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Identity of the connection curvature tensor of almost manifold $C(\lambda)$

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Abstract

This paper aims to investigate the geometry of the projective curvature tensor and to obtain some identities for this tensor. Several three classes of nearly infinite $C(\lambda)$ are distinguished and studied.

Keywords: Projective curvature tensor, Riemannian curvature tensor, almost manifold $-C(\lambda)$.
2010 MSC: Please write mathematics subject classification of your paper here.

1. Introduction

D. Janssen and L. Vanheke [2] proposed the concept of almost manifolds $-C(\lambda)$. The authors defined such manifolds by the Riemann curvature tensor condition, and demonstrated that examples of almost manifold $-C(\lambda)$ are Sasakian cosymplectic manifolds and Kenmotsu manifolds [3]. Furthermore, Olchek and R. Roska [15] investigated such manifolds, and almost manifold $-C(\lambda)$ appear as a subclass of locally conformally almost cosymplectic manifolds in their work and then they studied f-Kenmotsu manifolds that are manifold $-C(\lambda)$ of constant curvature. Several researchers dealt with this class and yielded significant results such as [1, 2, 3, 6, 7, 11, 4, 5, 9]. In this paper, we study the geometry of the projective curvature tensor of a manifold $-C(\lambda)$ using the techniques of [16, 11], and it is organized as follows: The second section provides the necessary information on almost contact metric manifolds, specifically the adjoint G-structure of an almost contact metric manifold. In particular, we construct the adjoint G-structure of an almost contact metric manifold. The third (main) section defines an almost manifold $-C(\lambda)$ and obtains some additional Riemannian curvature tensor identities. We characterize three types of almost manifold $-C(\lambda)$ and give their local characterization in the fourth section.

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2. Preliminary information

Let M be a smooth manifold, $\dim M = 2n+1; X(M) - C^{\infty}(M)$ then the external differentiation operator comes from the module of smooth vector fields on a manifold $M; d-$. All manifolds, tensor fields, etc. Objects are supposedly to be smooth of class C^{∞}.

Definition 2.1. *[8, 12]. A triple of $(\eta, \pounds, \Phi)$ tensor fields on this manifold is almost contact structure on a manifold M, where η is a differential 1-form named a contact form of the structure, $\pounds$-is a vector field named a characteristic field, Φ is an endomorphism of the module $X(M)$, named structural endomorphism. Moreover,*

$$\eta(\pounds),\ 2)\eta^{o}\Phi = 0,\ 3)\Phi^{2} = -id + \eta \bigotimes \pounds \tag{2.1}$$

If, in addition, a fixed Riemannian structure $g = \langle .,. \rangle$ such that

$$(\Phi X, \Phi Y) = \langle x, y\rangle - \eta(X)\eta(Y); X, Y \in X(M) \tag{2.2}$$

a contact metric (in short, AC−) structure is almost the four $(\eta, \pounds, \Phi, g)$. The almost contact metric (in short, AC−) manifold is a nearly contact metric structure is fixed.

It is easy to check that the skew-symmetric is the tensor $\Omega(X, Y) = (X, \Phi Y)$, i.e. it is a 2-form on M. It is named the fundamental form structure.

It is well-known that the existence of an almost contact metric structure on a manifold requires that it be orientable and odd-dimensional. Assume $(\eta, \pounds, \Phi, g)$ is a contact metric structure on the almost manifold $M^{(2n+1)}$. Two mutually complementary projectors $m = \eta \bigotimes \pounds$ and $\ell = id - m = -\Phi^{2}$ [14, 13] are defined internally in module $X(M)$. As a result, $X(M) = \mathcal{L} \bigoplus \mathcal{M}$, where $\mathcal{L} = Im\Phi = ker\eta$ is the so-called contact distribution and $\mathcal{M} = Im\Updownarrow$ is the structural vector's linear hull (where $\Updownarrow$ and ℓ are projectors onto the submodules $\mathcal{M}$ and $\mathcal{L}$, respectively). It is obvious that the distributions are invariant with respect to Φ and mutually orthogonal. It is also obvious that $\tilde{\Phi}^{2} = -id, \langle \Phi, \tilde{\Phi}Y\rangle = \langle X, Y\rangle, X, Y \in \mathcal{L}$ where $\tilde{\Phi} = \Phi|_{\mathcal{L}}$. As a result, if $p \in M$, then an orthonormal frame $(p, e_0, e_1, ..., e_n, \Phi e_1, ..., \Phi e_n)$ is constructed in the tangent space $T_p(M)$,where $e_0 = \xi$. This type of frame is known as a materially adapted frame [19, 9]. On the other hand, let $\mathcal{L}^{c} = \mathcal{L} \bigoplus \mathcal{C}$ be the complexification of the distribution $\mathcal{L}$. Internally, it defines two mutually complementary projectors $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\bar{\sigma} = \frac{1}{2}(id + \sqrt{-1}\Phi)$ onto the properl submodules $\mathcal{D}_{\Phi}^{\sqrt{-1}}$ and $\mathcal{D}_{\Phi}^{-\sqrt{-1}}$ of the endomorphism, which correspond to the eigenvalues (-1) and $-(-1)$, respectively. Hence, it is possible to construct a frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., e_{\hat{n}})$ complexification of the space $T_p(M)$, where $\varepsilon_0 = \xi_p, \varepsilon_a = \sqrt{2}\sigma(e_a), \varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_a)$, consisting of the operator Φ_p's eigenvectors. This a frame is referred to as an A-frame [14, 13]. It is easy to imagine that the matrices of the tensor components Φ_p and g_p in the -frame take the form, respectively.

