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Representation of solutions of eight systems of difference equations via generalized Padovan sequences

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Abstract

We indicate that the systems of difference equations

$$x_{n+1}=f^{-1}\big(af\left(p_{n-1}\right)+bf\left(q_{n-2}\right)\big),\quad y_{n+1}=f^{-1}\big(af\left(r_{n-1}\right)+bf\left(s_{n-2}\right)\big),\quad n\in\mathbb{N}_0,$$

where the sequences p_n, q_n, r_n, s_n are some of the sequences x_n and y_n, $f:D_f\longrightarrow\mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f\subseteq\mathbb{R}$, initial values x_{-j}, y_{-j}, $j\in\{0,1,2\}$ are arbitrary real numbers in D_f and the parameters a,b are arbitrary complex numbers, with $b\neq 0$, can be solved in the explicit form in terms of generalized Padovan sequences.

Keywords: system of difference equations, solution of explicit form, Padovan number
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1. Introduction and Preliminaries

For the first time, remind that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, symbolize natural, non-negative integer, integer, real and complex numbers, respectively. If $k,l\in\mathbb{Z}$, $k\leq l$ the notation $i=\overline{k,l}$ stands for $\{i\in\mathbb{Z}:k\leq i\leq l\}$.

Nonlinear difference equations or systems of difference equations are a rich area of study in mathematics. For the last two decades, many papers on these equations or systems that are related to number sequences such as Fibonacci, Lucas, Padovan, Tetranacci, Horadam, Pell, Jacobsthal, and

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Jacobsthal-Lucas sequences and generalized of these number sequences, have been published (see, e.g., [3, 4, 5, 10, 13, 16, 21, 26, 27, 28, 31, 32, 33, 36]). In addition, other type difference equations or systems of difference equations were studied in [11, 12, 14, 17, 18, 20, 30, 34, 35, 38].

The equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} \pm cx_{n-q}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary positive real numbers, $k,\ l,\ p,\ q$ are non-negative integers and $a,\ b,\ c$ are positive constants, is some of the difference equations whose solutions are related to number sequences. Several mathematicians have been struggled positive solutions of concrete special cases of equation (1.1). Firstly, Elabbasy et al., in [3, 4], attained positive solutions of some special cases of equation (1.1) by using induction principle. One of the special cases is

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_{n-1} + x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

whose solutions are associated with the well known Padovan numbers in literature. In addition, the multi-dimensional expansion of the concrete some special cases of equation (1.1) can be seen in the literature (see, e.g., [2, 6, 7, 8, 9, 22, 23, 37]). The another equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where the parameters a, b, c and initial values x_{-1} and x_0 are complex numbers and $c \neq 0$, which is one of these equations. The solutions of equation (1.3) are related to number sequence, has dealt with in [24]. Unlike in (1.1), by using convenient transformation the equation in (1.3) reduce to the next third-order linear difference equation with constant coefficients

$$x_{n+1} = ax_n + bx_{n-1} + cx_{n-2}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

which has actually the general solution

$$x_n = x_0 S_n + x_{-1}\left(S_{n+1} - aS_n\right) + cx_{-2}S_{n-1}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $(S_n)_{n\geq -2}$ of equation (1.4) satisfying the initial values $S_{-2} = S_{-1} = 0,\ S_0 = 1$. Recently in [25], among other things, the next difference equation

$$x_n = f^{-1}\Big(af\left(x_{n-1}\right) + bf\left(x_{n-2}\right) + cf\left(x_{n-3}\right)\Big),\ n \in \mathbb{N}_0, \tag{1.6}$$

where $f : D_f \longrightarrow \mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f \subseteq \mathbb{R}$, parameters a, b, c and the initial values x_{-3}, x_{-2} and x_{-1} are real numbers, is a generalization of the equation (1.4). Morever, the authors have got the solution of the equation (1.6) in relation to the solution given in (1.5).
On the other hand, one of the popular topics for system of difference equations is also symmetric and close-to-symmetric systems such as

$$x_{n+1} = g\left(p_{n-k}, q_{n-l}\right), \quad y_{n+1} = g\left(r_{n-k}, s_{n-l}\right), \quad n \in \mathbb{N}_0, \tag{1.7}$$

where the sequences $p_n,\ q_n,\ r_n,\ s_n$ are some of the sequences x_n and y_n and $k,\ l$ are fixed natural numbers. There are some of studies which are some special cases of the system (1.7) in literature (see, for example, [1, 15, 19, 29, 39]).

Motivated by this line of investigations, here we indicate that the systems of difference equations

$$x_{n+1} = f^{-1}\Big(af\left(p_{n-1}\right) + bf\left(q_{n-2}\right)\Big), \quad y_{n+1} = f^{-1}\Big(af\left(r_{n-1}\right) + bf\left(s_{n-2}\right)\Big), \quad n \in \mathbb{N}_0, \tag{1.8}$$

where the sequences $p_n,\ q_n,\ r_n,\ s_n$ are some of the sequences x_n and y_n, $f : D_f \longrightarrow \mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f \subseteq \mathbb{R}$, the initial values $x_{-j},\ y_{-j},\ j \in \{0, 1, 2\}$ are arbitrary real numbers and the parameters and a, b are arbitrary complex numbers, can be solved. To do this, we will use the solutions given in (1.5) and the solutions obtained by rearranging these solutions.

2. Main Results

In this section, we consider the eight special cases of systems (1.8), where the sequences p_n, q_n, r_n, s_n are some of the sequences x_n and y_n, for $n \geq -2$, and initial values x_{-j}, y_{-j}, $j \in \{0, 1, 2\}$, are arbitrary real numbers.

