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On fluctuation analysis of different kinds of n-policy queues with single vacation

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Abstract

In this paper, we debate queueing systems with N-policy and single vacation. We consider these systems when the vacation times have Erlang distribution. Moreover, we adapted the input by studying two different kinds: first, an ordinary Poisson input, and second, type 2 geometric batch input. We derive the probability generating function of the number of units in the system in two cases by using fluctuation analysis.

Keywords: N-policy, single vacation, fluctuation theory, input batches, bulk input, marked delayed renewal process, delayed renewal process, point process, marked point process, random walk.
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1. Introduction

Many applications in industry such as digital communication, computer network, and inventory systems require the sever takes a rest after a bussy period and returns to his job when the system's units reach a specific threshold. For example, consider a production system, where production does not begin until some particularized raw N of stuff is stored in the system during the idle time [10]. The original investigation of batch arrival queue with N-policy was made by Lee and Srinivasan [19]. They performed a system to get the optimal stationary operating policy under a proper linear cost formation besides other inquiries. Following Lee et al. [18] have considered this type of widely throughout various manners. In particular, some viewpoints of this system have also been analyzed by Chae and Lee [21], Teghem [25], Medhi [23], Kalita and Choudhury [16], and Choudhury and Baruah [9].

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Numerous studies are dedicated to batch arrival systems under different vacation policies because of their interdisciplinary nature. Many scholars, involving Baba [4], Choudhury [7, 8], Lee et al. [20, 21], Rosenberg and Yechiali [24], Madan and Abu-Dayyeh [22] and Teghem [26], and others have analyzed batch arrival queue under different vacation policies.

In this work, we derive the probability generating function (in short, pgf) of the number of units in the N-policy queue with signal vacation. This work is done by using fluctuation theory when the vacation time has Erlang distribution, and the input is ordinary Poisson in the first case, and in the second one, it is type 2 geometric. The paper presents essential concepts related to our work in the second section, and then we begin with our goals in the third section.

2. Formal Description

One of the main objectives of this section is the analysis of the queueing process $\{Q(t); t > 0\}$ giving the cumulative number of units in the system at any time $t > 0$. This process, along with all other related processes, will be considered on a probability space $(\Omega,\ \mathcal{F}(\Omega), P)$. We define the queueing process as right continuous.

2.1. Preliminaries

Let us start with significant definitions that help to progress in our targets. The materials of this definition are from many different works corresponding to Dshalalow, see [2, 3, 5, 11, 16].

Definition 2.1. *Let $(\Omega,\ \mathcal{F}(\Omega), P)$ be a probability space, where $\mathcal{F}(\Omega)$ is σ-algebra, then any arbitrary monotone nondecreasing family $\{\mathcal{F}_t; t \geq 0\}$ of sub-σ-algebras in $\mathcal{F}(\Omega)$ is called filtration.*

Filtration is often associated with an extended history of a particular stochastic process.

Definition 2.2. *A stochastic process X_t is said to be $\mathcal{F}_t$-adapted if for each $t \geq 0$, the function $w \mapsto X(t, w)$ is $\mathcal{F}_t$-measurable, i.e., given a fixed t, for every Borel set $A \subseteq \mathbb{R}$, the set $\{w : X(t, w) \in A\}$ is an element of $\mathcal{F}_t$.*

Definition 2.3. *Let $(\Omega,\ \mathcal{F}(\Omega), \mathcal{F}_t, P)$ be a filtered probability space. A r.v. T is called a stopping time if for any $t \geq 0$, the event $\{T \leq t\}$ belongs to $\mathcal{F}_t$.*

Definition 2.4. *A point process $\{t_n; n = 1, 2, \ldots,\}$ on the positive real axis is an a.s. monotone increasing sequence of stopping times and so-call arrival times. Associated with point process $\{t_n\}$, the counting process N_t defined as*

$$N_t = N([0, t]) = \sum_{k=1}^{\infty} \mathbf{1}_{[0,t]}(t_k) = \sum_{k=1}^{\infty} \varepsilon_{t_k}([0, t]),$$

which gives the number of arrivals in the time interval $[0, t]$ and ε_a is a unit (or Dirac) mass, defined as

$$\varepsilon_a(A) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases}$$

where a is a real number, and A is a Borel set.