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix},\ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} \tag{2.3}$$

where I_n denotes the order n identity matrix . It has been demonstrated [14, 13] that the set of frames determines G-structure on M with structural the group $1 \times U(n)$ represented by matrices such as by matrices such as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$ when $A \in U(n)$. A G-structure is called adjoint [14, 13]. Let us *emphasize that the adjoint G-structure space is formed up of complex frames, which are frames of complexification of the corresponding tangent spaces. As a result, even when dealing with real tensors, when we speak about their components over the space of an adjoint G-structure, we are referring to components of complex extensions of these tensors. A tensor of this type is known as real [13]. A real tensor is defined as the sum of a pure complex tensor and its complex conjugate tensor [13]. Throughout this work, we will assume that the indices* $i, j, k, ...$ *have values ranging from* 0 *to* $2n$, *and the indices* $a, b, c, d, f, g, ...$ *have values ranging from* 1 *to* n, *and that* $\hat{a} = a + n, \hat{\hat{a}} = a, \hat{0} = 0$.

3. Almost manifold $-C(\lambda)$

Let $\{M^{(2n+1)}, \eta, \xi, \Phi, g\}$ be an AC-manifold.

Definition 3.1. *[10, 15] An almost manifold* $-C(\lambda)$ *is an almost contact metric manifold if its Riemannian curvature tensor satisfies the relation*

$$\begin{aligned}\langle R(Z,W)Y,X\rangle =& \langle R(\Phi Z,\Phi W)Y,X\rangle - \lambda\{g(X,Y)g(Y,Z) - g(X,Z)g(Y,W)\\ &- g(X\Phi,W)g(Y,\Phi Z) + g(X,\Phi Z)g(Y,\Phi W)\}\end{aligned} \tag{3.1}$$

where $X, Y, Z, W \in X(M)$, *and* λ *in* R.

Definition 3.2. *[10, 15] A normal almost manifold* $-C(\lambda)$ *is called manifold* $-C(\lambda)$.

Theorem 3.3. *[11] An AC -manifold is almost manifold* $-C(\lambda)$ *if and only if the components of its of Riemannian curvature tensor on the space of the adjoint G-structure satisfy the relations:* $R^a_{\hat{b}cd} = \lambda\delta^{ab}_{cd}, R^a_{0b0} = \lambda\delta^a_b, R^a_{bc\hat{d}}$, *by virtue of the Ricci satisfying identity,*

$$R^a_{bc\hat{d}} - R^a_{cb\hat{d}} = -\lambda\delta^{ad}_{bc} \tag{3.2}$$

where λ *is a real number,* $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$, *and the remaining components sub are obtained from the previous identity to the symmetry properties of the curvature tensor or are equal to zero. Using the Riemannian curvature tensor of the known components on the space of the adjoint G-structure, expressions for the components of the Ricci tensor of almost manifold* $-C$ *over the space of the adjoint G-structure* $S_{00} = 2\lambda n, S_{a\hat{b}} = S_{b\hat{a}} = R^b_{ca\hat{c}} + \lambda n\delta^a_b$ *were obtained using the formula* $S_{ij} = -R^k_{ijk}$ *in [11],. The other components are zero. It is easy to notice that*

$$\mathcal{X} = 2\lambda n + 2R^a_{ba\hat{b}} + 2\lambda n^2 \tag{3.3}$$

is scalar curvature almost manifold $-C(\lambda)$ *on the space of the associated G-structure*

Let $\{M^{2n+1}, \eta, \xi, \Phi, g\}$ *be an AC-manifold. [19], Weyl tensor projective recalls that the tensor or curvature tensor of a pseudo-Riemannian manifold* (M^n, g) *is defined as follows*

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}(\langle Q(Z),Y\rangle X - \langle Q(Z),X\rangle Y),$$

where $\mathcal{Q} \leftrightarrow \mathcal{Q}^i_j = g^{ik}r_{kj}$ *is the Ricci operator. Its disappearance is required and sufficient for the manifold* M^n *(locally) to admit a geodesic mapping onto the (pseudo) Euclidean space* R^n *(i.e. to be projectively flat). On the other hand, it is clear that zero of this tensor is equal to the fact that*

$$R(X,Y)Z = \frac{K}{n(n-1)}(\langle Z,Y\rangle X - \langle Z,X\rangle Y)$$

when is the scalar curvature of the metric g, that is equal to the curvature constancy of this manifold. That is, they are spaces with constant curvature and are only projectively [19]. In terms of its contravariant components, the projective curvature tensor can be written as

$$P^i_{jkl} = R^i_{jkl} - \frac{1}{n-1}(S_{lj}\delta^i_k - S_{kj}\delta^i_l), \tag{3.4}$$

or, in terms of its covariant components,

$$P_{ijkl} = R_{ijkl} - \frac{1}{n-1}(S_{lj}g_{ik} - S_{kj}g_{il}). \tag{3.5}$$

Proposition 3.4. 1. *The tensor P is antisymmetric with respect to the second pair of indices; i.e. $P_{ijkl} = -P_{ijlk}$.*
2. $P_{ijkl} + P_{iklj} + P_{iljk} = 0$.