2.1. *Case 1:* $p_n = x_n$, $q_n = x_n$, $r_n = x_n$, $s_n = y_n$

In this case, system (1.8) becomes

$$x_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2})\big), \quad y_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(y_{n-2})\big), \quad n \in \mathbb{N}_0. \tag{2.1}$$

Since f is "$1-1$", from (2.1)

$$f(x_{n+1}) = af(x_{n-1}) + bf(x_{n-2}), \quad f(y_{n+1}) = af(x_{n-1}) + bf(y_{n-2}), \quad n \in \mathbb{N}_0. \tag{2.2}$$

By using the changes of variables

$$f(x_n) = u_n, \quad \text{and} \quad f(y_n) = v_n, \quad n \geq -2, \tag{2.3}$$

system (2.2) is transformed to the following one

$$u_{n+1} = au_{n-1} + bu_{n-2}, \quad v_{n+1} = au_{n-1} + bv_{n-2}, \ n \in \mathbb{N}_0. \tag{2.4}$$

By taking $a = 0$, $b = a$, $c = b$ in (1.4) and $S_n = J_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5) the solution of the first equation in (2.4) as

$$u_n = u_0 J_{n+1} + u_{-1} J_{n+2} + bu_{-2} J_n, \ n \in \mathbb{N}_0. \tag{2.5}$$

By subtracting the second one from the first equations in (2.4), we have

$$u_{n+1} - v_{n+1} = b(u_{n-2} - v_{n-2}), \ n \in \mathbb{N}_0. \tag{2.6}$$

From (2.6) we see that the sequence $(u_n - v_n)_{n \geq -2}$ satisfies the following difference equation

$$w_n = bw_{n-3}, \ n \geq 1, \tag{2.7}$$

from which it follows that

$$u_{3n+i} - v_{3n+i} = b^{n+1}(u_{i-3} - v_{i-3}), \tag{2.8}$$

for $n \in \mathbb{N}_0$, $i \in \{1, 2, 3\}$.
From (2.5) and (2.8), we get

$$\begin{aligned} v_{3n+i} &= u_{3n+i} - b^{n+1}u_{i-3} + b^{n+1}v_{i-3}, \\ &= u_0 J_{3n+i+1} + u_{-1} J_{3n+i+2} + bu_{-2} J_{3n+i} - b^{n+1}u_{i-3} + b^{n+1}v_{i-3}, \end{aligned} \tag{2.9}$$

for $n \in \mathbb{N}_0$, $i \in \{1, 2, 3\}$.
Employing (2.5) and (2.9) in (2.3) and after some calculation, we obtain

$$x_n = f^{-1}\left[f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + bf(x_{-2}) J_n\right], \ n \geq -2, \tag{2.10}$$

$$y_{3n+1} = f^{-1}\big(f(x_0) J_{3n+2} + f(x_{-1}) J_{3n+3} + f(x_{-2})\big(bJ_{3n+1} - b^{n+1}\big) + b^{n+1}f(y_{-2})\big), \ n \geq -1, \tag{2.11}$$

$$y_{3n+2} = f^{-1}\big(f(x_0) J_{3n+3} + f(x_{-1})\big(J_{3n+4} - b^{n+1}\big) + bf(x_{-2}) J_{3n+2} + b^{n+1}f(y_{-1})\big), \ n \geq -1, \tag{2.12}$$

$$y_{3n+3} = f^{-1}\big(f(x_0)\big(J_{3n+4} - b^{n+1}\big) + f(x_{-1}) J_{3n+5} + bf(x_{-2}) J_{3n+3} + b^{n+1}f(y_0)\big), \ n \geq -1. \tag{2.13}$$

2.2. Case 2: $p_n = y_n,\ q_n = x_n,\ r_n = y_n,\ s_n = y_n$

In this case, we obtain the system

$$x_{n+1} = f^{-1}\big(af\left(y_{n-1}\right) + bf\left(x_{n-2}\right)\big), \quad y_{n+1} = f^{-1}\big(af\left(y_{n-1}\right) + bf\left(y_{n-2}\right)\big), \quad n \in \mathbb{N}_0. \tag{2.14}$$

Note that system (2.14) are obtained from equations (2.1) by interchanging letters x and y, from which all the statements concerning solutions to the equations follow from the corresponding statements in Case 1 by only interchanging letters x and y.
The general solutions to the system (2.14) is given

$$x_{3n+1} = f^{-1}\big(f\left(y_0\right)J_{3n+2} + f\left(y_{-1}\right)J_{3n+3} + f\left(y_{-2}\right)\left(bJ_{3n+1} - b^{n+1}\right) + b^{n+1}f\left(x_{-2}\right)\big),\ n \geq -1, \tag{2.15}$$

$$x_{3n+2} = f^{-1}\big(f\left(y_0\right)J_{3n+3} + f\left(y_{-1}\right)\left(J_{3n+4} - b^{n+1}\right) + bf\left(y_{-2}\right)J_{3n+2} + b^{n+1}f\left(x_{-1}\right)\big),\ n \geq -1, \tag{2.16}$$

$$x_{3n+3} = f^{-1}\big(f\left(y_0\right)\left(J_{3n+4} - b^{n+1}\right) + f\left(y_{-1}\right)J_{3n+5} + bf\left(y_{-2}\right)J_{3n+3} + b^{n+1}f\left(x_0\right)\big),\ n \geq -1, \tag{2.17}$$

$$y_n = f^{-1}\big(f\left(y_0\right)J_{n+1} + f\left(y_{-1}\right)J_{n+2} + bf\left(y_{-2}\right)J_n\big),\ n \geq -2. \tag{2.18}$$

2.3. Case 3: $p_n = y_n,\ q_n = y_n,\ r_n = x_n,\ s_n = y_n$

In this case, system (1.8) is expressed as

$$x_{n+1} = f^{-1}\big(af\left(y_{n-1}\right) + bf\left(y_{n-2}\right)\big), \quad y_{n+1} = f^{-1}\big(af\left(x_{n-1}\right) + bf\left(y_{n-2}\right)\big), \quad n \in \mathbb{N}_0. \tag{2.19}$$

Since f is "$1-1$", from (2.19)

$$f\left(x_{n+1}\right) = af\left(y_{n-1}\right) + bf\left(y_{n-2}\right), \quad f\left(y_{n+1}\right) = af\left(x_{n-1}\right) + bf\left(y_{n-2}\right), \quad n \in \mathbb{N}_0. \tag{2.20}$$

