Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2049-2055
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5342

$ii\delta_g$-closed set in topological spaces

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The purpose of this study is to determine if it is possible to use the class of δ_g-close sets and $\delta-$closed sets to find a new class of close sets, namely $ii\delta_g-$close sets and $ii\delta-$close sets. We use these new classes to arrive at the separation axiom, which are the separation axiom of $ii-T_{3/4}$. We also use the properties of $\delta\hat{g}-$closed sets to find new class of close sets, namely $ii\delta\hat{g}-$closed set. We use it to create a new type of separation axiom. The study found new closed sets, namely: $ii\delta\hat{g}-$closed sets is introduced for topological space. We will also prove that category falls between the class of $ii\delta-$closed set and the class of $iigs-$closed set. As well study the relationship between it and other closed sets, such as the closed set of type $iigs-$ as well as the closed set of type iig, and then, through these closed set, we will study new types of the axiom of separation, which are $ii-T_{3/4}$ space and $ii-\hat{T}_{3/4}$ space, and also clarify the relationship between the axiom of separation, and some of the closed and open sets previously studied. The study also included the relationship between the axiom of separation, of type $ii-T_{1/2}$ space and $ii-T_{3/4}$ space as well as the axiom of separation of type $ii-T_{3/4}$ space and $ii\hat{T}_{3/4}$ space. The study included some important proofs. The study also included a chart, showing the relationship between closed sets of types that were studied and can be used.

Keywords: $ii-$generalized closed sets, $ii\delta_g$-closed, $\hat{g}-$open sets, $ii\delta\hat{g}-$closed set, $ii-T_{3/4}$ and $ii-\hat{T}_{3/4}$ space

1. Introduction

Levine [8], Mashhour et all [12], Njastad [14]and Velicko [16] introduced semi- open set pre-open set, $\alpha-$ open set and $\delta-$ closed set respectively, Levine [9] introduced generalized closed (briefly $sg-$ closed set) and studied their basic properties.

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Received: March 2021 *Accepted:* July 2021

Bhattacharya and Lahiri [4] Arya and Nour [3], Maki et al [10, 11] Dontchev and Ganster[6] introduced semi generalized closebrid (efly $sg-$ closed set), generalized semi- closed set (briefly $gs-$ closed set), generalized $\alpha-$ closed set (briefly $g\alpha-$ closed set), $\alpha-$ generalized closed (briefly $\alpha g-$ closed set), and $\delta-$ generalized closed (briefly $\delta g-$ closed set) respectively. Navaneet [13] introduced $\hat{g}-$ closed set in topological space. The purpose of this present paper is to define a new class of closed set called $ii\delta\hat{g}-$closed and also we obtain some basic properties of $ii\delta\hat{g}-$ closed set in topological space. Applying these sets we obtain anew space which is called $ii-T_{3/4}$ and $ii-\hat{T}_{3/4}$ space.

2. Preliminaries

Definition 2.1. *The subset* B *of* (X,τ) *is known as*

1. *A closed generalized set (in short $g-$ closed) if $cl(B)\subset G$ at any time $B\subset G$ and G is open [9].*
2. *A $\alpha-$closed generalized set (in short $\alpha g-$ closed) if $\alpha cl(B)\subset G$ at any time $B\subset G$ and G is $\alpha-$open [11].*
3. *A semi–closed generalized set (in short $sg-$closed) if $scl(B)\subset G$ at any time $B\subset G$ and G is semi open[4]*
4. *A semi–closed generalized set (in short $gs-$closed) if $scl(B)\subset G$ at any time $B\subset G$ and G is open.*
5. *A $\alpha-$closed generalized set (in short $g\alpha-$closed) $\alpha cl(B)\subset G$ at any time $B\subset G$ and G is open [10].*
6. *A semi- pre closed generalized set (in short $gsp-$closed) if $spcl(B)\subset G$ at any time $B\subset G$ and G is open.[7]*
7. *A regular closed generalized set (in short $rg-$closed) $cl(B)\subset G$ at any time $B\subset G$ and G is regular.*
8. *A $\hat{g}-$closed set if $cl(B)\subset G$ at any time $B\subset G$ and G is semi open in (X,τ) [13].*