Proof . (1) $P_{ijkl} = R_{ijkl} - \frac{1}{n-1}(S_{lj}g_{ik} - S_{kj}g_{il}) = -R_{ijlk} + \frac{1}{n-1}(S_{kj}g_{ik} - S_{lj}g_{il}) = -P_{ijlk}$.
(2) is obvious. □

Proposition 3.5. *For a Riemannian manifold $MP_{ijkl} = -P_{jikl}$ if and only if M is an Einstein manifold.*

Proof . Taking (3.5) and the properties of the Riemannian curvature tensor in order to obtain:

$$P_{ijkl} = -P_{jikl} \Leftrightarrow S_{jl}g_{ik} - S_{jk}g_{il} = -S_{li}g_{jk} + S_{ki}g_{jl}.$$

The last equality is contracted with the object g^{ih}. Then we get $S_{lj}\delta^k_h - S_{kj}\delta^h_l = -S_{li}g_{jk}g^{ih} + S_{ki}g_{jl}g^{ih}$. We obtain the equality obtained by index k and h, then we obtain $S_{jl}(n-1) = -S_{jl} + Kg_{jl} \Rightarrow S_{jl} = \frac{K}{n}g_{jl}$, then M the Einstein manifold.

Conversely, let M be an Einstein manifold, i.e. $S_{ij} = \lambda g_{ij}$. Then $P_{ijkl} = R_{ijkl} - \frac{1}{n-1}(S_{lj}g_{ik} - S_{kj}g_{il}) = -R_{jikl} - \frac{1}{n-1}(\lambda g_{jl}g_{ik} - \lambda g_{kj}g_{il}) = -R_{jikl} + \frac{1}{n-1}(\lambda g_{kj}g_{il} - \lambda g_{jl}g_{ik}) = -R_{jikl} + \frac{1}{n-1}(S_{il}g_{jk} - S_{ik}g_{jl}) = -P_{jikl}$. □

Proposition 3.6. *For a Riemannian manifold M, the projective curvature tensor satisfies the equality $P_{ijkl} = P_{klij}$ if and only if the M-Einstein manifold.*

Proof . The proof is carried out in the same manner as the previous Proposition's proof. Let M be a nearly manifold $-C(\lambda)$ choosing the transaction Theorem 3.3, the components of the projective curvature tensor for almost manifold $-C(\lambda)$ on the space of the adjoint G -structure can be calculated:

$$\begin{aligned}
&(1)\ P^0_{a0\hat{b}} = R^0_{a0\hat{b}} - \frac{1}{2n}S_{a\hat{b}} = \frac{1}{2}\lambda\delta^a_b - \frac{1}{2n}R^b_{ca\hat{c}}.\\
&(2)\ P^0_{a\hat{b}0} = R^0_{a\hat{b}0} + \frac{1}{2n}S_{a\hat{b}} = \frac{1}{2n}R^b_{ca\hat{c}} - \frac{1}{2}\lambda\delta^a_b.\\
&(3)\ P^a_{bc\hat{d}} = R^a_{bc\hat{d}} - \frac{1}{2n}(S_{b\hat{d}}\delta^a_c - S_{bc}\delta^a_{\hat{d}}) = R^a_{bc\hat{d}} - \frac{1}{2n}(R^d_{hb\hat{h}}\delta^a_c + \lambda n\delta^a_c\delta^d_b).\\
&(4)\ P^a_{b\hat{d}c} = \frac{1}{2n}(R^d_{hb\hat{h}}\delta^a_c + \lambda n\delta^a_c\delta^d_b) - R^a_{bc\hat{d}}.\\
&(5)\ P^a_{\hat{b}cd} = R^a_{\hat{b}cd} - \frac{1}{2n}(S_{\hat{b}d}\delta^a_c - S_{\hat{b}c}\delta^a_d)\\
&\qquad = \lambda\delta^{ab}_{cd} - \frac{1}{2n}(R^b_{hd\hat{h}}\delta^a_c + \lambda n\delta^a_c\delta^b_d - R^b_{hc\hat{h}}\delta^a_d - \lambda n\delta^a_d\delta^b_c)\\
&\qquad = \frac{1}{2}\lambda\delta^{ab}_{cd} - \frac{1}{2n}(R^b_{hd\hat{h}}\delta^a_c - R^b_{hc\hat{h}}\delta^a_d).
\end{aligned} \tag{3.6}$$

as well as complex conjugate formulas the remaining components are all zero.