By using the changes of variables

$$f\left(x_n\right) = u_n, \quad n \geq -1, \quad \text{and} \quad f\left(y_n\right) = v_n, \quad n \geq -2, \tag{2.21}$$

system (2.20) is transformed to the following one

$$u_{n+1} = av_{n-1} + bv_{n-2}, \quad v_{n+1} = au_{n-1} + bv_{n-2},\ n \in \mathbb{N}_0. \tag{2.22}$$

Employing the first equation in (2.22) in the second one we get

$$v_{n+1} = bv_{n-2} + a^2v_{n-3} + abv_{n-4},\ n \geq 2. \tag{2.23}$$

Equation (2.23) can be written in the following form

$$v_{n+1} + av_{n-1} = a\left(v_{n-1} + av_{n-3}\right) + b\left(v_{n-2} + av_{n-4}\right),\ n \geq 2, \tag{2.24}$$

from (2.5), we can write the solution of equation (2.24) as

$$v_{n+1} + av_{n-1} = J_n\left(v_2 + av_0\right) + J_{n+1}\left(v_1 + av_{-1}\right) + bJ_{n-1}\left(v_0 + av_{-2}\right),\ n \geq -1. \tag{2.25}$$

By using (2.25) in (2.23), we obtain

$$v_{n+1} = a^2v_{n-3} + b\big(J_{n-3}\left(v_2 + av_0\right) + J_{n-2}\left(v_1 + av_{-1}\right) + bJ_{n-4}\left(v_0 + av_{-2}\right)\big),\ n \geq 1, \tag{2.26}$$

from equation (2.26) we get

$$v_n = a^2v_{n-4} + b\big(J_{n-4}\left(v_2 + av_0\right) + J_{n-3}\left(v_1 + av_{-1}\right) + bJ_{n-5}\left(v_0 + av_{-2}\right)\big),\ n \geq 2. \tag{2.27}$$

Equation (2.27) is separated to the following four ones

$$v_{4n+i} = a^2 v_{4(n-1)+i} + bJ_{4n+i-4}(v_2 + av_0) + bJ_{4n+i-3}(v_1 + av_{-1}) + b^2 J_{4n+i-5}(v_0 + av_{-2}), \quad (2.28)$$

for $n \in \mathbb{N}$, $i \in \{-2,-1,0,1\}$.
Telescoping summation of the equalities in (2.28) gives

$$\begin{aligned} v_{4n+i} &= (a^2)^n v_i + (v_2 + av_0) b \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i} + (v_1 + av_{-1}) b \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} \\ &+ b^2 (v_0 + av_{-2}) \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i-1}, \end{aligned} \quad (2.29)$$

for $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$.
From (2.29) and since

$$v_2 = au_0 + bv_{-1}, \quad \text{and} \quad v_1 = au_{-1} + bv_{-2}.$$

By using the definition of the $(J_n)_{n\in\mathbb{N}_0}$ sequence, we have

$$\begin{aligned} v_{4n+i} &= (a^2)^n v_i + (au_0 + bv_{-1} + av_0) b \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i} \\ &+ (au_{-1} + bv_{-2} + av_{-1}) b \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} + b^2 (v_0 + av_{-2}) \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i-1} \\ &= (a^2)^n v_i + b^2 v_{-2} \left(\sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} + a \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i-1} \right) \\ &+ bv_{-1} \left(a \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} + b \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i} \right) \\ &+ bv_0 \left(a \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i} + b \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i-1} \right) + abu_{-1} \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} \\ &+ abu_0 \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i} \\ &= (a^2)^n v_i + b^2 v_{-2} \left(\sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} + a \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i-1} \right) \\ &+ bv_{-1} \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+3} + bv_0 \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+2} + abu_{-1} \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i+1} \\ &+ abu_0 \sum_{j=0}^{n-1} (a^2)^{n-j-1} J_{4j+i}, \end{aligned} \quad (2.30)$$

for $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$.
By using (2.30) in the first equation in (2.22) and the definition of the $(J_n)_{n\in\mathbb{N}_0}$ sequence, it follows

that

$$\begin{aligned}
u_{4n+i+1} &= av_{4n+i-1}+bv_{4n+i-2}\\
&= a\Big[\left(a^2\right)^n v_{i-1}+b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i-2}\right)\\
&+ bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+2}+bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+1}+abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i}\\
&+ abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i-1}\Big]+b\Big[\left(a^2\right)^n v_{i-2}+b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i-1}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i-3}\right)\\
&+ bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+1}+bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i}+abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i-1}\\
&+ abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i-2}\Big]\\
&= \left(a^2\right)^n u_{i+1}+b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+2}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i}\right)+bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+4}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+3}+abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+2}+abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+i+1},
\end{aligned}\tag{2.31}$$

for $n\in\mathbb{N}_0$, $i\in\{-2,-1,0,1\}$.
From (2.30),(2.31), the definition of the $(J_n)_{n\in\mathbb{N}_0}$ sequence, we obtain

$$\begin{aligned}
u_{4n-1} &= \left(a^2\right)^n u_{-1}+b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-2}\right)+bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}+abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}+abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1}\\
&= b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-2}\right)+bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}+u_{-1}\left(\left(a^2\right)^n+ab\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}\right)+abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1},
\end{aligned}\tag{2.32}$$

$$\begin{aligned}
u_{4n} &= \left(a^2\right)^n u_0+b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1}\right)+bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}+abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}+abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}\\
&= b^2v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}+a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1}\right)+bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}+abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}+u_0\left(\left(a^2\right)^n+ab\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}\right),
\end{aligned}\tag{2.33}$$

$$\begin{aligned}
u_{4n+1} &= \left(a^2\right)^n u_1 + b^2 v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}\right) + bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+4}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3} + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}\\
&= v_{-2}\left(\left(a^2\right)^n b + b^2\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j}\right)\right)\\
&+ v_{-1}\left(\left(a^2\right)^n a + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+4}\right) + bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3}\\
&+ abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1},
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
u_{4n+2} &= \left(a^2\right)^n u_2 + b^2 v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}\right) + bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+5}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+4} + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}\\
&= b^2 v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}\right) + v_{-1}\left(\left(a^2\right)^n b + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+5}\right)\\
&+ v_0\left(\left(a^2\right)^n a + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+4}\right) + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+3}\\
&+ abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2},
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
v_{4n-2} &= \left(a^2\right)^n v_{-2} + b^2 v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-3}\right) + bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j} + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-2}\\
&= v_{-2}\left(\left(a^2\right)^n + b^2\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-3}\right)\right) + bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j} + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-2},
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
v_{4n-1} &= \left(a^2\right)^n v_{-1} + b^2 v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-2}\right) + bv_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1} + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1}\\
&= b^2 v_{-2}\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-2}\right) + v_{-1}\left(\left(a^2\right)^n + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+2}\right)\\
&+ bv_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j+1} + abu_{-1}\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j} + abu_0\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1}J_{4j-1},
\end{aligned} \tag{2.37}$$