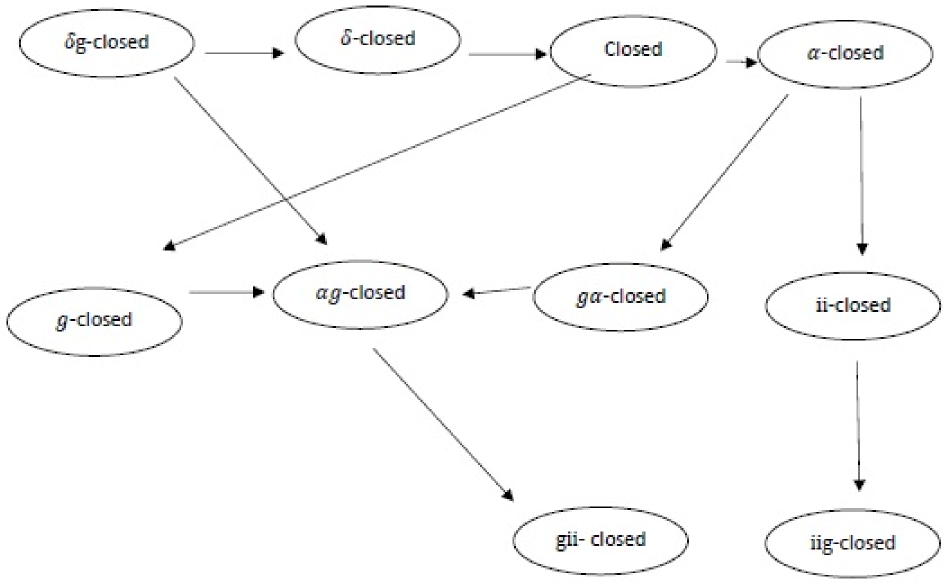
Definition 2.2. *The topology space (X,τ) is referred to as a:*

1. *if each $g-$closed set is closed [14].*
2. *$T_{1/2}-$space if each $g-$closed set is closed [5].*
3. *$T_{1/4}-$space if each finite subset $G\subset X$ and every point $y\notin G$, $\exists A\subset X$ such that $G\subset A$, $y\notin A$, and A is open or closed [2].*
4. *$T_{3/4}-$space if every $\delta g-$closed set is $\delta-$closed [6].*
5. *$T_{\alpha\hat{g}}$ space if every $\alpha\hat{g}-$closed set is $\alpha-$closed [1].*
6. *$ii-T_{1/2}$ space if every $iig-$closed is $ii-$closed.*
7. *$\hat{T}_{3/4}$ space if every $\delta\hat{g}-$closed set is $\delta-$closed set.*

Definition 2.3. *A subset B of (X,τ) is said to be*

1. *$\delta-$closed set if $B=\delta cl(B)$, where $\delta cl(B)=\{x\in X-intcl(G)\cap B\neq\phi\}$, $x\in G$ and $G\in\tau$ [15].*
2. *$\delta-$closed generalized set (in short $\delta g-$closed) if $\delta cl(B)\subset G$ at any time $B\subset G$ in addition $G\in\tau$ [15]*

The diagram that follows is an expansion of a previously well-known diagram. Not that any of the consequences are irreversible.



Theorem 2.4. *Let A be subset of s-regular space (X, τ)*

1. *A is δ-closed if and only if A is g-closed.*
2. *If in addition $T_{1/2}$ then A is $\delta g-$closed if and only if A is closed.*

The previous observations lead to the problem of finding the space (X, τ) in which the $g-$closed sets of (X, τ_s) are $\delta g-$closed in (X, τ). while we have not been able to completely resolve this problem, we offer partial solution, for that reason we will call the spaces with $T_{1/2}$ s-regularization almost weakly Hausdorff. Recall that a space is called weakly Hausdorff if its s-regularization is T_1. This point exclude topology on any infinite set gives an example of an almost weakly Hausdorff which is not weakly Hausdorff.