1. Consider the equalities $P^{a}_{00b} = 0, P^{\hat{a}}_{00b} = 0, P^{0}_{00b} = 0$, i.e. $P^{i}_{00b} = 0$, i.e. $P(\xi, \varepsilon_b)\xi = 0$. Since ε_b is the basis of the subspace $D_{\Phi}^{\sqrt{-1}}$, and the projection onto this subspace is the endomorphism $\pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, therefore the resulting equality written as $1) P(\xi, \Phi^2 X)\xi = 0; 2) P(\xi, \Phi X)\xi = 0; \forall X \in X(M)$. These equalities are equivalent to the following identity

$$P(\xi, \Phi^2 X)\xi = 0; \forall X \in X(M). \tag{3.7}$$

Since $\Phi^2 X = -X + \eta(X)\xi$, can be written in the form $P(\xi, X)\xi = 0; \forall X \in X(M)$. The procedure described above is known as the identity restoration procedure [14, 13].

2. Consider the inequalities $P^{0}_{0ab} = 0, P^{c}_{0ab} = 0, P^{\hat{c}}_{0ab} = 0$, i.e. $P^{i}_{0ab} = 0$, i.e. $P(\varepsilon_a, \varepsilon_b)\xi = 0$.Since ε_b is the basis of the subspace $D_{\Phi}^{\sqrt{-1}}$, the projection onto, this subspace is the endomorphism $\pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, therefore equality written as $P(\Phi^2 X + \sqrt{-1}\Phi X, \Phi^2 Y + \sqrt{-1}\Phi Y)\xi = 0, \forall X \in X(M)$. In the last equality we select the real and imaginary parts, then we get: $1) P(\Phi^2 X, \Phi^2 Y)\xi - P(\Phi Y, \Phi X)\xi = 0;\ 2) P(\Phi^2 Y, \Phi X)\xi + P(\Phi Y, \Phi^2 X)\xi = 0; \forall X, Y \in X(M)$ These equalities are equivalent to the following identity

$$P(\Phi^2 X, \Phi^2 Y)\xi - P(\Phi Y, \Phi X)\xi = 0; \forall X, Y \in X(M). \tag{3.8}$$

3. Applying the above-mentioned procedure for recovering identity to the equalities $P^{0}_{0a\hat{b}} = 0, P^{c}_{0a\hat{b}} = 0, P^{\hat{c}}_{0a\hat{b}} = 0$, we obtain

$$P(\Phi^2 X, \Phi^2 Y)\xi - P(\Phi Y, \Phi X)\xi = 0; \forall X, Y \in X(M). \tag{3.9}$$

From (3.8) and (3.9) we have

$$P(\Phi^2 X, \Phi^2 Y)\xi = P(\Phi Y, \Phi X)\xi = 0; \forall X, Y \in X(M). \tag{3.10}$$

4. If we apply the procedure for restoring the identity to the equality $P^{0}_{0ab} = 0, P^{c}_{a0b} = 0, P^{\hat{c}}_{a0b} = 0$, then we get

$$P(\xi, \Phi^2 X)\Phi^2 Y - P(\xi, \Phi X)\Phi Y = 0; \forall X, Y \in X(M). \tag{3.11}$$

5. Apply the recovery procedure $P^{0}_{a0\hat{b}} = \frac{1}{2}\lambda\delta^{b}_{a} - \frac{1}{2n}S_{a\hat{b}}, P^{c}_{\underline{a0\hat{b}}} = 0, P^{\hat{c}}_{a0\hat{b}} = 0$, i.e $P(\xi, \varepsilon_{\hat{b}})\varepsilon_a = \frac{1}{2}\lambda\langle\varepsilon_a, \varepsilon_{\hat{b}}\rangle\xi - \frac{1}{2n}S(\varepsilon_a, \varepsilon_{\hat{b}})\xi$. Since ε_a is the basis of the subspace $D_{\Phi}^{\sqrt{-1}}, \varepsilon_{\hat{a}}$. Endomorphisms are projections onto the subspaces that are based on the subspace $D_{\Phi}^{-1}.\pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$ and $\bar{\pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$ then the resulting equality can be written in $P(\xi, -\Phi^2 X + \sqrt{-1}\Phi X)(\Phi^2 Y + \sqrt{-1}\Phi Y) = \frac{1}{2}\lambda\langle\Phi^2 X + \sqrt{-1}\Phi X, \Phi^2 Y + \sqrt{-1}\Phi Y)\xi - \frac{1}{2n}S(\Phi^2 X + \sqrt{-1}\Phi X, \Phi^2 Y + \sqrt{-1}\Phi Y)\xi, \forall X, Y \in X(M)$.

We get the equivalent identity from the obtained equality of the real and imaginary parts.

$$\begin{aligned} P(\xi, \Phi^2 X)\Phi^2 Y + P(\xi, \Phi X)\Phi Y = \frac{1}{2}\lambda\langle\Phi^2 X, \Phi^2 Y\rangle\xi + &\frac{1}{2}\lambda\langle\Phi X, \Phi Y\rangle\xi \\ &- \frac{1}{2n}S(\Phi^2 X, \Phi^2 Y)\xi \\ &- \frac{1}{2n}S(\Phi X, \Phi Y)\xi; \forall X, Y \in X(M). \end{aligned} \tag{3.12}$$

