_1}, T_{n_2})) \leq P(d(n_1, n_2))$. Therefore, $T : N \to N$ is a P-contractions on a metric space. □

Theorem 2.4. *Let $(N, \|\cdot\|)$ be a Banach space and $T : N \to N$ be a continuous P-contractions on N. Then T has a unique stable fixed point in N.*

Proof . Let $\{s_n\}$ be a sequence in N such that $s_n = Tsn - 1, n = 1, 2, \ldots$. If $s_{t+1} = s_t$, for some $t = 0, 1, \ldots$, then $s_t = Ts_t$ and hence T has a fixed point. Thus, assume that $s_{n+1} \neq s_n$ and $q_n = \|s_{n+1} - s_n\|$, for all $n \in 0, 1, 2, \ldots$. Then $q_n > 0$, for all n. Since T is a P-contractions, there exists $u > 0$ and $P(q_n) = P(\|s_{n+1} - s_n\|)$. But $P(\|s_{n+1} - s_n\|) = P(\|Ts_n - Ts_{n-1}\|)$ which is implies that

$$P(q_n) = P(\|Ts_n - Ts_{n-1}\|) \leq P(\big\|(\|s_n - s_{n-1}\|)\big\|) - u = P(q_{n-1}) - u.$$

Thus

$$P(q_n) \leq P(q_{n-1}) - u \leq \ldots \leq P(q_0) - nu.$$

Thus, $\lim_{n\to\infty} P(q_n) = -\infty$ and $\lim_{n\to\infty} q_n = 0$, hence $\lim_{n\to\infty} q_n^m P(q_n) = 0$, for some $0 < m < 1$. Consequentially, $q_n^m P(q_n) - q_n^m P(q_0) \leq -q_n^m nu \leq 0$ and $\lim_{n\to\infty} nq_n^m = 0$ implies that there exists $r \in 1, 2, \ldots$ such that $q_n^m n \leq 1$, for all $n \geq r$. So $q_n \leq \frac{1}{n^{\frac{1}{m}}}$. We claim that s_n is a Cauchy sequence. Let $n, k \in \mathbb{N}$ such that $k > n \geq r$, then

$$\begin{aligned} \|s_n - s_k\| \leq&\|s_n - s_{n+1}\| + \|s_{n+1} - s_{n+2}\| + \ldots + \|s_{k-1} - s_k\| \\ =&q_n + q_{n+1} + \ldots + q_{k-1} \leq \frac{1}{n^{\frac{1}{m}}} + \frac{1}{(n+1)^{\frac{1}{m}}} + \ldots + \frac{1}{(k-1)^{\frac{1}{m}}} \\ \leq&\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{m}}}. \end{aligned}$$

Since $\{\frac{1}{i^{\frac{1}{m}}}\} \to 0$, $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{m}}} = 0$, hence $\|s_n - s_k\| \to 0$, so s_n is a Cauchy sequence in N, but N is a Banach space, then $s_n \to s_0 \in N$. Since T is continuous and $s_n \to s_0$,

$$\lim_{n\to\infty} s_{n+1} = s_0 = \lim_{n\to\infty} Ts_n = T(\lim_{n\to\infty} s_n) = Ts_0.$$

So, we have $s_0 = Ts_0$ and hence s is a fixed point of T. To prove that T has a unique fixed point, let s_1, s_2 are fixed points of T such that $s_1 \neq s_2$, then $Ts_1 = s_1$ and $Ts_2 = s_2$, hence $Ts_1 \neq Ts_2$ and $\|Ts_1 - Ts_2\| > 0$. Since T is a P-contractions, then there is $u > 0$ such that $u \leq P(\|s_1 - s_2\|) - P(\|Ts_1 - Ts_2\|) = 0$ which is a contradiction. Therefore T has a unique fixed point. Now, it remains only to prove that s_0 is stable, let $s \in N$, then T_s^n be a sequence in N and hence as above T_s^n is Cauchy sequence in N, thus T_s^n converges to s_0, therefore s_0 is a stable fixed point of T. □

