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Rational maps whose Julia sets are quasi circles

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Abstract

In this paper, we give a family of rational maps whose Julia sets are quasicircles also we the boundaries of I_0 , I_∞ are quasicircles , we have the family of complex rational maps are given by

$$\mathcal{Q}_\alpha(Z) = 2\alpha^{1-n}\ Z^n - \frac{z^n\left(z^{2n}-\alpha^{n+1}\right)}{z^{2n}-\alpha^{3n-1}}, \tag{0.1}$$

where $n \geq 2$ and $\alpha \in C\backslash\{0\}$, but $\alpha^{2n-2} \neq 1,\ \ \alpha^{1-n} \neq 1$.

Keywords: Julia Sets, Fatou Sets, Singular Perturbation, Quasi circles

1. Introduction

In general the rational maps have dynamics more complex than that of polynomials because each polynomial has a totally invariant superattracting focused at ∞. See [12, 13]. However, if we study dynamics behaviors of polynomials When studying the behavior of the rational maps close to the polynomials through the concept of perturbation. McMullen is the first who used the singular perturbation on Z^n(see [11]) he study the family of rational maps $F_d(Z) = Z^p + d/Z^l$ where $p \geq 2, l \geq 1$ and $d \in C\backslash\{0\}$, this map called the McMullen maps. The McMullen map has been studied by several authors. The authors in [3], we give The Escape Trichotomy Theorem for F_d by the orbits of the free critical points. After that through several people, they found a generalization of McMullen maps see [3, 6, 7, 14]. Fu and Yang [8], They studied the following maps

$$h_\lambda(Z) = \frac{z^d\left(z^{2d}-\alpha^{d+1}\right)}{z^{2d}-\alpha^{3d-1}},$$

where $d \geq 2$ and $\lambda \in C\backslash\{0\}$, such that $\lambda^{2n-2} \neq 1$. They got several things, including Julia sets is cantor circles or quasicircles or Sierpinski carpet according to the iterate of the free critical points. See [8, 9, 15].

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2. Background and the Main Result

For each a rational map R with degree $d \geq 2$ on $C_\infty = C \cup \{\infty\}$. Let R^k be the $k-th$ iteration of R, where $k \in N$. We define the Julia set as the closure of the set of all repelling periodic points. Also, The Julia set of R is the set of points at which the family of iterates $\{R^k : k \in N\}$ fails to be a normal family in the sense of Montel, denoted by $J(R)$. However, $C_\infty \backslash J(R)$ is the Fatou set of R $F(R)$. We call any connected component of the Fatou set is Fatou component. Now we study

$$\mathcal{Q}_\alpha(Z) = 2\alpha^{1-n}\ Z^n - \frac{z^n\left(z^{2n} - \alpha^{n+1}\right)}{z^{2n} - \alpha^{3n-1}},$$

where $n \geq 2$ and $\alpha \in C\backslash\{0\}$, but $\alpha^{2n-2} \neq 1,\ \ \alpha^{1-n} \neq 1$. If $n = 1,\ \alpha = 0\ \ or\ \ \alpha^{2n-2} = 1,\ \ \alpha^{1-n} = 1,\ \mathcal{Q}_\alpha$ degenerates to the polynomial $q_n(z) = -z^{-n}\ \ or\ \ z^n$, then the map $\mathcal{Q}_\alpha$ perturbed in the polynomial q_n. $\mathcal{Q}_\alpha$ have superattracting periodic orbits of 0 and ∞. The immediate basin of attraction of 0 and ∞ denoted by I_0 and I_∞ respectively. Since the degree of any rational maps is the maximal of degrees of Numerator and denominator , thus in (0.1) the degree of $\mathcal{Q}_\alpha$ is $3n$. By Corollary 2.7.2. in [1], any rational maps have the critical points $(2d-2)$, then the map $\mathcal{Q}_\alpha$ has $(6n-2)$ critical points (counted with multiplicity).
We offer the main result as follows :

Theorem 2.1. *Assume that the orbit of one free critical point p_α of $\mathcal{Q}_\alpha$ is attracted by ∞ (resp. 0), and $p_\alpha \in I_\alpha$ (resp. I_0), then $J(\mathcal{Q}_\alpha)$ is a quasicircle.* *(see Figure 1)*