Definition 2.5. *Let $\{t_1, t_2, \ldots\}$ be a point process on the positive real axis and let N_t be the associated counting processes defined on a filtered probability space $(\Omega,\ \mathcal{F}(\Omega), \mathcal{F}_t, P)$. The point and counting processes are called (ordinary) Poisson if N_t obeys the three postulates:*

(i) Independent increments property.
For $0 < s < t$, the increment $N_t - N_s$ is independent of $\mathcal{F}_s$.

(ii) Stationary increments property
Let $(a,\ b]$ be an interval such that $0 \leq a < b$. Then, $N_b - N_a \in [N_{b-a}]$.

(iii) There is a positive constant λ such that for each $t > 0$, the distribution of N_t is Poisson with parameter λt, i.e.

$$P = \{N_t = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, \ldots$$

The constant λ is called the rate or intensity of N_t. Obviously, λ is the mean number of arrivals in a unit time interval.

Definition 2.6 (Laplace-Stieltjes transform). *Let τ is a nonnegative r.v. then*

$$\beta(\theta) = E\left[e^{-\theta\tau}\right]$$

is called the Laplace-Stieltjes transform of r.v.τ.

Note that if $m(\theta)$ is the mgf (moment generating function) of $r.v.\tau$, then $\beta(\theta) = m(-\tau)$. Moreover, if τ is a nonnegative r.v. with $\beta(\theta)$ and independence of Poisson r.v. N_τ with parameter λ, then the joint transform

$$E\left[z^{N_\tau}e^{-\theta\tau}\right] = \beta(\theta + \lambda(1 - z))$$

Definition 2.7 (Marked Poisson Process). *Let N_t be Poisson counting process and $T := \{t_1, t_2, \ldots\}$ be the associated point process. Suppose $X = \{X_1, X_2, \ldots\}$ is a sequence of iid real-valued r.v.'s with a common pgf $a(z)$ s.t. X is independent of T, then*

$$\zeta = \sum_{k=1}^{\infty} X_k \varepsilon_{\tau_k}$$

is marked Poisson process. Consequently, this marked point process is with position independent marking, and it has independent and stationary increments. Moreover, its pgf is given as compound Poisson r.v. by

$$E\left[z^{\zeta}\right] = e^{\lambda(a(z)-1)\tau}$$

and the joint transform

$$E\left[z^{\zeta}e^{-\theta\tau}\right] = \beta\big(\theta + \lambda(1 - a(z))\big)$$

2.2. Queues With Bulk Input

The input is bulk if it is modeled as a marked process. Let us assume that our system has bulk input defined as a marked Poisson process (X, τ) driven by a Poisson point process $\tau = \{\tau_k : k = 1, 2, \ldots\}$ with the intensity λ of arriving points τ_k's. The arriving points τ_k's carry batches customers of random sizes X_k such that $X = \{X_k : k = 1, 2, ..\}$ is a sequence of *iid* $r.v's$ with the following pgf and expectation

$$a(z) = Ez^X = \sum_{j=0}^{\infty} a_j z^j, \quad a = EX, \quad a_0 = 0$$

Let us suppose that the Poisson marked process with position independent marking. That means $\{X_k\}$ is independent of the point process $\{\tau_k\}$. Let Q_n be the number of customers in the queue upon departure of the nth customer at the time $T_n,\ n=0,1,\dots$ Then $\{Q_n\}$ is a chain with the following transitions:

$$Q_{n+1}=\begin{cases}Q_n-1+V_{n+1}, & Q_n>0\\ X_{W_n}-1+V_{n+1}, & Q_n=0\end{cases}$$

where X_{W_n} is the first batch arriving after the time T_n. Apparently, $\{Q_n\}$ is a time-homogeneous Markov chain embedded in $\{Q(t)\}$ upon departure epochs. The TPM of $\{Q_n\}$ it is a $\triangle_2$ -matrix. Hence, the chain is irreducible and aperiodic. Moreover, we need to find the following

$$P_i\left(z\right)=E\left[z^{Q_1}\middle|Q_0=i\right]=\begin{cases}E\left[z^{i-1+V_1}\right], & i>0\\ E\left[z^{X_{W_n}-1+V_1}\right], & i=0\end{cases}=\begin{cases}z^{i-1}\beta\left(\lambda-\lambda a\left(z\right)\right), & i>0\\ z^{-1}a\left(z\right)\beta\left(\lambda-\lambda a\left(z\right)\right), & i=0\end{cases}$$