Definition 2.5. *Let N be a normed space and $T : N \to N$ be a map. Then T is called generalized P-contractions simply (G-P-contractions) if $P \in H$ and there is $u > 0$ such that if for each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$, then $u + P(\|T_{n_1} - T_{n_2}\|) \leq P(\|n_1 - n_2\|^*)$ where*

$$\|n_1 - n_2\|^* = \max\{\|n_1 - n_2\|, \|n_1 - T_{n_2}\|, \frac{1}{2}[\|n_1 - T_{n_2}\| + \|T_{n_1} - n_2\|]\}.$$

Example 2.6. *Let $N = \{\frac{1}{a^2} : a \in \mathbb{N}\} \cup \{0\}$ and $\|n_1 - n_2\| = |n_1 - n_2|$. Then N is a normed space. Define $T : N \to N$ by*

$$T_n = \begin{cases} \frac{1}{(a+1)^2} & if \quad n = \frac{1}{a^2} \\ 0 & if \quad n = 0 \end{cases}$$

Define

$$P(x) = \begin{cases} \frac{\ln x}{\sqrt{x}} & if \quad 0 < x < e^2 \\ x - e^2 + \frac{2}{e} & if \quad x \geq e^2 \end{cases}$$

Then, the function $P : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following conditions:
1− *If $s < t$ in $\mathbb{R}^+$, then $P(s) < P(T)$.*
2− *Let $\{s_n\}$ be a sequence in $\mathbb{R}^+$. Then $\lim_{n\to\infty} s_n = 0$ if $\lim_{n\to\infty} P(s_n) = -\infty$.*
3− *$\lim_{s\to 0^+} s^m P(s) = 0$, for some m, $0 < m < 1$ (when $m = \frac{2}{3}$).*

Hence $P \in H$. Therefore, T is a G-P-contraction, where $u = \ln 2 > 0$. If for each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$, then

$$u + P(\|T_{n_1} - T_{n_2}\|) \leq P(\|n_1 - n_2\|^*),$$

where $\sup \frac{\|T_{n_1} - T_{n_2}\|}{\|n_1 - n_2\|^*} = 1$ and

$$\|n_1 - n_2\|^* = \max\{\|n_1 - n_2\|, \|n_1 - T_{n_2}\|, \frac{1}{2}[\|n_1 - T_{n_2}\| + \|T_{n_1} - n_2\|]\}.$$

Proposition 2.7. *Every P-contractions on a normed space is a G-P-contraction.*

Proof . Let N be a normed space and $T : N \to N$ be a P-contractions and $P \in H$. Then there exists $u > 0$ such that if for each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$,

$$u + P(\|T_{n_1} - T_{n_2}\|) \leq P(\|n_1 - n_2\|) \leq P(\|n_1 - n_2\|).$$

Now let

$$\|n_1 - n_2\|^* = \max\left\{\|n_1 - n_2\|, \|n_1 - T_{n_2}\|, \frac{1}{2}\left[\|n_1 - T_{n_2}\| + \|T_{n_1} - n_2\|\right]\right\}.$$

Then $\|n_1 - n_2\| \leq \|n_1 - n_2\|^*$ and hence $P(\|n_1 - n_2\|) \leq P(\|n_1 - n_2\|^*)$. Therefore, $u + P(\|T_{n_1} - T_{n_2}\|) \leq P(\|n_1 - n_2\|^*)$ and T is a G-P-contractions. □

Proposition 2.8. *Every G-P-contractions on normed space is a Ċiric type generalized P-contractions on metric space.*

Proof . Let N be a normed space and $T : N \to N$ be a G-P-contractions and $P \in H$. Then there exists $u > 0$ such that if for each $n_1, n_2 \in N$ with $\|T_{n_1} - T_{n_2}\| > 0$, then

$$u + P(\|T_{n_1} - T_{n_2}\|) \leq P(\|n_1 - n_2\|^*)$$

where

$$\|n_1 - n_2\|^* = \max\left\{\|n_1 - n_2\|, \|n_1 - T_{n_2}\|, \frac{1}{2}\left[\|n_1 - T_{n_2}\| + \|T_{n_1} - n_2\|\right]\right\}.$$