Theorem 2.5. *In an almost weakly Hausdorff space (X, τ) the g-closed sets of (X, τ) are $\delta-$closed in (X, τ) and thus g-closed in (X, τ).*

3. $ii\delta g-$Closed Sets

Definition 3.1. *The subset B of (X, τ) is known as*

1. *A $ii-$closed generalized set (in short $iig-$closed) if $iicl(B) \subset G$ at any time $B \subset G$ and G is $ii-$open set.*
2. *A $ii-$generalized semi-closed set (in short $iigs-$closed) if $iiscl(B) \subset G$ at any time $B \subset G$ and G is $ii-$open set.*
3. *A $ii\alpha-$generalized closed set (in short $ii\alpha g-$closed) if X at any time $B \subset G$ and G is $ii\alpha-$open set.*
4. *A $ii-$open set closed set if $iicl(B) \subset G$ at any time $B \subset G$ and G is A $\hat{g}-$open set in (X, τ).*

Definition 3.2. *A subset B of (X, τ) is said to be*

1. *$ii\alpha-$closed set if $B = ii\delta cl(B)$, where $ii\delta cl(B) = \{x \in X - int(iicl(G)) \cap B \neq \phi\}$, $x \in G$ and $G \in \tau_{ii}$.*
2. *$ii\alpha-$generalized closed set (in short $ii\delta g-$closed) if $ii\delta cl(B) \subset G$ at any time $B \subset G$ and $G \in \tau_{ii}$.*

Theorem 3.3. *let (X, τ) be a Topology space then [13]*

1. *Each $ii\delta-$closed set is $ii\delta g-$closed set.*
2. *Each $ii\delta g-$closed set is $iig-$closed set.*
3. *Each δ generalized closed set is $g-$closed set and hence$\alpha g-$closed.*

The converse of the above theorem is not true, as shown in the following situation.

Example 3.4. *Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1, 2\}\}$ is a topological space defined by X. Let $B = \{1, 3\}$. Then B is $ii\delta g-$closed set since X is the sole $ii-$open superset of B. However, it is clear that B is not $ii\delta-$closed set.*

Definition 3.5. *Let B be a subset of a topology space (X, τ), then B is said to be $ii\delta\hat{g}-$closed set if $ii\delta cl(B) \subset G$ at any time $B \subset G$ and B is a $ii\hat{g}-$open subset in (X, τ).*

Theorem 3.6. *Each $ii\delta-$closed set is $ii\delta\hat{g}-$closed.*
Proof . *Let B is a $ii\delta-$closed set and let G be any $ii\hat{g}-$open set containing B, since B is $ii\delta-$closed, this mean that $B = ii\delta cl(B)$ for each subset B of X, consequently $ii\delta cl(B) \subset G$ and thus B is $ii\delta\hat{g}-$closed set.* □

The converse claim in the previous theorem is not always true, we give an example of $ii\delta\hat{g}-$closed set which is not $ii\delta-$closed set.

Example 3.7. *Let $X = \{1,\ 2,\ 3\}$ and $\tau = \{\phi,\ X,\ \{1\},\ \{2\},\ \{1, 2\},\ \{1, 3\}\}$, then*