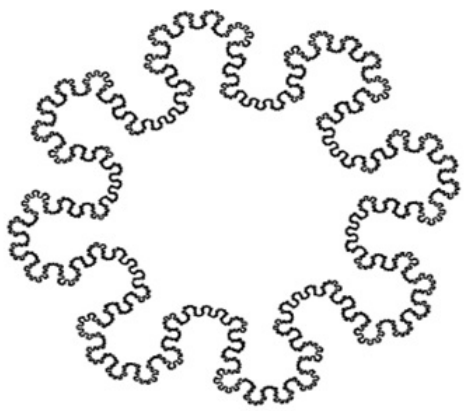

Figure 1:

Theorem 2.2. *The boundary of I_0 and The boundary of I_∞ are quasicircles if the one of the free critical orbits of $\mathcal{Q}_\alpha$ is attracted by either* 0 *or* ∞.

3. Preliminaries

In this section, we give the symmetric dynamical behaviors and the symmetric distribution of critical points for .

Lemma 3.1. *For each $\omega \in C$ such that satisfying $\omega^{2n} = 1$. Therefore $\mathcal{Q}^l_\alpha(\omega\ z) = \omega^{n^l}(z)\ \mathcal{Q}^l_\alpha(z)$ for $l \geq 1$.*

Proof . Suppose that

$$\mathcal{Q}_\alpha(z) = 2\alpha^{1-n}\ z^n - \frac{z^n\left(z^{2n} - \alpha^{n+1}\right)}{z^{2n} - \alpha^{3n-1}},$$

and calculation by use $\omega^{2n} = 1$, we have

$$\mathcal{Q}_\alpha(\omega\ z) = \omega^n\left(2\alpha^{1-n}\ z^n - \frac{z^n\left(z^{2n} - \alpha^{n+1}\right)}{z^{2n} - \alpha^{3n-1}}\right) = \omega^n\mathcal{Q}_\alpha(z).$$

Assume that $\mathcal{Q}^l_\alpha(\omega\ z) = \omega^{n^l}(z)\ \mathcal{Q}^l_\alpha(z)$ for some $l \geq 1$. Now we use the Law of Induction, then $\mathcal{Q}^{l+1}_\alpha(\omega\ z) = \mathcal{Q}_\alpha\left(\mathcal{Q}^l_\alpha(\omega\ z)\right) = \omega^{n^{l+1}}\ \mathcal{Q}^{l+1}_\alpha(z)$. □

Lemma 3.2. *For any $\eta(z) = \frac{\alpha^2}{z}$. Then $\mathcal{Q}_\alpha$ satisfies the equation $\eta \circ \mathcal{Q}_\alpha(z) = \mathcal{Q}_\alpha \circ \eta(z)$ for each $z \in C_\infty$.*

Proof . *We notice $\eta^{-1}(z) = \frac{\alpha^2}{z} = \eta(z)$. Then*

$$\begin{aligned}
\mathcal{Q}_\alpha \circ \eta(z) &= 2\alpha^{1-n}\ \left(\frac{\alpha^2}{z}\right)^n - \frac{\left(\frac{\alpha^2}{z}\right)^n\left(\left(\frac{\alpha^2}{z}\right)^{2n} - \alpha^{n+1}\right)}{\left(\frac{\alpha^2}{z}\right)^{2n} - \alpha^{3n-1}} \\
&= \frac{-\alpha^{6n} + \alpha^{3n+1}z^{2n} + 2\alpha^{5n+1} - 2\alpha^{4n}z^{2n}}{z^n\left(\alpha^{4n} - \alpha^{3n-1}z^{2n}\right)} \\
&= \alpha^2\frac{z^{2n} - \alpha^{3n-1}}{2\alpha^{1-n}z^{3n} - 2\alpha^{2n}z^n - z^n\left(z^{2n} - \alpha^{n+1}\right)} \\
&= \frac{\alpha^2}{\mathcal{Q}_\alpha(z)} = \eta \circ \mathcal{Q}_\alpha(z).
\end{aligned}$$

□

From Lemma 3.1 and Lemma 3.2, the orbits of points with the form $\omega^k\ z$, where $k = 0, 1, 2, \ldots, 2n-1$, or form (α^2/z) behave symmetry of the iteration of $\mathcal{Q}_\alpha$, e.g., if $\mathcal{Q}^l_\alpha(z)$ tends to 0 (or ∞), then $\mathcal{Q}^l_\alpha(\omega^k\ z)$ or $\mathcal{Q}^l_\alpha(\alpha^2/z)$ also tends to 0 (or ∞) or ∞ (or 0) respectively, for $1 \leq k \leq 2n-1$ as l tends to ∞ of $\mathcal{Q}_\alpha$. A Fatou component W is invariant if $f(W) \subseteq W$ and fixed if $f(W) = W$. If $f^p(W) = W$, we called W is periodic component for some $p \in N$.