$$\begin{aligned}
v_{4n} &= (a^2)^n v_0 + b^2 v_{-2}\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}\right) + bv_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}\\
&+ bv_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + abu_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1} + abu_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\\
&= b^2 v_{-2}\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}\right) + bv_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}\\
&+ v_0\left((a^2)^n + b\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\right) + abu_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1} + abu_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}, \qquad (2.38)
\end{aligned}$$

$$\begin{aligned}
v_{4n+1} &= (a^2)^n v_1 + b^2 v_{-2}\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\right) + bv_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+4}\\
&+ bv_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3} + abu_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + abu_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}\\
&= v_{-2}\left((a^2)^n b + b^2\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\right)\right) + bv_{-1}\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+4}\\
&+ bv_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3} + u_{-1}\left((a^2)^n a + ab\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\right)\\
&+ abu_0\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}, \qquad (2.39)
\end{aligned}$$

for $n \in \mathbb{N}_0$.
From (2.21), (2.32)-(2.39) and some calculation, we obtain

$$\begin{aligned}
x_{4n-1} &= f^{-1}\Bigg(b^2 f(y_{-2})\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-2}\right) + bf(y_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\\
&+ bf(y_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1} + f(x_{-1})\left((a^2)^n + ab\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\right)\\
&+ abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}\Bigg), \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
x_{4n} &= f^{-1}\Bigg(b^2 f(y_{-2})\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}\right) + bf(y_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}\\
&+ bf(y_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + abf(x_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}\\
&+ f(x_0)\left((a^2)^n + ab\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\right)\Bigg), \qquad (2.41)
\end{aligned}$$

$$\begin{aligned}
x_{4n+1} &= f^{-1}\Bigg(f(y_{-2})\left((a^2)^n b + b^2\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\right)\right)\\
&+ f(y_{-1})\left((a^2)^n a + b\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+4}\right) + bf(y_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2} + abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}\Bigg), \qquad (2.42)
\end{aligned}$$

$$
\begin{aligned}
x_{4n+2} &= f^{-1}\Bigg(b^2 f(y_{-2})\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}+a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}\right)\\
&+ f(y_{-1})\left((a^2)^n b+b\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+5}\right)+f(y_0)\left((a^2)^n a+b\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+4}\right)\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}+abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\Bigg),
\end{aligned}
\tag{2.43}
$$

$$
\begin{aligned}
y_{4n-2} &= f^{-1}\Bigg(f(y_{-2})\left((a^2)^n+b^2\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}+a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-3}\right)\right)\\
&+ bf(y_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}+bf(y_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}+abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-2}\Bigg),
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
y_{4n-1} &= f^{-1}\Bigg(b^2 f(y_{-2})\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}+a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-2}\right)\\
&+ f(y_{-1})\left((a^2)^n+b\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\right)+bf(y_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}+abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}\Bigg),
\end{aligned}
\tag{2.45}
$$

$$
\begin{aligned}
y_{4n} &= f^{-1}\Bigg(b^2 f(y_{-2})\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}+a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j-1}\right)\\
&+ bf(y_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}+f(y_0)\left((a^2)^n+b\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\right)\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}+abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\Bigg),
\end{aligned}
\tag{2.46}
$$

$$
\begin{aligned}
y_{4n+1} &= f^{-1}\Bigg(f(y_{-2})\left((a^2)^n b+b^2\left(\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}+a\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j}\right)\right)\\
&+ bf(y_{-1})\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+4}+bf(y_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+3}\\
&+ f(x_{-1})\left((a^2)^n a+ab\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+2}\right)+abf(x_0)\sum_{j=0}^{n-1}(a^2)^{n-j-1}J_{4j+1}\Bigg),
\end{aligned}
\tag{2.47}
$$

for $n \in \mathbb{N}_0$.

2.4. Case 4: $p_n = y_n,\ q_n = x_n,\ r_n = x_n,\ s_n = x_n$

In this case, system (1.8) is written as in the form

$$x_{n+1} = f^{-1}\big(af\left(y_{n-1}\right) + bf\left(x_{n-2}\right)\big), \quad y_{n+1} = f^{-1}\big(af\left(x_{n-1}\right) + bf\left(x_{n-2}\right)\big), \quad n \in \mathbb{N}_0. \tag{2.48}$$

Note that system (2.48) are obtained from equations (2.19) by interchanging letters x and y, from which all the statements concerning solutions to the equations follow from the corresponding statements in Case 3 by only interchanging letters x and y.
The general solutions to the system (2.48) is given

$$\begin{aligned} y_{4n-1} &= f^{-1}\Bigg(b^2 f\left(x_{-2}\right)\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-2}\right) + bf\left(x_{-1}\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2} \\ &+ bf\left(x_0\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} + f\left(y_{-1}\right)\left(\left(a^2\right)^n + ab\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j}\right) \\ &+ abf\left(y_0\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-1}\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.49}$$

$$\begin{aligned} y_{4n} &= f^{-1}\Bigg(b^2 f\left(x_{-2}\right)\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-1}\right) + bf\left(x_{-1}\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+3} \\ &+ bf\left(x_0\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2} + abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} \\ &+ f\left(y_0\right)\left(\left(a^2\right)^n + ab\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j}\right)\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.50}$$

$$\begin{aligned} y_{4n+1} &= f^{-1}\Bigg(f\left(x_{-2}\right)\left(\left(a^2\right)^n b + b^2\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j}\right)\right) \\ &+ f\left(x_{-1}\right)\left(\left(a^2\right)^n a + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+4}\right) + bf\left(x_0\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+3} \\ &+ abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2} + abf\left(y_0\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1}\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.51}$$