According to Abolnikov and Dukhovny (Abolnikov and Dukhovny, Markov chains with transition delta-matrix: ergodicity conditions, invariant probability measures and applications 1991), we need to check on $P_1^{'}\left(1-\right)$.

$$P_1^{'}\left(1-\right)=\beta^{'}\left(0\right)\left(-\lambda\right)a^{'}\left(1\right)=a\lambda b=\rho.$$

where $\beta^{'}\left(0\right)=-b$, $a^{'}\left(1\right)=a$ and ρ is the offered load with the condition $\rho<1$. Therefore, $\{Q_n\}$ is recurrent positive, and then this chain is ergodic. To find pgf of distribution of Q_n, we have

$$P\left(z\right)=\sum_{i=0}^{\infty}p_iP_i(z)=p_0z^{-1}a\left(z\right)\beta\left(\lambda-\lambda a\left(z\right)\right)+z^{-1}\beta\left(\lambda-\lambda a\left(z\right)\right)\sum_{i=1}^{\infty}p_iz^i$$

By solving the above, we get the generalized Pollaczed-Khinchine formula

$$P\left(z\right)=p_0\beta\left(\lambda-\lambda a\left(z\right)\right)\frac{a(z)-1}{z-\beta\left(\lambda-\lambda a\left(z\right)\right)}$$

To find p_0, we can use the fact $P\left(1-\right)=1$, then $L'Hospital$ rule to find $p_0=1-\lambda ab$.

2.3. Discrete fluctuation theory

A marked delayed renewal process with position dependent marking is defined as a random walk with this formula

$$\zeta=\sum_{k=1}^{\infty}X_k\varepsilon_{\tau_k}$$

and associated with the following delayed renewal point process

$$\xi=\sum_{k=1}^{\infty}\varepsilon_{\tau_k}$$

where X_k are the increments (marks) of the process that occur at the separate times τ_k. Because of delaying, the inter-renewal times are independent, and all except for $\tau_0=\triangle_0$ are identically distributed. The marks X_k are called position-dependent if X_k may depend on $\Delta_k=\tau_k-\tau_{k-1}$, but given this inter-renewal time, X_k is conditionally independent of all previous increments $X_i,\ i<k$, and of the previous inter-renewal time.

We assume that the sequence $\{\tau_n:\ n=0,1,2\}$ is a nondecreasing monotone. Consequently, there is no clustering, and this is leading us to the associated counting process $N_t=\sum_{k=0}^{\infty}\mathbf{1}_{[0,\ t]}$ is continuous in probability. Moreover, we presume that the marks X_k's are nonnegative integer-valued r.v. 's, and they have some joint transforms

$$\gamma\left(z,\theta\right)=Ez^{X_1}e^{-\Delta_1\theta}|z|\leq 1,\ \operatorname{Re}\theta\geq 0,$$
$$\gamma_0\left(z,\theta\right)=Ez^{\mathcal{X}_0}e^{\Delta_0\theta},\ |z|\leq 1,\ \operatorname{Re}\theta\geq 0,$$

We will focus on the behavior of this random walk when the marking component

$$A_k=X_0+\ldots+X_k$$

increases over fixed level N. To do that, we will introduce the following random index

$$\upsilon=\inf\ \{n:A_n=X_0+\ldots+X_n\geq N\}$$

at which the collective marks A_υ crosses threshold N. Similarly, the r.v.τ_υ will be the first passage time of this random walk.
We want to find the next joint transform:

$$\Phi_\nu=\Phi_\nu\left(u\ ,\upsilon,\ \vartheta,\ \theta\right)=E\xi^\nu u^{A_{\nu-1}}\upsilon^{A_\nu}e^{-\vartheta\tau_{\nu-1}-\theta\tau_{\nu-1}}. \tag{2.1}$$