Define a function $d : N \times N \to \mathbb{R}$ by $d(n_1, n_2) = \|n_1 - n_2\|$ for each $n_1, n_2 \in N$. Then (N, d) is a metric space. Consequentially, $d(T_{n_1}, T_{n_2}) > 0$ and thus

$$u + P(d(T_{n_1}, T_{n_2})) \leq P(M(n_1, n_2)),$$

where

$$M(n_1, n_2) = \max\left\{d(n_1, n_2), d(n_1, T_{n_2}), \frac{1}{2}\left[d(n_1, T_{n_2}) + d(T_{n_1}, n_2)\right]\right\}.$$

Therefore,$T : N \to N$ be a Ciric type generalized P-contractions on a metric space. While the converse is not true and the next example explains that. □

Example 2.9. *Let $N = \{s = (s_1, s_2, ...) : s_k \in \mathbb{R}, k = 1, 2, ...\}$. Define a function $d : N \times N \to \mathbb{R}$ by $d(s, t) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|s_k - t_k|}{(1+|s_k - t_k|)}$. Then, it is clear that (N, d) is a metric space. Let $T : N \to N$ be any Ciric type generalized P-contractions on a metric space N. Then, N is not normed space because $d(\alpha s, \alpha t) \neq |\alpha| d(s, t)$. Hence, $T : N \to N$ is not G-P-contractions on a normed space.*

Theorem 2.10. *Let $(N, \|\cdot\|)$ be a Banach space and $T : N \to N$ be a continuous G-P-contractions on N. Then T has a unique stable fixed point in N.*

Proof . Let $\{s_n\}$ be a sequence in N such that $s_n = Ts_{n-1}, n = 1, 2, ...$ If $s_{t+1} = s_t$ for some $t = 0, 1, ...,$ then $s_t = Ts_t$ and hence T has a fixed point. Thus, assume that $s_{n+1} \neq s_n$ and let $q_n = \|s_{n+1} - s_n\|$, for all $n \in \{0, 1, 2, ...\}$, then $q_n > 0$, for all n. Since T is a G-P-contractions, there exists $u > 0$ and $P(q_n) = P(\|s_{n+1} - s_n\|)$. But $P(\|s_{n+1} - s_n\|) = P(\|Ts_n - Ts_{n-1}\|)$ which implies that

$$\begin{aligned} P(q_n) =& P(\|Ts_n - Ts_{n-1}\|) \leq P\left(\|(\|s_n - s_{n-1}\|)|^*\right) - u \\ =& P(\max\{\|s_n - s_{n-1}\|, \|s_n - s_{n+1}\|\}) - u \\ =& P(\max\{q_{n-1}, q_n\}) - u. \end{aligned}$$

If $q_{n-1} \leq q_n$, for some $n \in \{1, 2, ...\}$, then $P(q_n) \leq P(q_n) - u$ this is a contradiction. Hence $q_{n-1} > q_n$, for all $n \in \{1, 2, ...\}$. So

$$P(q_n) \leq P(q_{n-1}) - u \leq ... \leq P(q_0) - nu.$$

Thus, $\lim_{n\to\infty} P(q_n) = -\infty$ and $\lim_{n\to\infty} q_n = 0$, hence $\lim_{n\to\infty} q_n^m P(q_n) = 0$, for some m, $0 < m < 1$. Consequentially $q_n^m P(q_n) - q_n^m P(q_0) \leq -q_n^m nu \leq 0$ and $\lim_{n\to\infty} nq_n^m = 0$ implies that there exists $r \in \{1, 2, ...\}$ such that $q_n^m n \leq 1$, for all $n \geq r$. So $q_n \leq \frac{1}{n^{\frac{1}{m}}}$. We claim that s_n is a Cauchy sequence, let $n, k \in \mathbb{N}$ such that $k > n \geq r$. Then

$$\begin{aligned} \|s_n - s_k\| \leq& \|s_n - s_{n+1}\| + \|s_{n+1} - s_{n+2} + ... + \|s_{k-1} - s_k\| \\ =& q_n + q_{n+1} + ... + q_{k-1} \leq \frac{1}{n^{\frac{1}{m}}} + \frac{1}{(n+1)^{\frac{1}{m}}} + ... + \frac{1}{(k-1)^{\frac{1}{m}}} \\ \leq& \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{m}}}. \end{aligned}$$