By (3.11), the last identity can be written in the form:

$$\begin{aligned} P(\xi,\Phi^2X)\Phi^2Y =& P(\xi,\Phi X)\Phi Y \\ =&\lambda\langle\Phi^2X,\Phi^2Y\rangle\xi+\lambda\langle\Phi X,\Phi Y\rangle\xi \\ &-\frac{1}{n}S(\Phi^2X,\Phi^2Y)\xi-\frac{1}{n}S(\Phi X,\Phi Y)\xi;\forall X,Y\in X(M) \end{aligned} \tag{3.13}$$

6. Similarly, from the equalities $P_a^0bc=0, P_a^dbc=0, P_a^{\hat{d}}bc=0$ we get the identity

$$P(\Phi^2X,\Phi^2Y)\Phi^2Z=P(\Phi^2X,\Phi Y)\Phi Z+P(\Phi X,\Phi^2Y)\Phi Z+P(\Phi X,\Phi Y)\Phi^2Z; X,Y,Z\in X(M). \tag{3.14}$$

7. Apply the identity recovery procedure to the equalities $P_{bc\hat{d}}^0=0, P_{bc\hat{d}}^a=R_{bc\hat{d}}^a-\frac{1}{2n}S_{b\hat{d}}\delta_c^a, P_{bc\hat{d}}^{\hat{a}}=0$, we get

$$\begin{aligned} &P(\Phi^2X,\Phi^2Y)\Phi^2Z+P(\Phi^2X,\Phi Y)\Phi Z-P(\Phi X,\Phi^2Y)\Phi Z+P(\Phi X,\Phi Y)\Phi^2Z \\ =&R(\Phi^2X,\Phi^2Y)\Phi^2Z+R(\Phi^2X,\Phi Y)\Phi Z-R(\Phi X,\Phi^2Y)\Phi Z+R(\Phi X,\Phi Y)\Phi^2Z \\ &-\frac{1}{2n}\{S(\Phi^2Z,\Phi^2Y)\Phi^2XS(\Phi^2Z,\Phi Y)\Phi X-S(\Phi Z,\Phi^2X)\Phi YS(\Phi Z,\Phi Y)\Phi^2X\}, \end{aligned} \tag{3.15}$$

for all $X,Y,Z\in X(M)$. It is worth noting that the following equalities hold for the Ricci tensor, which is almost manifold$-C(\lambda)$ [18].

$$\begin{aligned} &1)S(\xi,\xi)=2\lambda n; \\ &2)S(\xi,X)=2\lambda n\eta(X); \\ &3)S(\Phi^2X,\Phi^2Y)=S(\Phi X,\Phi Y); \\ &4)S(\Phi X,\Phi^2Y)=-S(\Phi^2X,\Phi Y), \end{aligned} \tag{3.16}$$

for all $X,Y\in X(M)$. Then identity 3.15, taking into account 3.14, 3.16, and also applying the identities of the Riemannian curvature of almost manifold $-C(\lambda)$ [17], can be written in the following form:

$$\begin{aligned} p(\Phi^2X,\Phi^2Y)\Phi^2Z-P(\Phi X,\Phi^2Y)\Phi Z =& R(\Phi^2X,\Phi^2Y)\Phi^2Z-R(\Phi X,\Phi^2Y)\Phi Z \\ &-\frac{1}{2n}\{S(\Phi^2Z,\Phi^2Y)\Phi^2X+S(\Phi^2Z,\Phi Y)\Phi X\}, \end{aligned} \tag{3.17}$$

for all $X,Y,Z\in X(M)$.

8. Consider the equalities: $-P_{b\hat{c}\hat{d}}^0=0, P_{b\hat{c}\hat{d}}^a=0, P_{b\hat{c}\hat{d}}^{\hat{a}}=R_{b\hat{c}\hat{d}}^{\hat{a}}-\frac{1}{2n}(S_{b\hat{d}}\delta_{\hat{c}}^{\hat{a}}-S_{b\hat{c}}\delta_{\hat{d}}^{\hat{a}}$ As in the previous case, we get:

$$\begin{aligned} P(\Phi^2X,\Phi^2Y)\Phi^2Z-P(\Phi X,\Phi Y)\Phi^2Z =& R(\Phi^2X,\Phi^2Y)\Phi^2Z-R(\Phi X,\Phi Y)\Phi^2Z \\ &-\frac{1}{2n}\{S(\Phi^2Z,\Phi^2Y)\Phi^2X-S(\Phi^2Z,\Phi^2X)\Phi^2Y+S(\Phi^2Z,\Phi X)\Phi Y\}, \end{aligned} \tag{3.18}$$

for all $X,Y,Z\in X(M)$. Since the manifold $-C(\lambda)$, the identity [17]

$$R(\Phi^2X,\Phi^2Y)\Phi^2Z-R(\Phi X,\Phi Y)\Phi^2Z=\lambda\{\Phi^2X\langle\Phi Y,\Phi Z\rangle-\Phi X\langle Y,\Phi Z\rangle-\Phi^2Y\langle\Phi X,\Phi Z\rangle+\Phi Y\langle X,\Phi Z\rangle\};$$

for all $X, Y, Z \in X(M)$, almost identifies 3.18 and takes the form:

$$\begin{aligned} &P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi Y)\Phi^2Z \\ =&\lambda\ \Phi^2X\langle\Phi Y,\Phi Z\rangle - \Phi X\langle Y,\Phi Z\rangle - \Phi^2Y\langle\Phi X,\Phi Z\rangle + \Phi Y\langle X,\Phi Z\rangle \\ &-\frac{1}{2n}\{S(\Phi^2Z,\Phi^2Y)\Phi^2X - S(\Phi^2Z,\Phi^2X)\Phi^2Y + S(\Phi^2Z,\Phi X)\Phi Y\}, \end{aligned} \tag{3.19}$$

for all $X, Y, Z \in X(M)$. The preceding theorem can be summarized as follows. □