Theorem 3.3. *[1] If the Fatou set $F(R)$ of R has two completely invariant components, then these are the only components of R.*

Corollary 3.4. *Suppose that W is a Fatou component of $\mathcal{Q}_\alpha$, then $W = \eta(W)$. In special case $\eta(I_0) = I_\infty$ and $\eta(I_\infty) = I_0$.*

Proof . *Suppose that W is a Fatou component of $\mathcal{Q}_\alpha$, then by use Lemma 3.2 $W = \eta(W)$, thus W fixed component and $W = \eta(W) = \eta^{-1}(W)$, it follows W is completely invariant . By Theorem 3.3, we have only two components I_0 and I_∞, and by Lemma 3.2 $\eta(z) = \alpha^2/z$, then $\eta(I_0) = I_\infty$ and $\eta(I_\infty) = I_0$.* □

Now, we have

$$\acute{\mathcal{Q}}_\alpha(z) = nz^{n-1}\frac{\left(3\alpha^{3n-1} - 4\alpha^n - \alpha^{n+1}\right)z^{2n} - z^{4n}\left(1 - 2\alpha^{1-n}\right) - \alpha^{4n} + 2\alpha^{5n-1}}{\left(z^{2n} - \alpha^{3n-1}\right)^2} \tag{3.1}$$

$$\acute{\mathcal{Q}}_{\alpha}(z)=nz^{n-1}\frac{\left(3\alpha^{3n-1}-4\alpha^{n}-\alpha^{n+1}\right)z^{2n}-z^{4n}\left(1-2\alpha^{1-n}\right)-\alpha^{4n}+2\alpha^{5n-1}}{\left(z^{2n}-\alpha^{3n-1}\right)^{2}}=0,$$

then either $z=0$ with multiplicity $n-1$ and from Lemma 3.2, ∞ is a critical point of $\mathcal{Q}_{\alpha}$ with multiplicity $n-1$, or

$$z^{4n}\left(1-2\alpha^{1-n}\right)-\left(3\alpha^{3n-1}-4\alpha^{n}-\alpha^{n+1}\right)z^{2n}+\alpha^{4n}-2\alpha^{5n-1}=0$$

$$p_{\alpha}^{2n}=\frac{3\alpha^{3n-1}+4\alpha^{4n}+\alpha^{n+1}\pm\sqrt{\left(3\alpha^{3n-1}+4\alpha^{4n}+\alpha^{n+1}\right)^{2}-4\left(1-2\alpha^{1-n}\right)\left(\alpha^{4n}-2\alpha^{5n-1}\right)}}{2\left(1+2\alpha^{1-n}\right)}$$

There are two roots

$$p_{\alpha}^{2n}=\frac{3\alpha^{3n-1}+4\alpha^{4n}+\alpha^{n+1}+\sqrt{\left(3\alpha^{3n-1}+4\alpha^{4n}+\alpha^{n+1}\right)^{2}-4\left(1-2\alpha^{1-n}\right)\left(\alpha^{4n}-2\alpha^{5n-1}\right)}}{2\left(1+2\alpha^{1-n}\right)}\tag{3.2}$$

and

$$p_{\alpha}^{2n}=\frac{3\alpha^{3n-1}+4\alpha^{4n}+\alpha^{n+1}-\sqrt{\left(3\alpha^{3n-1}+4\alpha^{4n}+\alpha^{n+1}\right)^{2}-4\left(1-2\alpha^{1-n}\right)\left(\alpha^{4n}-2\alpha^{5n-1}\right)}}{2\left(1+2\alpha^{1-n}\right)}\tag{3.3}$$

there are $4n$ critical points other than 0 and ∞. By use Lemma 3.1 and Lemma 3.2, we can written the critical points as form $Cr(\mathcal{Q}_{\alpha})=\{\omega_0^k p_{\alpha}\ ,\omega_0^k\ \alpha^2/p_{\alpha}:0\leq k\leq 2n-1\}$, where $\omega_0=\exp(i\pi/n)$, note that $\mathcal{Q}_{\alpha}$ has one free critical orbit. Let $W\subset C_{\infty}$ and $c\in C$. We define $cW=\{cz:/z\in\ W\}$.

