Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2153-2159
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.21753.2293

More on the Hermite-Hadamard inequality

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this article, we introduce the notion of (h, k)-convex functions and their operator form. Moreover, we derive Hermite–Hadamard-type, and Fejěr-type inequalities for this class.

Keywords: convex functions, Hermite-Hadamard inequality, operator convex functions
2010 MSC: 47A63, 15A60, 47B05, 47B10

1. Introduction and Preliminaries

Throughout this paper, we denote by I the closed interval $[a, b]$. A real-valued function f is called convex on I if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. If opposite the inequality holds, then f is said to be concave on I. The following inequality holds for any convex function $f : I \to \mathbb{R}$,

$$f\big(\frac{a+b}{2}\big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

The inequalities (1.1) was firstly discovered by Hermite in 1881 in the journal Mathsis. But this result was nowhere appeared for years. Beckenbach wrote that this inequality was proven by Hadamard in 1893 [2]. In 1974, Mitronovič found Hermite's note in Mathesis [9]. So (1.1) is called Hermite-Hadamard inequality. (In brief HH-inequality). Such inequality is very useful in many mathematical contexts and contributes as a tool for establishing some interesting estimations. In recent years many authors have been interested in giving some refinement and extensions of the Hermite-Hadamard inequality. In order to archive our results we need the following definitions and preliminary. Let H

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Received: November 2020 *Accepted:* May 2021

be a complex Hilbert space. We represent the set of all bounded linear operators on H by $B(H)$. If $A \in B(H)$ satisfies $A^* = A$, then A is called a self-adjoint operator. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted by $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H. Then we define $f(A) = \Phi(f)$ for all $f \in C(Sp(A))$ and we call it the continuous functional calculus for a self adjoint operator A. A continuous function $f : I \to \mathbb{R}$ is called operator convex, (operator concave), if

$$f\big(\lambda A + (1-\lambda)B\big) \leq (\geq) \lambda f(A) + (1-\lambda) f(B)$$

for all self-adjoint operators A and B with spectra in I.
As examples of such functions, we consider $f(t) = t^r$. It is operator convex on $(0,\infty)$, if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$, and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The function $f(t) = \ln t$ and the entropy function $f(t) = -t \ln t$ are operator concave on $(0,\infty)$. Note that $f(t) = e^t$ is not operator convex, but it is convex in real manner. For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.
Dragomir has proved operator convex function version of HH-inequality as below [3]

$$f\big(\frac{A+B}{2}\big) \leq \int_0^1 f\big(tA + (1-t)B\big)dt \leq \frac{f(A)+f(B)}{2}$$

where f is operator convex on I, and A, B are self-adjoint operators in $B(H)$ with their spectrums contained in I. Motivated by the above results, in this paper we extend the concept of convex function to $(h-k)$ convex function and provide some Hermite- Hadamard type inequalities generalizing and improving (1.1).

Firstly we give a simple proof of inequalities (1.1).

Theorem 1.1 (Hermite-Hadamard inequality). *Let $f : I \to \mathbb{R}$ be a convex function. Then the inequality* (1.1) *holds.*

Proof . The function $g(x) = f(x) + f(a+b-x)$ is convex on $[a,b]$ and symmetric. So

$$g\big(\frac{a+b}{2}\big) \leq g(x) \leq g(a)\big(= g(b)\big)$$

by integrating we arrive at

$$f\big(\frac{a+b}{2}\big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

as required. $\square$

Now, by dividing the interval $[a,b]$ into n subintervals with length $\frac{b-a}{n}$ and by HH-inequality for the subintervals $\big[a + \frac{i}{n}(b-a), a + \frac{i+1}{n}(b-a)\big]$ and some simple calculations, we have the following refinement for HH inequality.

Theorem 1.2. *Let f be convex on $[a,b]$. Then for $1 \neq n \in \mathbb{N}$, we have*

$$f\big(\frac{a+b}{2}\big) \leq \frac{1}{n}\sum_{k=1}^{n} f\big(\frac{(2k-1)a + \big(2(n-k)+1\big)b}{2n}\big)$$

$$\begin{aligned}
&\leq \int_0^1 f\big(ta+(1-t)b\big)dt \\
&\leq \frac{1}{n}\Big(\frac{f(a)+f(b)}{2}+\sum_{k=2}^{n} f\big(\frac{(k-1)a+(n-k+1)b}{n}\big)\Big) \\
&\leq \frac{f(a)+f(b)}{2} \qquad (1.2)
\end{aligned}$$

2. $(h-k)$-convex functions

In this section, we introduce $(h-k)$-convex functions, which are a generalization of convex functions.
Let J be an open interval in $\mathbb{R}$ and $(0,1)\subseteq J$.