$$\begin{aligned}
&\tau = \{\phi,\ X,\ \{1\},\ \{1,2\},\ \{1,3\}\}\\
&i-open = \{\phi,\ X,\ \{1\},\ \{2\}, \{3\}, \{2,1\},\ \{3,1\}\}\\
&int-open = \{\phi,\ X,\ \{1\},\ \{2\}, \{2,3\}, \{1,3\},\ \{1,2\}\\
&ii-open\ set = \{\phi,\ X,\ \{1\},\ \{2\},\ \{1,2\},\ \{1,3\}\}\\
&ii-closed\ set = \{\phi,\ X,\ \{3\},\ \{2\},\ \{2,3\},\ \{1,3\}\}\\
&ii\delta-closed\ set = \{\phi,\ X,\ \{2\}, \{1,3\}\}\\
&ii\delta\hat{g}-closed\ set = \{\phi,\ X,\ \{3\},\ \{2\},\ \{2,3\},\ \{1,3\}\}
\end{aligned}$$

Here $\{3\}$ is $ii\delta\hat{g}-$closed set, but not $ii\delta-$closed set in (X, τ).

Theorem 3.8. *Each $ii\delta\hat{g}-$closed set is $iigs-$closed set.*
Proof .*Suppose that B is $ii\delta\hat{g}-$closed set and let G be any $ii-$open set comprising B in (X, τ) because every $ii-$open set is $ii\hat{g}-$open set, $ii\delta cl(B) \subset G$ for every subset B of X. Since $iicl(B) \subset ii\delta cl(B) \subset G$, $iiscl(B) \subset G$. And thus B is $iigs-$closed.* □

As shown in the following example, the converse of the above Proposition is not true.

Example 3.9. *Suppose that $X = \{1, 2, 3\}$ and let $\tau = \{\phi, X, \{1\}, \{1, 3\}\}$, be a topology space of X then*

$$\begin{aligned}
&\tau_c = \{\phi, X, \{3\}, \{1,2\}\}\\
&ii-open\ set = \{\phi, X, \{1\}, \{1,3\}, \{1,2\}\}
\end{aligned}$$

Then the set $\{3\}$ is iigs-closed set but it is not $ii\delta\hat{g}-$closed set in (X, τ).

Theorem 3.10. *Each $ii\delta\hat{g}-$closed set is $iig-$closed set.*
Proof .*Suppose that B is $ii\delta\hat{g}-$closed set and let G be any $ii-$open set comprising B in (X,τ) because every $ii-$open set is $ii\hat{g}-$open set, B is $ii\delta\hat{g}-$closed set.*
$ii\delta cl(B) \subset G$ for every subset B of X. Since $iicl(B) \subset ii\delta cl(B) \subset G$, $iicl(B) \subset G$. And thus B is $iig-$closed set. □

As proven in the following case, the converse of the above Proposition is not true.

Example 3.11. *Suppose that $X = \{1,2,3\}$ and let $\tau = \{\phi, X, \{2\}, \{1,3\}\}$ is a topology space of X, then*

$$\tau_c = \{\phi, X, \{2\}, \{1,3\}\}$$
$$ii - open\ set = \{\phi, X, \{2\}, \{1,3\}\}$$

After that, there's the set $\{1\}$ is iig-closed set. However this is not the case $ii\delta\hat{g}-$closed set In a topological space (X,τ).

Theorem 3.12. *Each $ii\delta\hat{g}-$closed set is $ii\alpha g-$closed set.*
Proof . *That is correct $ii\alpha cl(B) \subset ii\delta cl(B)$ each and every subset B of X.* □

Example 3.13. *Let $X = \{1,2,3\}$ and let $\tau = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$ be a topology space of $\{1\}$ then, after that, there's the set $\{2\}$ is $ii\alpha g-$closed.But it does not represent $ii\delta\hat{g}-$closed set.*

Theorem 3.14. *Each $ii\delta\hat{g}-$closed set is $ii\delta g-$closed set.*
Proof .*Suppose that B is $ii\delta\hat{g}-$closed set and let G be any $ii-$open set comprising B in (X,τ). Because every $ii-$open set is $ii\hat{g}-$open set, $ii\delta cl(B) \subset G$ at any time $B \subset G, G$ is $ii\hat{g}-$open. As a result $ii\delta cl(B) \subset G, G$ is $ii-$open set. Consequently B $ii\delta g-$closed set.* □

As proven in the following case, the converse of the above Proposition is not true.