Theorem 3.7. *The following identities are equivalent to the projective curvature tensor of an almost manifold–$C(\lambda)$:*

1. $P(\xi,\Phi^2X)\xi = 0; 2)P(\xi,X)\xi = 0; 3)P(\Phi^2X,\Phi^2Y)\xi - P(\Phi Y,\Phi X)\xi = 0.$
2. $P(\Phi^2X,\Phi^2Y)\xi + P(\Phi Y,\Phi X)\xi = 0.$
3. $P(\Phi^2X,\Phi^2Y)\xi = P(\Phi Y,\Phi X)\xi = 0.$
6. $P(\xi,\Phi^2X)\Phi^2Y - P(\xi,\Phi X)\Phi Y =$
7. $P(\xi,\Phi^2X)\Phi^2Y = P(\xi,\Phi X)\Phi Y = \lambda\langle\Phi^2X,\Phi^2Y\rangle\xi + \lambda\langle\Phi X,\Phi Y\rangle\xi - \frac{1}{n}S(\Phi^2X,\Phi^2Y)\xi - \frac{1}{n}S(\Phi X,\Phi Y)\xi;$
8. $P(\Phi^2X,\Phi^2Y)\Phi^2Z = P(\Phi^2X,\Phi Y)\Phi Z + P(\Phi X,\Phi^2Y)\Phi Z + P(\Phi X,\Phi Y)\Phi^2Z;$
9. $P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi^2Y)\Phi Z = R(\Phi^2X,\Phi^2Y)\Phi^2Z - R(\Phi X,\Phi^2Y)\Phi Z - \frac{1}{2n}\{S(\Phi^2Z,\Phi^2Y)\Phi^2X + S(\Phi^2Z,\Phi Y)\Phi X\};$
10. $P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi Y)\Phi^2Z = \lambda\{\Phi^2X\langle\Phi Y,\Phi Z\rangle - \Phi X\langle Y,\Phi Z\rangle - \Phi^2Y\langle\Phi X,\Phi Z\rangle + \Phi Y\langle X,\Phi Z\rangle\} - \frac{1}{2n}\{S(\Phi^2Z,\Phi^2Y)\Phi^2X - S(\Phi^2Z,\Phi Y)\Phi X - S(\Phi^2Z,\Phi^2X)\Phi^2Y + S(\Phi^2Z,\Phi X)\Phi Y\}; X, Y, Z \in X(M)$

4. Classes of almost manifold $-C(\lambda)$

The identity

$$\begin{aligned} P(\xi,\Phi^2X)\Phi^2Y =&P(\xi,\Phi X)\Phi Y \\ &=\lambda\langle\Phi^2X,\Phi^2Y\rangle\xi + \lambda\langle\Phi X,\Phi Y\rangle\xi - \frac{1}{n}S(\Phi^2X,\Phi^2Y)\xi - \frac{1}{n}S(\Phi X,\Phi Y)\xi, \end{aligned} \tag{4.1}$$

for all $X, Y, Z \in X(M)$, is the first additional property of an almost manifold's projective curvature tensor of an almost manifold $C(\lambda)$.

Definition 4.1. *$C(\lambda)$ on manifold M that satisfies the first additional identity of projective curvature, or is a manifold of class CP_1 if*

$$P(\xi,\Phi^2X)\Phi^2Y = P(\xi,\Phi X)\Phi Y = 0; \quad X, Y, Z \in X(M). \tag{4.2}$$

Theorem 4.2. *Almost manifold $C(\lambda)$ is a manifold of class CP if and only if $P^0_{a0\hat{b}} = 0$, where $P^0_{a0\hat{b}}$ is the projective curvature tensor component for almost manifold $C(\lambda)$ on the space of the adjoint G -structure.*

Proof . Given almost manifold C- is a manifold of class CP_1. Then, according to definition 4.2, there is a space in which the identity is written $P(\xi,\Phi X)\Phi Y = 0; \forall X, Y \in X(M)$; in the attached G-structure written in the form of $P^0_{i0j}(\Phi Y)^i(\Phi X)^j\xi + P^c_{i0j}(\Phi Y)^i(\Phi X)^j\varepsilon_c + P^{\hat{c}}_{i0j}(\Phi Y)^i(\Phi X)^j\varepsilon_{\hat{c}} = 0$, which, when combined with 2.3 and 3.6, will take the form $P^0_{a0\hat{b}}\xi + P^0_{\hat{b}0a}\xi = 0$. i.e. $P^0_{a0\hat{b}} = 0$.