Definition 2.1. *Let h and $k : J \to \mathbb{R}$ be non-negative real functions with h and $k \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an $(h-k)$-convex function, if f is non-negative, and for all $(x,y)\in I\times I$ and $\lambda\in[0,1]$, we have*

$$f\big(\lambda x+(1-\lambda)y\big)\leq h(\lambda)f(x)+k(1-\lambda)f(y). \qquad (2.1)$$

It is clear that if $h(\lambda)=k(\lambda)=\lambda$, then all non-negative convex functions belong to $(h-k)$-convex ones.
Now, we are eager to present an $(h-k)$-convex version for HH-inequality.

Theorem 2.2 (Hermite- Hadamard-type inequality). *Let f be an $(h-k)$-convex function on I, then:*

$$\frac{f\big(\frac{a+b}{2}\big)}{h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)}\leq \frac{1}{b-a}\int_a^b f(x)dx\leq f(a)\int_0^1 h(t)dt+f(b)\int_0^1 k(t)dt \qquad (2.2)$$

with $h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)\neq 0$.

Proof . For the left hand side inequality, we have

$$\begin{aligned}
f\big(\frac{a+b}{2}\big) &= f\Big(\frac{1}{2}\big(ta+(1-t)b\big)+\frac{1}{2}\big((1-t)a+tb\big)\Big)\\
&\leq h\big(\frac{1}{2}\big)f\big(ta+(1-t)b\big)+k\big(\frac{1}{2}\big)f\big((1-t)a+tb\big),
\end{aligned}$$

for every $t\in[0,1]$. By integrating on $[0,1]$, we get

$$\int_0^1 f\big(\frac{a+b}{2}\big)dt\leq h\big(\frac{1}{2}\big)\int_0^1 f\big(ta+(1-t)b\big)dt+k\big(\frac{1}{2}\big)\int_0^1 f\big((1-t)a+tb\big)dt.$$

So, by $\int_0^1 f\big(ta+(1-t)b\big)dt=\int_0^1 f\big((1-t)a+tb\big)dt$, we have

$$f\big(\frac{a+b}{2}\big)\leq \Big(h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)\Big)\int_0^1 f\big(ta+(1-t)b\big)dt.$$

Now, for the right hand inequality, we have:

$$\begin{aligned}
f\big(ta+(1-t)b\big)&\leq h(t)f(a)+k(1-t)f(b),\\
f\big((1-t)a+tb\big)&\leq h(1-t)f(a)+k(t)f(b),
\end{aligned}$$

for every $t \in [0,1]$. By summing them and integrating on $[0,1]$, we arrive at

$$\int_0^1 (f\big(ta+(1-t)b\big)+f\big((1-t)a+tb\big))dt \leq f(a)\int_0^1 (h(t)+h(1-t))dt + f(b)\int_0^1 (k(t)+k(1-t))dt.$$

So,

$$\int_0^1 f\big(ta+(1-t)b\big)dt \leq f(a)\int_0^1 h(t)dt + f(b)\int_0^1 k(t)dt.$$

This completes the desired proof. □

At this point, we review some various versions of convexity which have already introduced:

i) We say that $f : I \to \mathbb{R}$ is a p-function [5] or that f belong to the class $p(I)$, if f is non-negative and for each $x, y \in I$ and $t \in [0,1]$

$$f\big(tx+(1-t)y\big) \leq f(x)+f(y)$$

these functions are $(h-k)$-convex function with $h(t) = k(t) = 1$. For these functions, HH-inequality has the form:

$$\frac{1}{2}f\big(\frac{a+b}{2}\big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(a)+f(b)$$

ii) Let $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is named s-convex (in the second sense) [8], or f belongs to k_s^2, if for each $x, y \in (0,\infty)$ and $t \in [0,1]$

$$f\big(tx+(1-t)y\big) \leq t^s f(x) + (1-t)^s f(y).$$

These functions are $(h-k)$-convex with $h(t) = k(t) = t^s$. HH-inequality has the following form for them:

$$2^{(s-1)}f\big(\frac{a+b}{2}\big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}, (s \in (0,1])$$

see [4].

iii) Let $h : J \to \mathbb{R}$ be a non-negative function and $h \not\equiv 0$ and $(0,1) \subset J$. A non-negative function $f : I \to \mathbb{R}$ is called an h-convex function, if it satisfies in following inequality:

$$f\big(tx+(1-t)y\big) \leq h(t)f(x) + h(1-t)f(y)$$

for any $x, y \in I$, and $t \in (0,1)$ [10].