Example 3.15. *Let $X = \{1,2,3\}$ and let $\tau = \{\phi, X, \{3\}, \{1,2\}\}$ be a topology space of $\{X\}$ then, after that, there's the set $\{1\}$ is $ii\alpha g-$closed.But it does not represent $ii\delta\hat{g}-$closed set in (X,τ).*

4. Characterisation

Theorem 4.1. *The union is finite of $ii\delta\hat{g}-$closed set, is $ii\delta\hat{g}-$closed set.*
Proof .*Suppose that be a limited group of $ii\delta\hat{g}-$closed subset of a topology space (X,τ). After that, for each $ii\delta\hat{g}-$open set G_j in $ii\hat{T}_{3/4}$ X comprising A_j ,$ii\delta cl(A_j) \subset G_j$, $j = \{1,2,\ldots,m\}$. Thus, $\cup_{j=1}^{m} A_j \subset \cup_{j=1}^{m} G_j = Z$. Because of the arbitrary union of $ii\hat{g}-$open sets of topological space (X,τ) is additionally $ii\hat{g}-$open set in (X,τ), Z is $ii\hat{g}-$open set in (X,τ). Moreover $\cup_{j=1}^{m} ii\delta cl(A_j) = ii\delta cl\left(\cup_{j=1}^{m} A_j\right) \subset Z$. Therefore $\cup_{j=1}^{m} A_j$ is $ii\delta\hat{g}-$closed set in (X,τ).* □

Theorem 4.2. *If B be a $ii\delta\hat{g}-$closed in (X,τ). Then $ii\delta cl(B) - B$. There isn't a single non-empty set in it. $ii\hat{g}-$closed set.*
Proof .*Assume that B is $ii\delta\hat{g}-$closed set, suppose that V is a $ii\hat{g}-$closed set containing in $ii\delta cl(B) - B$. At this time V^c is $ii\hat{g}-$open of a topological space (X,τ) such that $B \subset V^c$. Also $V \subset ii\delta cl(B) - B$. Therefore $V \subset (ii\delta cl(B))^c \cap (ii\delta cl(B)) = \phi$. Thus $V = \phi$.* □

Theorem 4.3. *Let B be a $ii\hat{g}-$open set, $ii\delta\hat{g}-$closed set of a topological space (X,τ), then B is an $ii\delta-$closed subset of a topological space (X,τ).*
Proof . *Because B is $ii\hat{g}-$open set, also B is $ii\hat{g}-$closed set. This leads to $ii\delta cl(B) \subset B$. Thus B is $ii\delta-$closed.* □

Theorem 4.4. *A $ii\hat{g}-$closed and $ii\delta\hat{g}-$closed set's intersection is always $ii\delta\hat{g}-$closed.*
Proof . *Assume that B is $ii\delta\hat{g}-$closed, let V is $ii\delta-$closed. If G is $ii\hat{g}-$open set and $B \cap V \subset G$. Therefore $B \subset (G \cup V)^c$. At this time $ii\delta cl(B \cap V) \subset ii\delta cl(B) \cap V \subset G$.*
Hence $B \cap V$ is $ii\delta\hat{g}-$closed. □

5. Applications

Definition 5.1. *If every $ii\delta g-$closed set is a $ii\delta-$closed set, a space (X,τ) is called $ii-T_{3/4}$ space.*

Theorem 5.2. *In a $ii-T_{3/4}$ space, every $ii\delta\hat{g}-$closed set is a $ii\delta-$closed*
Proof . *Let X be $ii-T_{3/4}$ space. Assume that B is $ii\delta\hat{g}-$closed set in (X,τ). Every $ii\delta\hat{g}-$closed set is a $ii\delta g-$closed, as we know. Because X is $ii-T_{3/4}$ space, B is $ii\delta-$closed.* □