Lemma 3.5. *I_0 and I_{∞} have the symmetry $(2n-fold)$, if $z\in I_{\infty}$ or I_0, then $\omega z\in I_{\infty}$ or I_0, respectively, and $\omega^{2n}=1$. We discuss the case I_0.*
Proof . *For any $W\subset I_0$ be define as $\{z\in I_0:\omega z\in I_0\}$, W is non-empty and open set since I_0 consist of small neighborhood of 0. Now if $W\neq 0$, suppose that $z_0\in I_0\cap\partial W$, that is $z_0\in\partial W$ we have $z_0\in I_0$ and $\omega z_0\in I_0$. Hence $\omega z_0\in\partial I_0$ since $\omega z_0\in\partial W$ and $W\subset I_0$. Then $\mathcal{Q}_{\alpha}^l(z_0)\to 0$ whereas $\mathcal{Q}_{\alpha}^l(z_0)\not\to 0$ as $l\to\infty$. But $\mathcal{Q}_{\alpha}^l(z_0)=\omega^{n^l}\mathcal{Q}_{\alpha}^l(z_0)\to 0$. This is a contradiction. Therefore $\omega I_0=I_0$. Similarly we can proof that $\omega I_{\infty}=I_{\infty}$.* $\square$

Using the same steps and technique as the source [10], the results can be generali to the following Lemma.

Lemma 3.6. *Let W be a Fatou component of $\mathcal{Q}_{\alpha}$. Let z_0 and $\omega^{k_0}z_0$ belong to W, where $\omega^{2n}=1$ and $\omega^k\neq 1$. Then $\omega^k z_0\in W$ for each integer k.Then W has the symmetry $(2n-fold)$ and surround 0.*

Proposition 3.7. *Assume that $\alpha\in\Re$($\Re$ Real numbers). Then $\tau_{|\alpha|}=\{z\in C:|z|=|\alpha|\}$ be the round circle and $\mathcal{Q}_{\alpha}:\tau_{|\alpha|}\to\tau_{|\alpha|}$. Moreover, $\tau_{|\alpha|}\subset J(\mathcal{Q}_{\alpha})$ if the free critical orbits are attracted by ∞ or 0.*
Proof .*Assume that $z=|\alpha|e^{i\theta}$, where $\theta\in[0,2\pi)$. Then*

$$|\mathcal{Q}_{\alpha}(z)|\leq 2|\alpha|^{1-n}\ |\alpha|^n-|\alpha|^n\star\frac{|\alpha^{2n}e^{2ni\theta}-\alpha^{n+1}|}{|\alpha^{2n}e^{2ni\theta}-\alpha^{3n-1}|}=2|\alpha|-|\alpha|=|\alpha|.$$

This means that $\mathcal{Q}_{\alpha}(\tau_{|\alpha|})\subset\tau_{|\alpha|}$. If α is real, we have $|\alpha|\neq 1$, since $\alpha^{2n-2}\neq 1$, $\alpha^{1-n}\neq 1$, by use the definition of $\mathcal{Q}_{\alpha}$. Now we have two cases.

- *Case 1* *if* $|\alpha| > 1$, *we have* $|\alpha|^{\frac{n+1}{2n}} < |\alpha| < |\alpha|^{\frac{3n-1}{2n}}$, *that is* α^{3n-1} *is large such that by* (0.1) $\mathcal{Q}_\alpha$ *has* $3n$ *roots and no poles in* $D_{|\alpha|} = \{z \in C : |z| < |\alpha|\}$. *From the Argument Theorem, thus* $\mathcal{Q}_\alpha(\tau_{|\alpha|})$ *around the origin* $3n$ *times anticlockwise. Hence,* $\mathcal{Q}_\alpha(\tau_{|\alpha|}) = \tau_{|\alpha|}$