To verify a directed value of the functional Φ_ν, first, we propose the secondary set of random indices

$$\{\nu(k)=\inf\{n:X_0+\ldots+X_n>k\},\ k=0,1,\ldots\}$$

as well as the set of the functionals

$$\left\{\Phi_{\nu(k)}=E\xi^{\nu(k)}u^{A_{\nu(k)-1}}\upsilon^{A_{\nu(k)}}e^{-\vartheta_{\tau_{\nu(k)-1}}-\theta_{\tau_{\nu(k)}}},\ k=0,1,\ldots\right\} \tag{2.2}$$

Observing from (2.14) and (2.16) that

$$\nu=\nu\left(N-1\right). \tag{2.3}$$

Next, we define the operator

$$D_k\left\{f\left(k\right)\right\}(x):=\sum_{k=0}^{\infty}x^kf\left(k\right)\left(1-x\right),\ \|x\|<1. \tag{2.4}$$

And the inverse operator can return f, if we use it for every k:

$$\mathcal{D}_x^k\left(D_p\left\{f(p)\right\}(x)\right)=f(k),\qquad k=0,1,\ldots \tag{2.5}$$

where the inverse $\mathcal{D}^k$ is given as

$$k\longmapsto\mathcal{D}_x^k\varphi\left(x,\ y\right)=\begin{cases}\lim_{x\to 0}\frac{1}{k!}\frac{\partial^k}{\partial x^k}\left[\frac{1}{1-x}\varphi\left(x,\ y\right)\right], & k\geq 0\\ 0 & k<0\end{cases} \tag{2.6}$$

Therefore, we can restore $\Phi_{\nu(N-1)}=\Phi_\nu$ by employing $\mathcal{D}^{N-1}$ to $D_p\Phi_{\nu(p)}$.
In the following theorem, we will discuss the important features of the inverse operator $\mathcal{D}^k$ which are indicated in the subsequent work.

Theorem 2.8 (Properties of Operator $\mathcal{D}^\mathbf{k}$). *Let $\mathcal{D}^k$ be the inverse operator of D_k as above equation* (2.6)*, then the following properties are true* *(i)* $\mathcal{D}^k$ *is a linear functional.*

(ii) $\mathcal{D}_x^k(\mathbf{1}(x)) = \mathbf{1}$, *where* $\mathbf{1}(x) = \mathbf{1}$ *for all* $x \in \mathbb{R}$

(iii) Let g *be an analytic function at zero. Then, it holds true that*

$$\mathcal{D}_x^k\left(x^j g(x)\right) = \mathcal{D}_x^{k-j} g(x). \tag{2.7}$$

(iv) In particular of (iii), if $j = k$, *we have*

$$\mathcal{D}_x^k\left(x^k g(x)\right) = g(0). \tag{2.8}$$

(v) Let $a(x) = \sum_{i=0}^{\infty} a_i x^i$. *Then*

$$D_x^k(a(x)) = \sum_{i=0}^{k} a_i \text{ and } \mathcal{D}_x^k(a(xy)) = \sum_{i=0}^{k} a_i y^i \tag{2.9}$$

(vi) For any real number b *it holds true that*

$$D_x^k\left\{\frac{1}{1-bx}\right\} = \begin{cases} \frac{1-b^{k+1}}{1-b}, & b \neq 1 \\ k+1, & b = 1 \end{cases} \tag{2.10}$$

(vii) For any real number a *and for a positive integer* n, *except for* $a = n = 1$, *it holds true that*

$$\mathcal{D}_x^k\left\{\frac{1}{(1-ax)^n}\right\} = \begin{cases} \sum_{j=0}^{k} \binom{n+j-1}{j} a^j & \text{except for } a = n = 1 \\ k+1, & a = n = 1 \end{cases} \tag{2.11}$$

(viii) For two real numbers a *and* b *it holds*

$$\mathcal{D}_x^k\left\{\frac{1}{1-bx}\frac{1}{(1-ax)^n}\right\} = \begin{cases} \frac{1}{1-bx}\sum_{j=0}^{k} \binom{n+j-1}{j}\left(a^j - b^{k+1}\left(\frac{a}{b}\right)^j\right), & b \neq 1 \\ \sum_{j=0}^{k} \binom{n+j-1}{j} a^j (k-j+1), & b = 1 \end{cases} \tag{2.12}$$

The following theorem is essential to our further works.