Ofcourse, h-convex functions are included in $(h-k)$-convex functions, with $h(t) = k(t)$. And HH-inequality is as below for them: $\Big(h\big(\frac{1}{2}\big) \neq 0\Big)$

$$\frac{f\big(\frac{a+b}{2}\big)}{2h\big(\frac{1}{2}\big)} \leq \int_0^1 f\big(ta+(1-t)b\big)dt \leq \Big(\int_0^1 h(t)dt\Big)\big(f(a)+f(b)\big) \quad (a, b \in I).$$

The function $f(x) = \sqrt{x}$ is an h-convex one, with $h(t) = \sqrt{t}$, on $I \subseteq [0,\infty)$. It is worth to noting that f is not convex.

Let w be a symmetric non-negative function on I,that is. $w(x) = w(a+b-x)$ for all $x \in I$. Fejěr proved the following inequalities, that is a generalization of HH-inequality: [6]

$$f\big(\frac{a+b}{2}\big)\int_a^b w(x)dx \leq \frac{1}{b-a}\int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{2.3}$$

where f is convex on $[a,b]$.
For proving (2.3), we can follow the method in proof of HH-inequality. In the next theorem, we prove an $(h-k)$-convex function, version of (2.3).

Theorem 2.3 (Fejěr-type inequality). *Let f be an $(h-k)$-convex function on $[a,b]$ and let $w(x)$ be non-negative symmetric integrable function on $[a,b]$. Then*

$$\begin{aligned}\frac{f\big(\frac{a+b}{2}\big)}{h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)}\int_0^1 w(ta+(1-t)b)dt &\leq \int_0^1 f\big(ta+(1-t)b\big)w(ta+(1-t)b)dt\\ &\leq \Big[f(a)\int_0^1 h(t)w(ta+(1-t)b)dt\\ &\quad + f(b)\int_0^1 k(t)w(ta+(1-t)b)dt\Big].\end{aligned} \tag{2.4}$$

Proof . We have $w(ta+(1-t)b) = w((1-t)a+tb)$ and,

$$\begin{aligned}f(ta+(1-t)b)w(ta+(1-t)b) &\leq (h(t)f(a)+k(1-t)f(b))w(ta+(1-t)b)\\ f((1-t)a+tb)w(1-t)a+tb) &\leq h(1-t)f(a)+k(t)f(b))w((1-t)a+tb)\end{aligned}$$

So,

$$\begin{aligned}&\int_0^1 [f\big(ta+(1-t)b+f((1-t)a+tb)\big)]w(ta+(1-t)b)dt\\ &\quad \leq f(a)\int_0^1 (h(t)+h(1-t))w(ta+(1-t)b)dt + f(b)\int_0^1 (k(t)+k(1-t))w(ta+(1-t)b)dt.\end{aligned}$$

Therefore, the right hand side inequality proved. Now, for the left hand side inequality we can write:

$$f(\frac{a+b}{2})w(ta+(1-t)b) = f(\frac{1}{2}(ta+(1-t)b)+\frac{1}{2}((1-t)a+tb))w(ta+(1-t)b)$$

then,

$$\begin{aligned}f\big(\frac{a+b}{2}\big)\int_0^1 w(ta+(1-t)b)dt \leq h(\frac{1}{2})&\int_0^1 f\big(ta+(1-t)b\big)w(ta+(1-t)b)dt\\ &+ k(\frac{1}{2})\int_0^1 f\big((1-t)a+tb\big)w((1-t)a+tb)dt.\end{aligned} \tag{2.5}$$

So we get the desired result. □

3. Operator Convex Functions, Main Theorems

In this section, we state and prove the operator version of the inequalities (2.2).

Definition 3.1. *We say that the continuous real function f on I, is operator $(h-k)$-convex on I, if*

$$f\big(\lambda A+(1-\lambda)B\big)\le h(\lambda)f(A)+k(1-\lambda)f(B).$$

For any self-adjoint operators A, B in $B(H)$, with spectra in I, and all $\lambda\in[0,1]$.

Remark 3.2. *If f is an operator $(h-k)$-convex function, then for any unit $x\in H$, we define the real-valued function*

$$\varphi_{x,A,B}(t)=\langle f\big(tA+(1-t)B\big)x,x\rangle,$$

where A and B are self-adjoint operators with spectra in I. Since f is an operator $(h-k)$- convex, then for $\alpha,\beta\in[0,1]$, we have

$$\begin{aligned}\varphi_{x,A,B}\big(t\alpha+(1-t)B\big)&=\Big\langle f\Big[\big(t\alpha+(1-t)B\big)A+\Big(1-\big(t\alpha+(1-t)B\big)\Big)B\Big]x,x\Big\rangle\\&=\langle f\big(t[\alpha A+(1-\alpha)B]+(1-t)[BA+(1-B)B]x,x\rangle\\&\le\langle[h(t)f\big(\alpha A+(1-B)B\big)+k(1-t)f\big(BA+(1-B)B\big)]x,x\rangle\\&=h(t)\langle f\big(\alpha A+(1-\alpha)B\big)x,x\rangle+k(1-t)\langle f\big(BA+(1-B)B\big)x,x\rangle\\&=h(t)\varphi_{x,A,B}(\alpha)+k(1-t)\varphi_{x,A,B}(B).\end{aligned}$$

Showing that $\varphi_{x,A,B}$ is an $(h-k)$-convex function on $[0,1]$.