- *Case 2* *if* $0 < |\alpha| < 1$, *we have* $|\alpha|^{\frac{n+1}{2n}} < |\alpha| < |\alpha|^{\frac{3n-1}{2n}}$. *Note that if* α^{n+1} *is large and* α^{3n-1} *is small such that by* (0.1), *then we have*

$$\mathcal{Q}_\alpha(z) = 2\alpha^{n-1}z^n - \frac{z^n\left(z^{2n} - \alpha^{n+1}\right)}{z^{2n}}.$$

Since α^{3n-1} *is small* $\mathcal{Q}_\alpha$ *has* n *roots and* $2n$ *poles in* D_α, *thus* $\mathcal{Q}_\alpha(\tau_{|\alpha|})$ *around the origin* n *times clockwise. Therefore,* $\mathcal{Q}_\alpha(\tau_{|\alpha|}) = \tau_|\alpha|$. *Therefore* $\mathcal{Q}_\alpha : \tau_{|\alpha|} \to \tau_{|\alpha|}$ *is a surjection in the two cases. Suppose that* $z_0 \in \tau_{|\alpha|} \subset F(\mathcal{Q}_\alpha)$, *then* $\mathcal{Q}^l_\alpha(z_0) \to 0$ *or* ∞ *as* $l \to \infty$. *However, on the other side,* $\mathcal{Q}^l_\alpha(z_0) \in \mathcal{Q}_\alpha(\tau_{|\alpha|}) = \tau_{|\alpha|}$ *for each* $l \geq 0$, *that is contradict.*
Hence, $\tau_\alpha \subset J(\mathcal{Q}_\alpha)$

□

Example 3.8. *If* n *is odd and* $\mathcal{Q}_\alpha(-\alpha) = -\alpha$, *then* τ_α *is not contained* $J(\mathcal{Q}_\alpha)$.
If $\mathcal{Q}_\alpha(-\alpha) = -\alpha$. *By* (3.1), *we have*

$$\begin{aligned}
\acute{\mathcal{Q}}_\alpha(z) &= nz^{n-1}\frac{-z^{4n} + \left(3\alpha^{3n-1} - \alpha^{n+1}\right)z^{2n}\left(1 - 2\alpha^{1-n}\right) - \alpha^{4n}}{\left(z^{2n} - \alpha^{3n-1}\right)^2} + 2n\alpha^{1-n}z^{n-1} \\
\acute{\mathcal{Q}}_\alpha((-\alpha)) &= n(-\alpha)^{n-1}\frac{-(-\alpha)^{4n} + \left(3\alpha^{3n-1} - \alpha^{n+1}\right)(-\alpha)^{2n}\left(1 - 2\alpha^{1-n}\right) - \alpha^{4n}}{\left((-\alpha)^{2n} - \alpha^{3n-1}\right)^2} + 2n\alpha^{1-n}(-\alpha)^{n-1} \\
&= -n\alpha^{n-1}\frac{\alpha^{4n} - 3\alpha^{5n-1} + \alpha^{3n+1} - \alpha^{4n}}{\left(\alpha^{2n} + \alpha^{3n-1}\right)^2} - 2n \\
&= \frac{n\alpha^{2n-2} - 4n\alpha^{n-1} - 3n}{1 + 2\alpha^{n-1} + \alpha^{2n-2}}
\end{aligned}$$

If $\acute{\mathcal{Q}}_\alpha((-\alpha)) = 0$, *then* $n\alpha^{2n-2} - 4n\alpha^{n-1} - 3n = 0$, *or* $\alpha^{2n-2} - 4\alpha^{n-1} - 3 = 0$, *thus, either* $\alpha = -(0.6457)^{\frac{1}{n-1}}$ *or* $\alpha = 4.6457^{\frac{1}{n-1}}$. *Hence* $-\alpha$ *is superattracting fixed point of* $\mathcal{Q}_\alpha$, *but this is not attract to* 0 *or* ∞. *Then* $\tau_{|\alpha|} \not\subset J(\mathcal{Q}_\alpha)$.