Theorem 2.9 (The Key Fluctuation Theorem). *Let the following functionals be given as*

$$\gamma := \gamma(xu\upsilon,\ \vartheta+\theta), \qquad \gamma_0 := \gamma_0(xu\upsilon,\ \vartheta+\theta), \tag{2.13}$$

$$\Gamma := \gamma(x\upsilon,\theta), \qquad \Gamma^1 := \gamma(\upsilon,\theta), \tag{2.14}$$

$$\Gamma_0 := \gamma_0(x\upsilon,\theta), \qquad \Gamma_0^1 := \gamma_0(\upsilon,\ \theta). \tag{2.15}$$

Then, it holds true that

$$\Phi^*(x) = D_p(\Phi_{\nu(p)}(x)) = \Gamma_0^1 - \Gamma_0 + \frac{\gamma_0\xi}{1-\gamma\xi}\left(\Gamma^1 - \Gamma\right), \tag{2.16}$$

and the functional Φ_ν *meets the following*

$$\begin{aligned}\Phi_\nu = \Phi_\nu(u,\upsilon,\ \vartheta,\ \theta) &= E\xi^\nu u^{A_{\nu-1}} \upsilon^{A_\nu} e^{-\vartheta\tau_{\nu-1}-\theta\tau_{\nu-1}}, \\ &= \mathcal{D}_x^{N-1}\left(\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0\xi}{1-\gamma\xi}\left(\Gamma^1 - \Gamma\right)\right)\end{aligned} \tag{2.17}$$

Corollary 2.10. *Let* $\xi = u = 1,\ \vartheta = 0$, *then*

$$\Phi_\nu = E\upsilon^{A_\nu} e^{-\theta\tau_\nu} = \gamma_0(\upsilon,\theta) - (1-\gamma(\upsilon,\ \theta))\,\mathcal{D}_x^{M-1}\left(\frac{\gamma_0(x\upsilon,\ \theta)}{1-\gamma(x\upsilon,\ \theta)}\right) \tag{2.18}$$

and if $\tau_0 = \Delta_0 = 0$ *and* $X_0 = A_0 = i \geq 0$, *then* $\gamma_0(\upsilon,\theta) = \upsilon^i$ *and*

$$\Phi_\nu = E\upsilon^{A_\nu} e^{-\theta\tau_\nu} = \upsilon^i - \upsilon^i(1-\gamma(\upsilon,\ \theta))\,\mathcal{D}_x^{M-1}\left(x^i \frac{1}{1-\gamma(x\upsilon,\ \theta)}\right) \tag{2.19}$$

2.4. An N-Policy Queue

The server turns out to be idle in the system if the queue declines to zero. Nevertheless, the next busy period does not begin with the early reaching batch unless it meets a certain positive number N. The server returns maintenance, and the queue size A_v and τ_v can be established by the formula

$$\Phi_v = Ez^{A_v}e^{-\theta\tau_v} = z^i - z^i\left(1-\Gamma^1\right)\mathcal{D}_x^{N-1-i}\left(\frac{1}{1-\Gamma}\right)$$

Since the number of units in the queue is zero, then $i = 0$.

$$\Phi_v = Ez^{A_v}e^{-\theta\tau_v} = 1-\left(1-\gamma\left(z,\theta\right)\right)\mathcal{D}_x^{N-1}\left(\frac{1}{1-\gamma\left(zx,\theta\right)}\right)$$

where

$$\gamma\left(z,\theta\right) = a\left(z\right)\frac{\lambda}{\lambda+\theta}$$

and the marginal transform

$$\alpha(z) = Ez^{A_v} = 1-\left[1-a\left(z\right)\right]\ \mathcal{D}_x^{N-1}\left(\frac{1}{1-a\left(xz\right)}\right)$$

and to create Kendall's formula, we start with

$$Q_{n+1} = \begin{cases} A_v - 1 + V_1, & Q_n = 0 \\ Q_{n-1} - 1 + V_1, & Q_n > 0 \end{cases}$$

to get the subsequent

$$P_i\left(z\right) = E\left[z^{Q_1}\middle|Q_0 = i\right] = \begin{cases} z^{-1}\alpha\left(z\right)\beta\left(\lambda-\lambda a\left(z\right)\right), & i = 0 \\ z^{i-1}\beta\left(\lambda-\lambda a\left(z\right)\right), & i > 0 \end{cases}$$