$$\begin{aligned} y_{4n+2} &= f^{-1}\Bigg(b^2 f\left(x_{-2}\right)\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+3} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1}\right) \\ &+ f\left(x_{-1}\right)\left(\left(a^2\right)^n b + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+5}\right) + f\left(x_0\right)\left(\left(a^2\right)^n a + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+4}\right) \\ &+ abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+3} + abf\left(y_0\right)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2}\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.52}$$

$$\begin{aligned} x_{4n-2} &= f^{-1}\Bigg(f(x_{-2})\left(\left(a^2\right)^n + b^2\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-1} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-3}\right)\right) \\ &+ bf(x_{-1})\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} + bf(x_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j} \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-1} + abf(y_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-2}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.53}$$

$$\begin{aligned} x_{4n-1} &= f^{-1}\Bigg(b^2 f(x_{-2})\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-2}\right) \\ &+ f(x_{-1})\left(\left(a^2\right)^n + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2}\right) + bf(x_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j} + abf(y_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-1}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.54}$$

$$\begin{aligned} x_{4n} &= f^{-1}\Bigg(b^2 f(x_{-2})\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j-1}\right) \\ &+ bf(x_{-1})\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+3} + f(x_0)\left(\left(a^2\right)^n + b\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2}\right) \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1} + abf(y_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.55}$$

and

$$\begin{aligned} x_{4n+1} &= f^{-1}\Bigg(f(x_{-2})\left(\left(a^2\right)^n b + b^2\left(\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2} + a\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j}\right)\right) \\ &+ bf(x_{-1})\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+4} + bf(x_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+3} \\ &+ f(y_{-1})\left(\left(a^2\right)^n a + ab\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+2}\right) + abf(y_0)\sum_{j=0}^{n-1}\left(a^2\right)^{n-j-1} J_{4j+1}\Bigg),\ n\in\mathbb{N}_0. \end{aligned} \tag{2.56}$$

2.5. Case 5: $p_n = y_n$, $q_n = x_n$, $r_n = y_n$, $s_n = x_n$

In this case, system (1.8) becomes

$$x_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(x_{n-2})\big),\quad y_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(x_{n-2})\big),\quad n\in\mathbb{N}_0. \tag{2.57}$$

First, note that from the equations in (2.57) it immediately follows that

$$x_n = y_n,\ \text{for all } n\in\mathbb{N}. \tag{2.58}$$

Employing (2.58) in the first equation in (2.57) , we have,

$$x_{n+1} = y_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2})\big),\ n\geq 2. \tag{2.59}$$

By using formula (2.10) with forward shifted initial values, we have

$$x_n = f^{-1}\big(f(x_2) J_{n-1} + f(x_1) J_n + bf(x_0) J_{n-2}\big),\ n \geq -1. \tag{2.60}$$

Using the following equalities

$$x_1 = f^{-1}\big(af(y_{-1}) + bf(x_{-2})\big),\quad x_2 = f^{-1}\big(af(y_0) + bf(x_{-1})\big), \tag{2.61}$$

in (2.60) we obtain

$$x_n = y_n = f^{-1}\big((af(y_0) + bf(x_{-1})) J_{n-1} + (af(y_{-1}) + bf(x_{-2})) J_n + bf(x_0) J_{n-2}\big),\ n \in \mathbb{N}. \tag{2.62}$$

2.6. Case 6: $p_n = x_n$, $q_n = y_n$, $r_n = x_n$, $s_n = y_n$

In this case, system (1.8) is written as in the form

$$x_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(y_{n-2})\big),\quad y_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(y_{n-2})\big),\quad n \in \mathbb{N}_0. \tag{2.63}$$

Note that system (2.63) are obtained from equations (2.57) by interchanging letters x and y, from which all the statements concerning solutions to the equations follow from the corresponding statements in Case 5 by only interchanging letters x and y.
The general solutions to the system (2.63) is given

$$y_n = x_n = f^{-1}\big((af(x_0) + bf(y_{-1})) J_{n-1} + (af(x_{-1}) + bf(y_{-2})) J_n + bf(y_0) J_{n-2}\big),\ n \in \mathbb{N}. \tag{2.64}$$

2.7. Case 7: $p_n = y_n$, $q_n = y_n$, $r_n = y_n$, $s_n = x_n$

In this case, system (1.8) becomes

$$x_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(y_{n-2})\big),\quad y_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(x_{n-2})\big),\quad n \in \mathbb{N}_0. \tag{2.65}$$

Since f is "$1-1$", from (2.65)

$$f(x_{n+1}) = af(y_{n-1}) + bf(y_{n-2}),\quad f(y_{n+1}) = af(y_{n-1}) + bf(x_{n-2}),\quad n \in \mathbb{N}_0. \tag{2.66}$$

By using the changes of variables

$$f(x_n) = u_n,\quad \text{and}\quad f(y_n) = v_n,\quad n \geq -2, \tag{2.67}$$

system (2.66) is transformed to the following one

$$u_{n+1} = av_{n-1} + bv_{n-2},\quad v_{n+1} = av_{n-1} + bu_{n-2},\ n \in \mathbb{N}_0. \tag{2.68}$$

Employing the first equation in (2.68) in the second one we get

$$v_{n+1} = av_{n-1} + abv_{n-4} + b^2 v_{n-5},\ n \geq 3. \tag{2.69}$$

Equation (2.69) can be written in the following form

$$v_{n+1} + bv_{n-2} = a(v_{n-1} + bv_{n-4}) + b(v_{n-2} + bv_{n-5}),\ n \geq 3, \tag{2.70}$$

from (2.5), we can write the solution of equation (2.70) as

$$v_{n+1} + bv_{n-2} = J_{n-1}(v_3 + bv_0) + J_n(v_2 + bv_{-1}) + bJ_{n-2}(v_1 + bv_{-2}),\ n \in \mathbb{N}_0. \tag{2.71}$$