4. The Proof of Main Results

We study a sufficient and necessary condition for $J(\mathcal{Q}_\alpha)$ is a quasicircle.
Let $f : D \to D'$ be an orientation-preserving homeomorphism between open sets in the plane. If f is continuously differentiable, then it is K-quasiconformal if the derivative of f at every point maps circles to ellipses with eccentricity bounded by K.
A simple closed curve is quasicircle if is equal to the image of the unit circle for a quasiconformal homeomorphism map from $C_\infty \to C_\infty$.

Lemma 4.1. *[3] Assume that the rational map is hyperbolic, it has exactly two Fatou components. Then the Julia set is a quasicircle.*

Corollary 4.2. *Suppose that* $\mathcal{Q}_\alpha$ *is hyperbolic map have exactly two Fatou component* I_0 *and* I_∞, *then* $J(\mathcal{Q}_\alpha)$ *is quasicircle.*
Proof . *By Lemma 4.1 and by Lemma 3.2,* I_0 *and* I_∞ *are contain in* $F(\mathcal{Q}_\alpha)$. *Then* $J(\mathcal{Q}_\alpha)$ *is quasicircle* □

Proposition 4.3. *Assume that $p_\alpha \in I_0$ or I_∞, then the $J(\mathcal{Q}_\alpha)$ is quasicircle.*
Proof . *Assume that $p_\alpha \in I_0$. Then by Lemma 3.2 $\alpha^2/\ p_\alpha \in I_\infty$, also by Lemma 3.1 $\omega^j\ p_\alpha \in I_0$ and $\omega^j\alpha^2\ p_\alpha \in I_\infty$ for $0 \leq j \leq 2n-1$, we assume that $\mathcal{Q}_\alpha^{-1}(I_0)$ has the unique component say, I_0 and $\mathcal{Q}_\alpha^{-1}(I_\infty)$ has the unique component say, I_∞. If the degree of the restriction of $\mathcal{Q}_\alpha$ is m and $\mathcal{Q}_\alpha : I_0 \to I_0$ is proper, then $n \leq m$ because n is the local degree of $\mathcal{Q}_\alpha$ at 0. Now in I_0 the preimages of 0 have elements other than 0 by Lemma 3.1, it follows $3n \leq m$ i.e. $3n = m$ because $3n$ is the degree of $\mathcal{Q}_\alpha$, implies $n = m$ or $3n = m$. We prove that $n = m$ is not correct. From assumption $p_\alpha \in I_0$ that $(2n-1)$ free critical points of $\mathcal{Q}_\alpha\{\omega^j\ p_\alpha : 1 \leq j \leq 2n-1\}$ lies in I_0 by Lemma 3.5 From the definition of $\mathcal{Q}_\alpha$ in (3.1), thus there exist preimages $2n$ at least for $\mathcal{Q}_\alpha(p_\alpha)\{\omega^j\ p_\alpha : 1 \leq j \leq 2n-1\}$ in I_0 (counted with multiplicity). Hence $2n \leq m$, which is contradict with $n = m$. Therefore $3n = m$ and $\mathcal{Q}_\alpha^{-1}(I_0)$ has the unique component I_0, also prove $\mathcal{Q}_\alpha^{-1}(I_\infty)$ has the unique component I_∞. We assume that only there are I_0 and I_∞ contain in $F(\mathcal{Q}_\alpha)$. Now, note that there were either superattracting basins or parabolic basins contain one critical point at least, which is contradict because each of the critical points lie in I_0 and I_∞. If there were either Herman rings or Siegel disks, thus $J(\mathcal{Q}_\alpha)$ contains one critical value at least and is contradicted. By Corollary 4.2, thus $\mathcal{Q}_\alpha$ is hyperbolic. According to Lemma 4.1, $J(\mathcal{Q}_\alpha)$ is a quasicircle.* □

Proposition 4.4. *Suppose that $|\alpha|$ is large enough. Then $J(\mathcal{Q}_\alpha)$ is a quasicircle.*
Proof .*From (3.2), we have*

$$p_\alpha^{2n} = \frac{3\alpha^{3n-1} + 4\alpha^{4n} + \alpha^{n+1} + \sqrt{\left(3\alpha^{3n-1} + 4\alpha^{4n} + \alpha^{n+1}\right)^2 - 4\left(1-2\alpha^{1-n}\right)\left(\alpha^{4n} - 2\alpha^{5n-1}\right)}}{2\left(1+2\alpha^{1-n}\right)}.$$