Theorem 5.3. *The following conditions are identical for a topology space (X,τ)*

(i) *(X,τ) is $ii-T_{3/4}$ space.*
(ii) *Every singleton $\{y\}$ has one of two states: $iig-$closed or $iig-$open.*

Proof .
(i) $\Longrightarrow$ (ii) Assume that $\{y\}$ is not a $iig-$closed. This means $X-\{y\}$ is a type of $ii\delta g-$closedclosed set of (X,τ). Since the (X,τ) is $ii-T_{3/4}$ space, this means$X-\{y\}$ is an $ii\delta g-$closed set of (X,τ), this means $\{y\}$ is $iig-$open set.
(ii) $\Longrightarrow$ (i) Assume that B be an $ii\delta g-$closed set of (X,τ), let $\{y\} \in ii\delta cl(B)$. By (2), $\{y\}$ either $iig-$closed or $ii\delta-$open.

- *Case (1) Suppose that $\{y\}$ be $iig-$closed. If we suppose that $y \in B$ we'll get $y \in ii\delta cl(B) - B$, which isn't possible according. Consequently $y \in B$.*
- *Case (2) Suppose that $\{y\}$ be $ii\delta-$open. We know that $y \in ii\delta cl(B)$, then $\{y\} \cap B \neq \phi$. This demonstrates $\{y\} \in B$.*

As a result, in both circumstances, we have $ii\delta cl(B) \subset B$. And $B \subset ii\delta cl(B)$. Therefor $B = ii\delta cl(B)$ or a similar expression B is $ii\delta-$closed. Therefore (X,τ) is $ii-T_{3/4}$ space. □

Corollary 5.4. *Every $ii-T_{3/4}$ space is $ii-T_{1/2}$ space.*
Proof . *Suppose that (X,τ) is $ii-T_{3/4}$, let B be $iig-$closed set, then B is $ii\delta g-$closed set. Therefore B is $ii\delta-$closed set. Then B is $ii-$closed set. Thus (X,τ) is $ii-T_{1/2}$, space.* □

Definition 5.5. *If every $ii\delta\hat{g}-$closed set is a $ii\delta-$closed set, a topology space (X,τ) is called $ii-\hat{T}_{3/4}$ space*

Theorem 5.6. *Each $ii-T_{3/4}$ space is $ii-\hat{T}_{3/4}$ space.*
Proof . *Assume that (X,τ) is $ii-T_{3/4}$ space and let B is $ii\delta g-$closed. Since every $ii\delta\hat{g}-$closed is $ii\delta g-$closed. Then $ii\delta g-$closed Since (X,τ) is $ii-T_{3/4}$ space, then $ii\delta-$closed set. Therefore (X,τ) is $ii-\hat{T}_{3/4}$ space.* □

6. Conclusion

The separation properties are very important for the topological space. Through this article, some new closed sets have been reached, by using the *ii*-open set. Through these aforementioned sets, new separation axioms have been reached, which are: $ii - \hat{T}_{3/4}$ space and $ii - T_{3/4}$ space. The relationship between the sets that was studied and some relationships between the axioms of the new separation and some of the axioms of the previous chapter were also clarified. In section 2 we introduced sum of definitions we have used in this study. In Section 3 we have given some basic definitions which by their properties we have been able to find new axioms of separation. In Section 4, the $ii\delta\hat{g}-$closed set was studied and the relationship between it and the previous sets in the study was found. In Section 5, the properties of these sets are used to find new separation axioms called $ii - T_{3/4}$ space and $ii - \hat{T}_{3/4}$ space We were able to prove some theorems that explain the relationship of these axioms to some and their relationship to the sets that were studied.

Open problems

It is possible to use some properties of open sets and to arrive at other closed sets and find the relationship, between them and thus through their properties it is possible to reach new types of axioms of separation.

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