By using (2.71) in (2.69), we obtain

$$v_{n+1} = b^2 v_{n-5} + a\Big(J_{n-3}\left(v_3 + bv_0\right) + J_{n-2}\left(v_2 + bv_{-1}\right) + bJ_{n-4}\left(v_1 + bv_{-2}\right)\Big),\ n \geq 3, \tag{2.72}$$

from equation (2.72) we get

$$v_n = b^2 v_{n-6} + a\Big(J_{n-4}\left(v_3 + bv_0\right) + J_{n-3}\left(v_2 + bv_{-1}\right) + bJ_{n-5}\left(v_1 + bv_{-2}\right)\Big),\ n \geq 4. \tag{2.73}$$

Equation (2.73) is separated to the following six ones

$$v_{6n+i} = b^2 v_{6(n-1)+i} + aJ_{6n+i-4}\left(v_3 + bv_0\right) + aJ_{6n+i-3}\left(v_2 + bv_{-1}\right) + abJ_{6n+i-5}\left(v_1 + bv_{-2}\right), \tag{2.74}$$

for $n \in \mathbb{N}$, $i \in \{-2,-1,0,1,2,3\}$.
Telescoping summation of the equalities in (2.74) gives

$$\begin{aligned} v_{6n+i} &= \left(b^2\right)^n v_i + \left(v_3 + bv_0\right) a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2} + \left(v_2 + bv_{-1}\right) a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+3} \\ &+ \left(v_1 + bv_{-2}\right) ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+1}, \end{aligned} \tag{2.75}$$

for $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1,2,3\}$.
From (2.75) and since

$$v_3 = av_1 + bu_0 = a^2 v_{-1} + abu_{-2} + bu_0, \quad v_2 = av_0 + bu_{-1}, \quad \text{and} \quad v_1 = av_{-1} + bu_{-2}.$$

By using the definition of the $(J_n)_{n\in\mathbb{N}_0}$ sequence, we have

$$\begin{aligned} v_{6n+i} &= \left(b^2\right)^n v_i + \left(a^2 v_{-1} + abu_{-2} + bu_0 + bv_0\right) a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2} \\ &+ \left(av_0 + bu_{-1} + bv_{-1}\right) a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+3} + \left(av_{-1} + bu_{-2} + bv_{-2}\right) ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+1} \\ &= \left(b^2\right)^n v_i + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+1} \\ &+ v_{-1} \left(a^3 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2} + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+3} + a^2 b \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+1} \right) \\ &+ v_0 \left(ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2} + a^2 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+3} \right) \\ &+ u_{-2} \left(a^2 b \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2} + ab^2 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+1} \right) \\ &+ abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+3} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2} \\ &= \left(b^2\right)^n v_i + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+1} \\ &+ av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+6} + av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+5} + abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+4} \\ &+ abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+3} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+i+2}, \end{aligned} \tag{2.76}$$

for $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1,2,3\}$.
By using (2.76) in the first equation in (2.68) and the definition of the $(J_n)_{n\in\mathbb{N}_0}$ sequence, it follows that

$$\begin{aligned}
u_{6n+i} &= av_{6n+i-2}+bv_{6n+i-3} \\
&= a\Big[\left(b^2\right)^n v_{i-2}+ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i-1}+av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+4} \\
&+ av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+3}+abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+2}+abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+1} \\
&+ abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i}\Big]+b\Big[\left(b^2\right)^n v_{i-3}+ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i-2} \\
&+ av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+3}+av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+2}+abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+1} \\
&+ abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i}+abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i-1}\Big] \\
&= \left(b^2\right)^n u_i+ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+1}+av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+6} \\
&+ av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+5}+abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+4}+abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+i+3} \\
&+ abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}P_{6j+i+2},
\end{aligned} \tag{2.77}$$

for $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1,2,3\}$.
From (2.76),(2.77), the definition of the $(J_n)_{n\in\mathbb{N}_0}$ sequence, we obtain

$$\begin{aligned}
u_{6n-2} &= \left(b^2\right)^n u_{-2}+ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j-1}+av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}+av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} \\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}+abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}+abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} \\
&= ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j-1}+av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}+av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} \\
&+ u_{-2}\left(\left(b^2\right)^n+ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right)+abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} \\
&+ abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j},
\end{aligned} \tag{2.78}$$

$$\begin{aligned}
u_{6n-1} &= \left(b^2\right)^n u_{-1} + ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}\\
&= ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + u_{-1}\left(\left(b^2\right)^n + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right)\\
&+ abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}, \qquad (2.79)
\end{aligned}$$

$$\begin{aligned}
u_{6n} &= \left(b^2\right)^n u_0 + ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\\
&= ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\\
&+ u_0\left(\left(b^2\right)^n + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right), \qquad (2.80)
\end{aligned}$$

$$\begin{aligned}
u_{6n+1} &= \left(b^2\right)^n u_1 + ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\\
&= v_{-2}\left(\left(b^2\right)^n b + ab^2\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right) + v_{-1}\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)\\
&+ av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}, \qquad (2.81)
\end{aligned}$$

$$\begin{aligned}
u_{6n+2} &= \left(b^2\right)^n u_2 + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} + av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} \\
&+ abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} + abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} \\
&= ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} + v_{-1} \left(\left(b^2\right)^n b + a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} \right) \\
&+ v_0 \left(\left(b^2\right)^n a + a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} \right) + abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} \\
&+ abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4}, \ n \in \mathbb{N}_0,
\end{aligned} \tag{2.82}$$

$$\begin{aligned}
u_{6n+3} &= \left(b^2\right)^n u_3 + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+9} + av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} \\
&+ abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} + abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} \\
&= ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} + v_{-1} \left(\left(b^2\right)^n a^2 + a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+9} \right) \\
&+ v_0 \left(\left(b^2\right)^n b + a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} \right) + u_{-2} \left(\left(b^2\right)^n ab + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} \right) \\
&+ abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5},
\end{aligned} \tag{2.83}$$

$$\begin{aligned}
v_{6n-2} &= \left(b^2\right)^n v_{-2} + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j-1} + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} + av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} \\
&+ abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+2} + abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+1} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j} \\
&= v_{-2} \left(\left(b^2\right)^n + ab^2 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j-1} \right) + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} \\
&+ av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} + abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+2} \\
&+ abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+1} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j},
\end{aligned} \tag{2.84}$$