Thus $|p_\alpha| \approx |\alpha|^{\frac{3n-1}{2n}}$, if $|\alpha|$ is large enough. Since $n \geq 2$, thus $\alpha^{\frac{3n-1}{2n}} \geq |\alpha|^{\frac{5}{4}} > |\alpha|^{\frac{6}{5}}$. Define $\beta = \{z : |z| > |\alpha|^{\frac{6}{5}}\}$. Now, if $|\alpha|$ is large enough and $z \in \beta$, then

$$\begin{aligned}\mathcal{Q}_\alpha(z) \geq& 2|\alpha|^{n-1}|z|^n - \frac{|z|^n\left(|z|^{2n} - |\alpha|^{n+1}\right)}{|z|^{2n} - |\alpha|^{3n-1}} > 2|\alpha|^{n-1}|\alpha|^{\frac{6n}{5}} - \frac{|\alpha|^{\frac{6n}{5}}\left(|\alpha|^{\frac{12n}{5}} - |\alpha|^{n+1}\right)}{|\alpha|^{\frac{12n}{5}} - |\alpha|^{3n-1}}\\ >& 2|\alpha|^{\frac{n+5}{5}} - \frac{1}{2}|\alpha|^{\frac{6}{5}} > 2|\alpha|^{\frac{6}{5}}\end{aligned}$$

This means that $\mathcal{Q}_\alpha(\beta) \subset \beta$. Therefore β is contained in I_∞. Conversely p_α holds $|p_\alpha| \approx |\alpha|^{\frac{3n-1}{2n}} > |\alpha|^{\frac{6}{5}}$. It follows that $p_\alpha \in \beta \subset I_\infty$ if $|\alpha|$ is large enough. By Proposition 4.3, $J(\mathcal{Q}_\alpha)$ is a quasicircle □

Proposition 4.5. *Assume that $\alpha \in \Re$. Then is a quasicircle iff $J(\mathcal{Q}_\alpha) = \tau_{|\alpha|}$ if $\alpha > 1$.*
Proof . *Suppose that $\alpha > 1$, we have $\mathcal{Q}_\alpha(\tau_\alpha) = \tau_{|\alpha|}$ and in the round disk $D_{|\alpha|} = \{z : \ |z| < |\alpha|\}$, $\mathcal{Q}_\alpha$ has no poles and $3n$ roots from Proposition 3.7. Hence $\mathcal{Q}_\alpha(D_\alpha) = D_{|\alpha|}$. Thus, $D_\alpha \subset F(\mathcal{Q}_\alpha)$. From Lemma 3.2, $C_\infty \backslash \bar{D}_{|\alpha|} \subset F(\mathcal{Q}_\alpha)$ In special case $D_\alpha \subset I_0$, also $C_\infty \backslash \bar{D}_{|\alpha|} \subset I_\infty$. It follows that $J(\mathcal{Q}_\alpha) = \tau_{|\alpha|}$ because $I_0 \cap I_\infty = \phi$.*
Conversely, we assume that $0 < |\alpha| < 1$. And assume that $J(\mathcal{Q}_\alpha)$ is a quasicircle, thus $J(\mathcal{Q}_\alpha) = \tau_{|\alpha|}$ because $\mathcal{Q}_\alpha(\tau_\alpha) = \tau_{|\alpha|}$. Hence, $D_\alpha = I_0$, and $\mathcal{Q}_\alpha : D_{|\alpha|} \to D_{|\alpha|}$ has a degree $3n$ for the covering map. Conversely, if $0 < |\lambda| < 1$, $D_{|\alpha|}$, include $2n$ poles and n roots for $\mathcal{Q}_\alpha$, which is a contradict. Hence if $0 < |\lambda| < 1$, $J(\mathcal{Q}_\alpha)$ is not quasicircle. □

Let $A_0 = \mathcal{Q}_\alpha^{-1}(I_0)\backslash I_0$ be the first preimage of I_0 and $A_\infty = \mathcal{Q}_\alpha^{-1}(I_\infty)\backslash I_\infty$ the first preimage of I_∞. If $J(\mathcal{Q}_\alpha)$ is not a quasicircle, by proposition 4.3. It follows one of the free critical points not lies in I_0 or I_∞, thus one of the free critical points lies in A_0 and A_∞. So A_0 and A_∞ are both non-empty. Let $U \subset X$ be an open set of a topological space X and $V \Subset U$an open, compactly contained set (i.e., $\bar{V}$ is compact and $\bar{V} \subset U$).