Conversely, let for an almost manifold $C(\lambda)$ –and $P^0_{a0\hat{b}} = 0$. Since for almost manifold $C(\lambda)$ -we obtain $P^c_{a0\hat{b}} = 0$ and $P^{\hat{c}}_{a0\hat{b}} = 0$, using the procedure for reconstructing the identities to the equalities $P^i_{a0\hat{b}} = 0$ we get $P(\xi,\Phi^2X)\Phi^2Y = P(\xi,\Phi X)\Phi Y = 0; \forall X, Y \in X(M)$. □

Theorem 4.3. *Almost manifold $C(\lambda)$, which is a manifold of the class CP_1, must be an Einstein manifold.*

Proof . Let M be almost manifold $C(\lambda)$ - the manifold of class CP_1, that is, $P^0_{a0\hat{b}} = 0$. By virtue of 3.6: 1, this is equivalent to $\frac{1}{2}\lambda\delta^b_a - \frac{1}{2n}R^b_{ca\hat{c}} = 0$, i.e.

$$R^b_{ca\hat{c}} = \lambda n\delta^b_a. \tag{4.3}$$

Then for the Ricci tensor we have $S_{00} = 2\lambda n, S_{a\hat{b}} = S_{\hat{b}a} = R^b_{ca\hat{c}} + \lambda n\delta^b_a = 2\lambda n\delta^b_a$, i.e. $S_{ij} = 2n\lambda g_{ij}$ i.e. M- Einstein manifold.□

The second additional property of the projective curvature tensor of an almost manifold $C(\lambda)$ is called identity

$$\begin{aligned} P(\Phi^2X, \Phi^2Y)\Phi^2Z - P(\Phi X, \Phi^2Y)\Phi Z =& R(\Phi^2X, \Phi^2Y)\Phi^2Z - R(\Phi X, \Phi^2Y)\Phi Z \\ & - \frac{1}{2n}\{S(\Phi^2Z, Phi^2Y)\Phi^2X + S(\Phi^2Z, \Phi Y)\Phi X\}, \end{aligned} \tag{4.4}$$

for all $X, Y, Z \in X(M)$. □

Definition 4.4. *The $C(\lambda)$ on manifold M that satisfies the second additional identity of projective curvature, or is a manifold of classCP_2 if*

$$P(\Phi^2X, \Phi^2Y)\Phi^2Z - P(\Phi X, \Phi^2Y)\Phi Z = 0; \quad \forall X, Y, Z \in X(M). \tag{4.5}$$

Theorem 4.5. *Almost manifold $C(\lambda)$ is a manifold of class CP_2 if and only if $P^a_{bc\hat{d}} = 0$, where $P^a_{bc\hat{d}}$ is the projective curvature tensor component for almost manifold $C(\lambda)$ on the space of the adjoint G-structure.*

Proof . The proof is the same as in Theorem 4.2. □

Theorem 4.6. *Almost manifold $C(\lambda)$, which is a manifold of the class CP_2, must be a manifold of pointwise constant Φ-holomorphic sectional curvature with $c = 2\lambda$*

Proof . Consider M is almost manifold $C(\lambda)$ - of class CP_2, that is, $P^a_{bc\hat{d}} = 0$ By virtue of 3.6: 3, this is equivalent to the fact that $P^a_{bc\hat{d}} = R^a_{bc\hat{d}} - \frac{1}{2n}(R^d_{hb\hat{h}}\delta^a_c + \lambda n\delta^a_c\delta^d_b) = 0$, i.e.

$$R^a_{bc\hat{d}} = \frac{1}{2n}\left(R^d_{hb\hat{h}}\delta^a_c + \lambda n\delta^a_c\delta^d_b\right). \tag{4.6}$$

$R^b_{ca\hat{c}} = \lambda n\delta^b_a$ is obtained by reducing equality 4.9 with respect to the indices a and c. This means that the manifold is an Einstein manifold. Taking equality into account, $R^b_{ca\hat{c}} = \lambda n\delta^b_a$ identity 4.9 takes the form $R^a_{bc\hat{d}} = \lambda\delta^a_c\delta^d_b$.

The obtained equality is symmetrized first with respect to indices a and d, and then with respect to indices b and c, yielding $R^{(ad}_{(}(bc)) = \lambda\tilde{\delta}^{ad}_{bc}$,that is, $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$. The manifold of point constant holomorphic sectional curvature $c = 2\lambda$ [18] is the manifold of point constant.□

We call the identity

$$\begin{aligned} P(\Phi^2X, \Phi^2Y)\Phi^2Z - P(\Phi X, \Phi Y)\Phi^2Z =& \lambda\{\Phi^2X\langle\Phi Y, \Phi Z\rangle - \Phi X\langle Y, \Phi Z\rangle - \Phi^2Y\langle\Phi X, \Phi Z\rangle + \Phi Y\langle X, \Phi Z\rangle\} \\ & - \frac{1}{2n}\{S(\Phi^2Z, \Phi^2Y)\Phi^2X - S(\Phi^2Z, \Phi Y)\Phi X \\ & - S(\Phi^2Z, \Phi^2X)\Phi^2Y + S(\Phi^2Z, \Phi X)\Phi Y\}, \end{aligned} \tag{4.7}$$

for all $X, Y, Z \in X(M)$. The projective curvature tensor's third additional property is almost manifold $C(\lambda)$. □