$$\begin{aligned}
v_{6n-1} &= \left(b^2\right)^n v_{-1} + ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}\\
&= ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + v_{-1}\left(\left(b^2\right)^n + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right)\\
&+ av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\\
&+ abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}, \qquad (2.85)
\end{aligned}$$

$$\begin{aligned}
v_{6n} &= \left(b^2\right)^n v_0 + ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\\
&= ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}\\
&+ v_0\left(\left(b^2\right)^n + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right) + abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}, \qquad (2.86)
\end{aligned}$$

$$\begin{aligned}
v_{6n+1} &= \left(b^2\right)^n v_1 + ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + av_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7} + av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}\\
&+ abu_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\\
&= ab^2v_{-2}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + v_{-1}\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)\\
&+ av_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + u_{-2}\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right)\\
&+ abu_{-1}\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abu_0\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}, \qquad (2.87)
\end{aligned}$$

$$\begin{aligned}
v_{6n+2} &= \left(b^2\right)^n v_2 + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} + av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} \\
&+ abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} + abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} \\
&= ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} \\
&+ v_0 \left(\left(b^2\right)^n a + a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} \right) + abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} \\
&+ u_{-1} \left(\left(b^2\right)^n b + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} \right) + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4}, \qquad (2.88)
\end{aligned}$$

and

$$\begin{aligned}
v_{6n+3} &= \left(b^2\right)^n v_3 + ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} + av_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+9} + av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} \\
&+ abu_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} + abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} + abu_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} \\
&= ab^2 v_{-2} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} + v_{-1} \left(\left(b^2\right)^n a^2 + a \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+9} \right) \\
&+ av_0 \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+8} + u_{-2} \left(\left(b^2\right)^n ab + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+7} \right) \\
&+ abu_{-1} \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+6} + u_0 \left(\left(b^2\right)^n b + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} \right), \qquad (2.89)
\end{aligned}$$

for $n \in \mathbb{N}_0$. From (2.67), (2.78)-(2.89) and some calculation, we obtain

$$\begin{aligned}
x_{6n-2} &= f^{-1} \Bigg(ab^2 f(y_{-2}) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j-1} + af(y_{-1}) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} + af(y_0) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} \\
&+ f(x_{-2}) \left(\left(b^2\right)^n + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+2} \right) + abf(x_{-1}) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+1} \\
&+ abf(x_0) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j} \Bigg), \ n \in \mathbb{N}_0, \qquad (2.90)
\end{aligned}$$

$$\begin{aligned}
x_{6n-1} &= f^{-1} \Bigg(ab^2 f(y_{-2}) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j} + af(y_{-1}) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+5} + af(y_0) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+4} \\
&+ abf(x_{-2}) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+3} + f(x_{-1}) \left(\left(b^2\right)^n + ab \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+2} \right) \\
&+ abf(x_0) \sum_{j=0}^{n-1} \left(b^2\right)^{n-j-1} J_{6j+1} \Bigg), \ n \in \mathbb{N}_0, \qquad (2.91)
\end{aligned}$$

$$
\begin{aligned}
x_{6n} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}+af(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}P_{6j+6}+af(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\\
&+ abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}+abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\\
&+ f(x_0)\left(\left(b^2\right)^n+ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right)\Bigg),\ n\in\mathbb{N}_0, \qquad (2.92)
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+1} &= f^{-1}\Bigg(f(y_{-2})\left(\left(b^2\right)^n b+ab^2\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right)+f(y_{-1})\left(\left(b^2\right)^n a+a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)\\
&+ af(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}+abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}+abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\Bigg),\ n\in\mathbb{N}_0, \qquad (2.93)
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+2} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}+f(y_{-1})\left(\left(b^2\right)^n b+a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+8}\right)\\
&+ f(y_0)\left(\left(b^2\right)^n a+a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)+abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}+abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\Bigg),\ n\in\mathbb{N}_0, \qquad (2.94)
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+3} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}+f(y_{-1})\left(\left(b^2\right)^n a^2+a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+9}\right)\\
&+ f(y_0)\left(\left(b^2\right)^n b+a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+8}\right)+f(x_{-2})\left(\left(b^2\right)^n ab+ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}+abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\Bigg),\ n\in\mathbb{N}_0, \qquad (2.95)
\end{aligned}
$$

$$
\begin{aligned}
y_{6n-2} &= f^{-1}\Bigg(f(y_{-2})\left(\left(b^2\right)^n+ab^2\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j-1}\right)+af(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\
&+ af(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}+abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\\
&+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}+abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j}\Bigg),\ n\in\mathbb{N}_0, \qquad (2.96)
\end{aligned}
$$

$$\begin{aligned} y_{6n-1} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + f(y_{-1})\left(\left(b^2\right)^n + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right)\\ &+ af(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\\ &+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.97}$$

$$\begin{aligned} y_{6n} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + af(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}\\ &+ f(y_0)\left(\left(b^2\right)^n + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right) + abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\\ &+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.98}$$

$$\begin{aligned} y_{6n+1} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + f(y_{-1})\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)\\ &+ af(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + f(x_{-2})\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right)\\ &+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.99}$$

$$\begin{aligned} y_{6n+2} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + af(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+8}\\ &+ f(y_0)\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right) + abf(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6}\\ &+ f(x_{-1})\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right) + abf(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\Bigg), n\in\mathbb{N}_0, \end{aligned} \tag{2.100}$$

and

$$\begin{aligned} y_{6n+3} &= f^{-1}\Bigg(ab^2 f(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + f(y_{-1})\left(\left(b^2\right)^n a^2 + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+9}\right)\\ &+ af(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+8} + f(x_{-2})\left(\left(b^2\right)^n ab + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right)\\ &+ abf(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + f(x_0)\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right)\Bigg), n\in\mathbb{N}_0. \end{aligned} \tag{2.101}$$