Thus, the pgf of this system

$$P\left(z\right) = \sum_{i=0}^{\infty}p_iP_i(z) = p_0\alpha\left(z\right)z^{-1}\beta\left(\lambda-\lambda a\left(z\right)\right) + z^{-1}\beta\left(\lambda-\lambda a\left(z\right)\right)\sum_{i=1}^{\infty}p_iz^i$$

By making only some steps, we find

$$P\left(z\right) = p_0\beta\left(\lambda-\lambda a\left(z\right)\right)\frac{\alpha\left(z\right)-1}{z-\beta\left(\lambda-\lambda a\left(z\right)\right)}$$

and

$$p_0 = \frac{1-\lambda ab}{\alpha} = \frac{1-\rho}{\alpha}$$

where

$$\alpha := EA_v = \alpha'(z)|_z = 1$$

2.5. An N-Policy Queue with the Single Vacation

In this system with bulk input and N-Policy, the server goes on a single vacation trip when the queue reaches to zero. Moreover, he returns to resume entirely if there are N units at least in the buffer. It is clear that the random walk process explaining the scheme on the idle period is now delayed. To formalize this system, we have

$$\Phi_v = Ez^{A_v}e^{-\theta\tau_v} = \Gamma_0^1 - \left(1-\Gamma^1\right)\mathcal{D}_x^{N-1}\left(\frac{\Gamma_0}{1-\Gamma}\right)$$
$$= \gamma_0\left(z,\theta\right) - \left(1-\gamma\left(z,\theta\right)\right)\mathcal{D}_x^{N-1}\left(\frac{\gamma_0\left(xz,\theta\right)}{1-\gamma\left(xz,\theta\right)}\right),$$

where

$$\gamma_0\left(z,\theta\right) = G\left(\theta+\lambda-\lambda a\left(z\right)\right)$$

and

$$G\left(\theta\right) = Ee^{-\theta\tau_0}$$

The latter equation is LST of the single vacation time. The complement state is the same as that in the N-Policy system. We suppose that our busy times are exponentially distributed and independent of the bulk input. That means

$$\gamma\left(z,\theta\right) = \frac{\lambda}{\lambda+\theta}a\left(z\right)$$

Therefore, we have

$$\alpha\left(z\right) = Ez^{A_v} = G\left(\lambda-\lambda a\left(z\right)\right) - \left[1-a\left(z\right)\right]\mathcal{D}_x^{N-1}\left\{\frac{G(\lambda-\lambda a(xz))}{1-a\left(xz\right)}\right\}$$

and to give Kendall's formula, we begin with

$$Q_{n+1} = \begin{cases} A_v - 1 + V_1, & Q_n = 0 \\ Q_n - 1 + V_1, & Q_n > 0 \end{cases}$$

to find the following

$$P_i\left(z\right) = E\left[z^{Q_1}\middle|Q_0=i\right] = \begin{cases} z^{-1}\alpha\left(z\right)\beta\left(\lambda-\lambda a\left(z\right)\right), & i=0 \\ z^{i-1}\beta\left(\lambda-\lambda a\left(z\right)\right), & i>0 \end{cases}$$

Thus, the pgf of this model

$$P\left(z\right) = p_0\alpha\left(z\right)z^{-1}\beta\left(\lambda-\lambda a\left(z\right)\right) + z^{-1}\beta\left(\lambda-\lambda a\left(z\right)\right)\sum_{i=1}^{\infty}p_iz^i$$