Now we are ready to prove the operator version of the inequalities (2.2).

Theorem 3.3. *Let f be an operator $(h-k)$-convex function. Then*

$$\frac{1}{h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)}f\big(\frac{A+B}{2}\big)\le\int_0^1 f\big(tA+(1-t)B\big)dt\le f(A)\cdot\int_0^1 h(t)dt+f(B)\int_0^1 k(t)dt$$

for self-adjoint operators $A,B\in B(H)$, with spectra contained in I.

Proof . By Remark 3.2 the function

$$\varphi_{x,A,B}(t)=\langle f\big(tA+(1-t)B\big)x,x\rangle$$

is $(h-k)$-convex on $[0,1]$. By Theorem 2.2.

$$\frac{\varphi_{x,A,B}\big(\frac{1}{2}\big)}{h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)}\le\int_0^1\varphi_{x,A,B}(t)dt\le\varphi_{x,A,B}(0)\int_0^1 h(t)dt+\varphi_{x,A,B}(1)\int_0^1 k(t)dt.$$

So

$$\begin{aligned}\frac{1}{h\big(\frac{1}{2}\big)+k\big(\frac{1}{2}\big)}\langle f\big(\frac{A+B}{2}\big)x,x\rangle&\le\int_0^1\langle f\big(tA+(1-t)B\big)x,x\rangle dt\\&\le\Big(\langle f(A)x,x\rangle\int_0^1 h(t)dt+\langle f(B)x,x\rangle\int_0^1 k(t)dt\Big)\end{aligned}$$

for any unit x in H, and self-adjoint operators A and B with spectra in I.
But f is continuous. Hence

$$\int_0^1\langle f\big(tA+(1-t)B\big)x,x\rangle dt=\Big\langle\Big(\int_0^1 f\big(tA+(1-t)B\big)dt\Big)x,x\Big\rangle$$

and the proof is complete. □

Theorem 3.4. *Let f and g be two $(h-k)$-convex operators on I. Then for any self-adjoint operators A and B with spectra in I, we have the inequality*

$$\int_0^1 \langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle dt$$
$$\leq \Big(\int_0^1 \big(h(t)\big)^2 dt\Big) M_A(A,B) + \Big(\int_0^1 \big(k(t)\big)^2 dt\Big) M_B(A,B) + \Big(\int_0^1 h(t)k(1-t)dt\Big) N(A,B)$$

' where

$$\begin{aligned} M_A(A,B) &= \langle f(A)x,x\rangle\langle g(A)x,x\rangle, \\ M_B(A,B) &= \langle f(B)x,x\rangle\langle g(B)x,x\rangle, \\ N(A,B) &= \langle f(A)x,x\rangle\langle g(B)x,x\rangle + \langle g(A)x,x\rangle\langle f(B)x,x\rangle. \end{aligned}$$

Proof .

$$\begin{aligned} &\langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle \\ &\quad\leq [h(t)\langle f(A)x,x\rangle + k(1-t)\langle f(B)x,x\rangle][h(t)\langle g(A)x,x\rangle + k(1-t)\langle g(B)x,x\rangle] \\ &\quad= h^2(t)\langle f(A)x,x\rangle\langle g(A)x,x\rangle + k^2(1-t)\langle f(B)x,x\rangle\langle g(B)x,x\rangle \\ &\qquad+ h(t)k(1-t)\langle f(B)x,x\rangle\langle g(A)x,x\rangle + h(t)k(1-t)\langle f(A)x,x\rangle\langle g(A)x,x\rangle \end{aligned}$$

by integrating, the proof is complete. ☐ Now, we apply Theorem 3.4 for h-convex case

$$\int_0^1 \langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle dt \leq \Big(\int_0^1 \big(h(t)\big)^2 dt\Big) M(A,B) + \Big(\int_0^1 h(t)h(1-t)dt\Big) N(A,B)$$

where $M(A,B) = M_A(A,B) + M_B(A,B)$ [1] and Theorem 3.4 for convex case

$$\int_0^1 \langle f\big(tA+(1-t)B\big)x,x\rangle\langle g\big(tA+(1-t)B\big)x,x\rangle dt \leq \frac{1}{3}M(A,B) + \frac{1}{6}N(A,B).$$

Acknowledgements

The authors wish to express their heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.

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