2.8. Case 8: $p_n = x_n$, $q_n = y_n$, $r_n = x_n$, $s_n = x_n$

In this case, we obtain the system

$$x_{n+1} = f^{-1}\big(af\left(x_{n-1}\right) + bf\left(y_{n-2}\right)\big), \quad y_{n+1} = f^{-1}\big(af\left(x_{n-1}\right) + bf\left(x_{n-2}\right)\big), \quad n \in \mathbb{N}_0. \tag{2.102}$$

Note that system (2.102) are obtained from equations (2.65) by interchanging letters x and y, from which all the statements concerning solutions to the equations follow from the corresponding statements in Case 7 by only interchanging letters x and y.
The general solutions to the system (2.102) is given

$$\begin{aligned} y_{6n-2} &= f^{-1}\Bigg(ab^2 f\left(x_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j-1} + af\left(x_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + af\left(x_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} \\ &+ f\left(y_{-2}\right)\left(\left(b^2\right)^n + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right) + abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} \\ &+ abf\left(y_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j}\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.103}$$

$$\begin{aligned} y_{6n-1} &= f^{-1}\Bigg(ab^2 f\left(x_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + af\left(x_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + af\left(x_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} \\ &+ abf\left(y_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + f\left(y_{-1}\right)\left(\left(b^2\right)^n + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right) \\ &+ abf\left(y_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.104}$$

$$\begin{aligned} y_{6n} &= f^{-1}\Bigg(ab^2 f\left(x_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + af\left(x_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}P_{6j+6} + af\left(x_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} \\ &+ abf\left(y_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} \\ &+ f\left(y_0\right)\left(\left(b^2\right)^n + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right)\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.105}$$

$$\begin{aligned} y_{6n+1} &= f^{-1}\Bigg(f\left(x_{-2}\right)\left(\left(b^2\right)^n b + ab^2\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\right) + f\left(x_{-1}\right)\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right) \\ &+ af\left(x_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + abf\left(y_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} \\ &+ abf\left(y_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3}\Bigg), \ n \in \mathbb{N}_0, \end{aligned} \tag{2.106}$$

$$\begin{aligned} y_{6n+2} &= f^{-1}\Bigg(ab^2 f(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + f(x_{-1})\left(\left(b^2\right)^n b + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+8}\right) \\ &+ f(x_0)\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right) + abf(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5} + abf(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.107}$$

$$\begin{aligned} y_{6n+3} &= f^{-1}\Bigg(ab^2 f(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + f(x_{-1})\left(\left(b^2\right)^n a^2 + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+9}\right) \\ &+ f(x_0)\left(\left(b^2\right)^n b + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+8}\right) + f(y_{-2})\left(\left(b^2\right)^n ab + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+7}\right) \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} + abf(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.108}$$

$$\begin{aligned} x_{6n-2} &= f^{-1}\Bigg(f(x_{-2})\left(\left(b^2\right)^n + ab^2\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j-1}\right) + af(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} \\ &+ af(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abf(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + abf(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.109}$$

$$\begin{aligned} x_{6n-1} &= f^{-1}\Bigg(ab^2 f(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j} + f(x_{-1})\left(\left(b^2\right)^n + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right) \\ &+ af(x_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} + abf(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2} + abf(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.110}$$

$$\begin{aligned} x_{6n} &= f^{-1}\Bigg(ab^2 f(x_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+1} + af(x_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+6} \\ &+ f(x_0)\left(\left(b^2\right)^n + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+5}\right) + abf(y_{-2})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+4} \\ &+ abf(y_{-1})\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+3} + abf(y_0)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1}J_{6j+2}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.111}$$

$$\begin{aligned} x_{6n+1} &= f^{-1}\Bigg(ab^2 f\left(x_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+2} + f\left(x_{-1}\right)\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+7}\right) \\ &+ af\left(x_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+6} + f\left(y_{-2}\right)\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+5}\right) \\ &+ abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+4} + abf\left(y_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+3}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.112}$$

$$\begin{aligned} x_{6n+2} &= f^{-1}\Bigg(ab^2 f\left(x_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+3} + af\left(x_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+8} \\ &+ f\left(x_0\right)\left(\left(b^2\right)^n a + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+7}\right) + abf\left(y_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+6} \\ &+ f\left(y_{-1}\right)\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+5}\right) + abf\left(y_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+4}\Bigg),\ n\in\mathbb{N}_0, \end{aligned} \tag{2.113}$$

and

$$\begin{aligned} x_{6n+3} &= f^{-1}\Bigg(ab^2 f\left(x_{-2}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+4} + f\left(x_{-1}\right)\left(\left(b^2\right)^n a^2 + a\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+9}\right) \\ &+ af\left(x_0\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+8} + f\left(y_{-2}\right)\left(\left(b^2\right)^n ab + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+7}\right) \\ &+ abf\left(y_{-1}\right)\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+6} + f\left(y_0\right)\left(\left(b^2\right)^n b + ab\sum_{j=0}^{n-1}\left(b^2\right)^{n-j-1} J_{6j+5}\right)\Bigg),\ n\in\mathbb{N}_0. \end{aligned} \tag{2.114}$$

3. Conclusion

In this study, we have consider the following two-dimensional system of difference equations

$$x_{n+1} = f^{-1}\big(af\left(p_{n-1}\right) + bf\left(q_{n-2}\right)\big), \quad y_{n+1} = f^{-1}\big(af\left(r_{n-1}\right) + bf\left(s_{n-2}\right)\big), \quad n\in\mathbb{N}_0,$$

where the sequences p_n, q_n, r_n, s_n are some of the sequences x_n and y_n, $f : D_f \longrightarrow \mathbb{R}$ is a “$1-1$” continuous function on its domain $D_f \subseteq \mathbb{R}$, initial values x_{-j}, y_{-j}, $j \in \{0, 1, 2\}$ are arbitrary real numbers in D_f and the parameters a, b are arbitrary complex numbers, with $b \neq 0$. We have obtained the explicit form of solutions of the eight special cases of aforementioned system by using suitable transformation. The solutions of the eight special cases of aforementioned system are associated with generalized Padovan sequences.

Open Problem: The aforementioned two-dimensional system can extend to the three-dimensional system of difference equations which is variable coefficients or constant coefficients. Special cases of three-dimensional system can be solve in explicit form.

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