By doing few procedures, we see that

$$P\left(z\right) = p_0\beta\left(\lambda-\lambda a\left(z\right)\right)\frac{\alpha\left(z\right)-1}{z-\beta\left(\lambda-\lambda a\left(z\right)\right)}$$

and

$$p_0 = \frac{1-\lambda ab}{\alpha} = \frac{1-\rho}{\alpha}$$

where

$$\alpha := EA_v = \alpha'\left(z\right)\Big|_{z=1}$$

3. Applications on an N-Policy Queue with Single Vacation

In this section, we will discuss the mean goals of our work. First of all, we want to derive pgf of the number of units for this system when the input is ordinary Poisson, and the vacation time is Erlang with parameter $(r,\ \mu)$. Second of all, we obtain the input batches are type 2 geometric with vacation time is Erlang $(r,\ \mu)$.

3.1. Ordinary Poisson input with Erlang vacation time

We will obtain an explicit formula for pgf of this model when the input is ordinary Poisson, and the vacation time is Erlang with parameter $(r,\ \mu)$. Consequently, we note

$$a(z) = z) \quad \text{and} \quad G\left(\lambda - \lambda a\left(z\right)\right) = \left[\frac{\mu}{\mu + \left(\lambda - \lambda z\right)}\right]^{r} = \frac{1}{\left[1 - \frac{\lambda}{\mu}\left(z{-}1\right)\right]^{r}}$$

To find $\alpha(z)$, we need to derive it by using property (viii) in Theorem 2.8

$$\begin{aligned}
\mathcal{D}_x^{N-1}\left[\frac{G(\lambda - \lambda a(zx))}{1 - a(zx)}\right] &= \mathcal{D}_x^{N-1}\left[\frac{1}{\left[1 - \frac{\lambda}{\mu}\left(zx - 1\right)\right]^{r}}\ \frac{1}{1 - zx}\right] \\
&= \frac{1}{1-z}\sum_{j=0}^{N-1}\binom{r+j-1}{j}\left[\left(\frac{\lambda}{\mu}z\right)^{j} - z^{N-1+1}\left(\frac{\frac{-\lambda}{\mu}z}{z}\right)^{j}\right] \\
&= \frac{1}{1-z}\sum_{j=0}^{N-1}\binom{r+j-1}{j}\left(\frac{\lambda}{\mu}\right)^{j}\left(z^{j} - z^{N}\right)
\end{aligned}$$

So, we get the following formula

$$\alpha\left(z\right) = \sum_{j=0}^{\infty}\binom{r+j-1}{j}\left(\frac{\lambda}{\mu}\right)^{j}\left(z{-}1\right)^{j} - \sum_{j=0}^{N-1}\binom{r+j-1}{j}\left(\frac{\lambda}{\mu}\right)^{j}\left(z^{j} - z^{N}\right)$$

To catch α, we have

$$\alpha = \alpha' \ \left(z\right)\big|_{z=1} = \sum_{j=0}^{N-1}\left(\frac{r+j-1}{j}\right)\left(\frac{\lambda}{\mu}\right)^{j}\left(j - N\right)$$

To find $P(z)$, we need to obtain the LST given below

$$\beta\left(\lambda - \lambda z\right) = \frac{\mu}{\mu + \left(\lambda - \lambda z\right)} = \frac{\mu}{\mu - \lambda\left(z{-}1\right)} = \frac{1}{1 - \frac{\lambda}{\mu}\left(z{-}1\right)} \quad \text{and}$$

$$\frac{\beta\left(\lambda - \lambda z\right)}{z{-}\beta\left(\lambda - \lambda z\right)} = \frac{1}{\frac{z}{\beta(\lambda-\lambda z)} - 1} = \frac{1}{\left(z{-}1\right)\left(1 - \frac{\lambda}{\mu}z\right)}$$

So, the pgf of the model is given as

$$P\left(z\right) = p_0\frac{\alpha\left(z\right) - 1}{\left(z{-}1\right)\left(1 - \frac{\lambda}{\mu}z\right)}$$

where p_0 is given as below

$$p_0 = \frac{1-\frac{\lambda}{\mu}}{\alpha}$$

3.2. Type 2 Geometric Input Batches with Erlang Vacation Time

We keep working on our assumption that the vacation time is Erlang with parameter $(r,\ \mu)$. However, we focus on system input by presuming the input batches are distributed type 2 geometrically. We will gain precise expression for pgf of this model. We start with the following

$$G(\theta) = \left(\frac{\mu}{\mu+\theta}\right)^r \quad \text{and} \quad a(z) = \frac{p}{1-qz} \Longrightarrow 1-a(z) = \frac{q(1-z)}{1-zq}$$