Since $\{\frac{1}{i^{\frac{1}{m}}}\} \to 0$, $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{m}}} = 0$. Hence $\|s_n - s_k\| \to 0$, so s_n is a Cauchy sequence in N, but N is Banach space, then s_n converges to $s_0 \in N$. Since T is continuous and

$$\lim_{n\to\infty} s_{n+1} = s_0 = \lim_{n\to\infty} Ts_n = T\big(\lim_{n\to\infty} s_n\big) = Ts_0,$$

we have $s_0 = Ts_0$ and hence s_0 is a fixed point of T. To prove that T has a unique fixed point, let s_1, s_2 be fixed points of T such that $s_1 \neq s_2$, then $Ts_1 = s_1$ and $Ts_2 = s_2$. Hence $Ts_1 \neq Ts_2$ and $\|Ts_1 - Ts_2\| > 0$. Since T is a G-P-contractions, there exists $u > 0$ such that $u \leq P(\|s_1 - s_2\|) - P(\|Ts_1 - Ts_2\|) = 0$ which is a contradiction. Therefore, T has a unique fixed point. The stability of the fixed point is follows from the fact every sequence in N is Cauchy and hence it converges. □

3. Conclusions

This paper presents new fixed point theorems namely G-P-contraction on Banach spaces as a generalization of P-contraction which are defined on Banach space. The main results of the presented study are as follows:
1− Every P-contractions on normed space is a P-contractions on metric space.
2− Every continuous P-contractions defined on Banach space has a unique stable fixed point.
3− Every P-contractions on normed space is a G-P-contractions.
4− Every G-P-contractions on normed space is a C̀iric type generalized P-contractions on metric space.
5− Every continuous G-P-contractions defined on Banach space has a unique stable fixed point.

References

[1] E. Altiparmak and I. Karahan, *Fixed point results in partially ordered partial bv(S)-metric spaces*, Int. J. Nonlinear Anal. Appl. 12(2) (2021) 35–52.

[2] A. Amini-Harandi, *Fixed and coupled fixed points of a new type set-valued contractive mappings in complete metric spaces*, Fixed Point Theory Appl. 2012(1) (2012) 94.

[3] A.V. Baboli and M.B. Ghaemi, *A common fixed point theorem via measure of noncompactness*, Int. J. Nonlinear Anal. Appl. 12(2) (2021) 293–296.

[4] S. Banach, *Sur les opérationes dans les ensembles abstraits et leur application aux équation intégrales*, Fund. Math. 3(1) (1922) 133–181.

[5] S.N. Elaydi, *Discrete Chaos*, CRC Press, 2007.

[6] D. Gulick, *Encounters with Chaos and Fractals*, CRC Press, 2012.

[7] T. Hamaizia and A. Aliouche, *A nonunique common fixed point theorem of Rhoades type in b -metric spaces with applications*, Int. J. Nonlinear Anal. Appl. 12(2) (2021) 399–413.

[8] D. Kumar, A. Tomar, S. Chandok, and R. Sharma, *Almost α-F-contraction , fixed points and applications*, Int. J. Nonlinear Anal. Appl. 12(2)(2021) 375–386.

[9] G. Mınak, A. Helvacı, and I. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results*, Filomat 28(6) (2014) 1143–1151.

[10] N. S. Mohseni, S. H. P. Masiha, and H. Mamghaderi, *On single valued contractions of Banach and Kannan type in g-metric spaces endowed with a graph*, Int. J. Nonlinear Anal. Appl. 12(2) (2021) 297–304.

[11] O. Popescu and G. Stan, *Two fixed point theorems concerning F-contraction in complete metric spaces*, Sym. 12(1) (2020) 1–10.

[12] J.J. Rotman, *An Introduction to Algebraic Topology*, Springer-Verlag New York Inc. 1988.