Theorem 4.6. *[5]*

(i) Every polynomial-like mapping $f : U' \to U$ of degree d is hybrid equivalent to a polynomial p of degree d.

(ii) If $K_f$is connected, p is unique up to conjugation by an affine map.

Proposition 4.7. *Both ∂I_0, ∂I_∞, and each of the preimages of them is quasicircles around* 0.

Proof . *For each closed set $V = C_\infty \setminus (I_0 \cup I_\infty)$ amidst I_0 and I_∞ divided into closed sets V_1, V_2, V_3 between I_∞ and A_0, A_0 and A_∞ and A_∞ and I_0 (see Figure (2)). For any smooth simple closed curve $\Gamma \subset A_) \subset V$ around* 0*. We assume that in V_3 the preimage of Γ is smooth simple closed curve around* 0*. Note that V, includes no critical values. Therefore in V_3 the preimage of Γ contains of finitely many smooth simple closed curves. Assume that Γ_3 is not around* 0*. Therefore in V_3, Γ_3 can disfigure to a point. It follows that in V, $\Gamma = \mathcal{Q}_\alpha(\Gamma_3)$ can also disfigure to a point. Which is contradicted because $\Gamma \subset V$is around* 0*. Hence in V_3 the preimage of Γ are smooth simple closed curves around* 0 *and* ∞*. In V_3, there are two components for $\mathcal{Q}_\alpha^{-1}(\Gamma)$, thus between two simple closed curves, the annular region include either poles or roots , this is impossible. Then, $\mathcal{Q}_\alpha^{-1}(\Gamma) \cap V_3$ is a smooth simple closed curve around* 0*, say $\mathfrak{J}$. Let $\gamma \subset C$ be a simple closed curve and assume that the bounded component of $C\setminus\gamma$ is γ^{int}. We remark that in $\Gamma^{int}$is the Jordan disk include $\mathfrak{J}^{int}$is compactly contained. By Theorem 4.6, Therefore $\mathcal{Q}_\alpha : \mathfrak{J}^{int} \to \Gamma^{int}$ is quasiconformally equivalent to $q_n(z) = z^n$. We know that $J(q_n) = S^1$. Then ∂I_0 is a quasicircle. Similarly, also I_∞ is a quasicircle. Because each of the preimages of I_0 and I_∞ are include in $V_1 \cup V_2 \cup V_3$, It follows that each of the preimages of I_0 and I_∞ are quasicircles around* 0. □

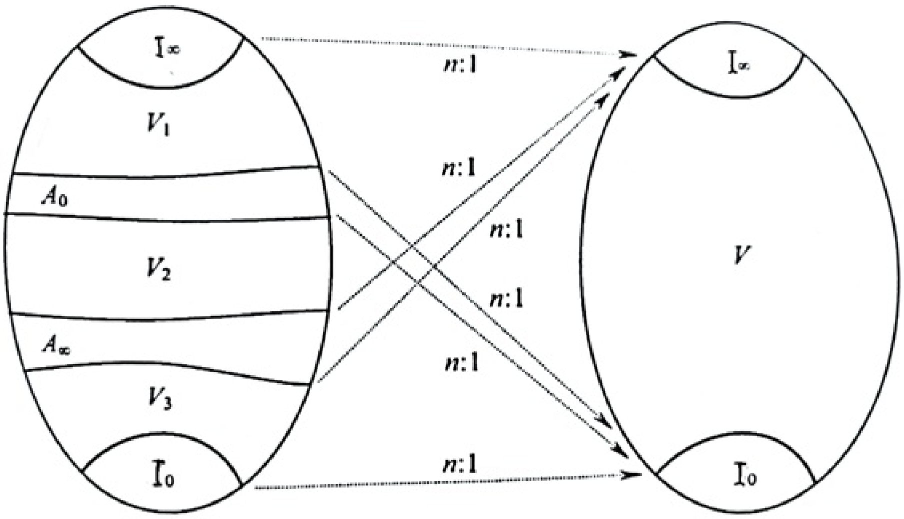

Figure 2:

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