So, we have the LST as below

$$G(\lambda-\lambda a(z)) = \frac{1}{(1+\frac{\lambda}{\mu}q)^r}\frac{(1-zq)^r}{\left[1-\frac{(1+\frac{\lambda}{\mu})q}{1+\frac{\lambda}{\mu}q}z\right]^r}$$

To get $\alpha(z)$, we require to obtain it by using some properties (v) and (viii) in Theorem 2.8

$$\mathcal{D}_x^{N-1}\left[\frac{G(\lambda-\lambda a(zx))}{[1-a(zx)]}\right] = \mathcal{D}_x^{N-1}\left[\frac{1}{(1+\frac{\lambda}{\mu}q)^r}\frac{(1-zq)^r}{[1-\frac{(1+\frac{\lambda}{\mu})q}{1+\frac{\lambda}{\mu}q}z]^r}\frac{1-xzq}{q(1-xz)}\right]$$

$$= \frac{1}{\left(1+\frac{\lambda}{\mu}q\right)^r q}\frac{1}{1-z}\sum_{i=0}^{N-1}(-qz)^i\sum_{j=0}^{N-i-1}\binom{r+j-1}{j}\left(\frac{\left(1+\frac{\lambda}{\mu}\right)q}{\left(1+\frac{\lambda}{\mu}q\right)}\right)^j\left(z^j-z^{N-i}\right)$$

Hence, we obtain the following result

$$\alpha(z) = \frac{1}{\left(1+\frac{\lambda}{\mu}q\right)^r}\left[\frac{(1-zq)^r}{(1-\frac{(1+\frac{\lambda}{\mu})q}{(1+\frac{\lambda}{\mu}q)}z)^r} - \frac{1}{(1-zq)}\sum_{i=0}^{N-1}(-1)^i\sum_{j=0}^{N-i-1}\binom{r+j-1}{j}\star\right.$$
$$\left.\left(\frac{\left(1+\frac{\lambda}{\mu}\right)q}{\left(1+\frac{\lambda}{\mu}q\right)}\right)^j\left(z^{j+i}-z^N\right)\right]$$

To find α, we think of

$$\alpha = \left.\alpha'(z)\right|_{z=1} = \frac{r\left(p-\frac{\lambda}{\mu}q^2\right)}{\left(1+\frac{\lambda}{\mu}q\right)p^3} - \frac{1}{p}\sum_{i=0}^{N-1}\sum_{j=0}^{N-i-1}\binom{r+j-1}{j}(-1)^i q^{i+j}\frac{\left(1+\frac{\lambda}{\mu}\right)^j}{\left(1+\frac{\lambda}{\mu}q\right)^{r+j}}(j+i-N)$$

To obtain $P(z)$, we want to get the LST given below

$$\frac{\beta(\lambda-\lambda a(z))}{z-\beta(\lambda-\lambda a(z))} = \frac{1-zq}{(z-1)}\ \frac{1}{1-\left(1+\frac{\lambda}{\mu}\right)qz}$$

Therefore, the pgf of version is presented the same as

$$P(z) = p_0 \frac{1 - zq}{1 - \left(1 + \frac{\lambda}{\mu}\right) qz} \frac{\alpha(z) - 1}{(z-1)}$$

where p_0 is provided such as follow

$$p_0 = \frac{1 - \frac{\lambda q}{\mu p}}{\alpha}$$

4. Conclusion

In this article, we obtain directed expressions of pgf of the number of units in the N-policy queue with a single vacation corresponding to different input and vacation times. Firstly, we assume that the input of the line is ordinary Poisson, and the vacation time of the server has Erlang distribution. Secondly, we suggest that the input is type 2 geometric and the same later vacation time. This work is done using fluctuation theory.

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