Definition 4.7. *$C(\lambda)$ on manifold M that satisfies the second additional identity of projective curvature, or is a manifold of classCP_3 if*

$$P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi Y)\Phi^2Z = 0; X,Y,Z \in X(M). \tag{4.8}$$

Theorem 4.8. *Almost manifold $C(\lambda)$ is a manifold of class CP if and only if $P^{\hat{a}}_{b\hat{c}\hat{d}} = 0$, where $P^{\hat{a}}_{b\hat{c}\hat{d}}$ is the projective curvature tensor component for almost manifold $C(\lambda)$ on the space of the adjoint G -structure.*

Proof . The proof is the same as in Theorem 4.2. □

Theorem 4.9. *Almost manifold $C(\lambda)$, which is a manifold of the class CP_3, must be an Einstein manifold*

Proof . Let M be an almost manifold C - of class CP_3, i.e., $PP^{\hat{a}}_{b\hat{c}\hat{d}} = 0$ is 0.This equivalents to $\frac{1}{2}\lambda\delta^{ab}_{cd} - \frac{1}{2n}(R^b_{hd\hat{h}}\delta^a_c - R^b_{hc\hat{h}}\delta^a_d) = 0$ according to 3.6: 5.Then we have $S_{00} = 2\lambda n, S_{a\hat{b}} = S_{\hat{b}a} = R^b_{ca\hat{c}} + \lambda n\delta^b_a = 2\lambda n\delta^b_a$, i.e. $S_{ij} = 2n\lambda g_{ij}$ for the Ricci tensor, which is M-Einstein manifold. □

References

[1] A. Akbar, *Some results on almost $C(\lambda)$-manifolds*, Int. J. Math. Sci. Engin. Appl. 7(1) (2013) 255–260.

[2] A. Akbar and A. Sarkar, *On the conharmonic and concircular curvature tensors of almost $C(\lambda)$-manifolds*, Int. J. Adv. Math. Sci. 1(3) (2013) 134–138.

[3] A. Akbar and A. Sarkar, *Almost $C(\lambda)$-manifolds admitting Wz curvature tensor*, J. Rajashtan Acad. Phys. Sci. 13(1) (2014) 31–38.

[4] S.R. Ashoka, C.S. Bagewadi and G. Ingalahalli, *Curvature tensor of almost $C(\lambda)$-manifolds*, Malaya J. Mat. 2(1) (2014) 10–15.

[5] S.R. Ashoka, C.S. Bagewadi and G. Ingalahalli, *A study on ricci solitons in almost $C(\lambda)$-manifolds*, Sohag J. Math. 3(2) (2016) 83–88.

[6] M. Atçeken and U. Yildirim, *On an almost $C(\lambda)$ manifold satisfying certain conditions on the concircular curvature tensor*, Pure Appl. Math. J. 4 (1-2) (2015) 31–34.

[7] M. Atçeken and U. Yildirim, *Almost Cla manifolds satisfying certain curvature conditions*, Adv. Stud. Contemp. Math. 26(3) (2016) 567–578.

[8] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag, 1976.

[9] V.V. Chaturvedi and B.K. Gupta, *C-Bochner curvature tensor on almost $C(\lambda)$ manifolds*, Palestine J. Math. 8(2) (2019) 258–265.

[10] D. Janssen and L. Vanhecke, *Almost contact structure sand curvature tensors*, Kodai Math. J. 4 (1981) 1–27.

[11] S. V. Kharitonova, *Almost $C(\lambda)$-manifolds*, Founda. Mat. 16(2) (2010) 139–146.

[12] V.F. Kirichenko, *Methods of generalized Hermitian geometry in the theory of almost contact manifolds*, J. Soviet Math. 42(5) (1988) 1885–1919.

[13] V.F. Kirichenko, and A.R. Rustanov, *Differential geometry of quasi-Sasakian manifolds*, Sbornik Math. 193(8) (2002) 1173–1201.

[14] V.F. Kirichenko, *Differential-Geometric Structures on Manifolds*, Odessa, 2013.

[15] Z. Olszak and R. Rosca, *Normal locally confomal almost cosymplectic manifolds*, Pub. Math. Debrecen 39(3-4) (1991) 315–323.

[16] A.R. Rustanov, S.V. Kharitonova and O.N. Kazakova, *About two classes of almost $C(\lambda)$ manifolds*, Bull. Orenburg State Univ. 178(3) (2015) 228–231.

[17] A.R. Rustanov, E.A. Polkina and S.V. Kharitonova, *Identities of the Riemannian curvature tensor of almost SS manifolds*, News of Universities, North Caucasus region, Natural Sciences. In the press, 2021.

[18] A. R. Rustanov, E. A. Polkina and S. V. Kharitonova, *On some aspects of the geometry of almost $C(\lambda)$ manifolds*, Bull. Higher Educ. Instit. North Caucasus Region. Natural Sci. 2020(3) (2020) 19–24.

[19] N.S. Sinyukov, *Geodesic Mappings of Riemannian Spaces*, Nauka , 1979.