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Superstability of the p-radical functional equations related to Wilson equation and Kim's equation

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Abstract

In this paper, we solve and investigate the superstability of the p-radical functional equations related to the following Wilson and Kim functional equations

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y),$$
$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)f(y),$$

where p is an odd positive integer and f is a complex valued function. Furthermore, the results are extended to Banach algebras.

Keywords: stability, superstability, radical functional equation, cosine functional equation, Wilson functional equation, Kim functional equation.
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1. Introduction

In 1940, the stability problem of the functional equation was conjectured by Ulam [24]. In 1941, Hyers [14] obtained a partial answer for the case of additive mapping in this problem.

Thereafter, the stability of the functional equation was improved by Bourgin [9] in 1949, Aoki [3] in 1950, Th. M. Rassias [23] in 1978 and Găvruta [13] in 1994.

In 1979, Baker*et al.*[7] announced the *superstability* as the new concept as follows: If f satisfies $|f(x+y)-f(x)f(y)|\le\epsilon$ for some fixed $\epsilon>0$, then either f is bounded or f satisfies the exponential functional equation $f(x+y)=f(x)f(y)$.

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D'Alembert [1] in 1769 (see Kannappen's book [16]) introduced the cosine functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{C}$$

and which superstability was proved by Baker [6] in 1980.

Baker's result was generalized by Badora [4] in 1998 to a noncommutative group under the Kannappen condition [15]: $f(x+y+z)=f(x+z+y)$, and it again was improved by Badora and Ger [5] in 2002 under the condition $|f(x+y)+f(x-y)-2f(x)f(y)|\le\varphi(x)$ or $\varphi(y)$.

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{W}$$

$$f(x+y)+f(x-y)=2g(x)f(y), \tag{K}$$

in which (W) is called the Wilson equation, and (K) arised by Kim was appeared in Kannappan and Kim's paper ([17]).

The superstability of the cosine (C), Wilson (W) and Kim (K) function equations were founded in Badora, Ger, Kannappan and Kim ([8, 17, 18, 21]).

In 2009, Eshaghi Gordji and Parviz [12] introduced the radical funtional equation related to the quadratic functional equation

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y). \tag{R}$$

In [20], Kim introduced the trigonometric functional equation as the Pexider-type's as following:

$$f(x+y)+f(x-y)=\lambda f(x)f(y), \tag{1.1}$$

$$f(x+y)+f(x-y)=\lambda f(x)g(y), \tag{1.2}$$

$$f(x+y)+f(x-y)=\lambda g(x)f(y), \tag{1.3}$$

$$f(x+y)\pm f(x-y)=\lambda g(x)h(y),$$

$$f(x+y)\pm g(x-y)=\lambda h(x)k(y).$$

Recently, Almahalebi*et al.*[2] obtained the superstability in Hyer's sense for the p-radical functional equations related to Wilson equation and Kim's equation.

The aim of this paper is to solve and investigate the superstability in Gavurta's sense for the p-radical functional equations related to Wilson and Kim's equations as following:

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)f(y), \tag{C_r}$$

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)f(y), \tag{C_r^λ}$$

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)g(y), \tag{W_r}$$

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y), \tag{W_r^λ}$$

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2g(x)f(y). \tag{K_r}$$

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)f(y). \tag{K_r^λ}$$

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+=[0,\infty)$ and $\mathbb{C}$ be the field of complex numbers. We may assume that f is a nonzero function, ε is a nonnegative real number, $\varphi:\mathbb{R}\to\mathbb{R}_+$ is a given nonnegative function and p is an odd nonnegative integer.

2. Superstability of the p-radical Wilson equation(W_r^λ) and Kim's eqaution(K_r^λ).

In this section, we find a soulution and investigate the superstability of p-radical functional equations related to the Wilson tpe equation(W_r^λ) and the Kim type equation (K_r^λ).

In the following lemmas, we obtain a solution of the functional equations (C_r^λ), (W_r^λ) and (K_r^λ), which check can be easy.

Lemma 2.1. *A function $f : \mathbb{R} \to \mathbb{C}$ satisfies (C_r^λ) if and only if $f(x) = F(x^p)$ for all $x \in \mathbb{R}$, where F is a solution of* (1.1)*. In particular, for the case $\lambda = 2$, a function $f : \mathbb{R} \to \mathbb{C}$ satisfies (C_r) if and only if $f(x) = \cos(x^p)$ for all $x \in \mathbb{R}$, namely, F is a solution of* (C)

Lemma 2.2. *A function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies (W_r^λ) if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of* (1.2)*. In particular, for the case $\lambda = 2$, a function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies (W_r) if and only if $f(x) = F(x^p) = \sin(x^p)$ and $g(x) = G(x^p) = \cos(x^p)$, where F and G are solutions of equation (W).*

Lemma 2.3. *A function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies the functional equation (K_r^λ) if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of* (1.3)*. In particular, for the case $\lambda = 2$, a function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies (K_r) if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of (K).*

Now we investigate the superstability of the Wilson equation (W_r^λ) and the Kim's equation (K_r^λ).

Theorem 2.4. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x) f(y)\right| \le \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \ \ and \ \ \varphi(x). \end{cases} \tag{2.1}$$

Then
(i) either f is bounded or g satisfies (C_r^λ),
(ii) either g(or f) is bounded or g satisfies (C_r^λ), and f and g satisfy (K_r^λ) and (W_r^λ).

Proof . (i) Assume that f is unbounded. Then we can choose $\{y_n\}$ such that $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$.

Putting $y = y_n$ in (2.1) and dividing both sides by $\lambda f(y_n)$, we have

$$\left|\frac{f\left(\sqrt[p]{x^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda f(y_n)} - g(x)\right| \le \frac{\varphi(x)}{\lambda f(y_n)}. \tag{2.2}$$

As $n \to \infty$ in (2.2), we get

$$g(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda f(y_n)} \tag{2.3}$$

for all $x \in \mathbb{R}$.

Replacing y by $\sqrt[p]{y^p+y_n^p}$ and $\sqrt[p]{y^p-y_n^p}$ in (2.1), we obtain

$$|(f\left(\sqrt[p]{x^p+(y^p+y_n^p)}\right)+f\left(\sqrt[p]{x^p-(y^p+y_n^p)}\right)-\lambda g(x)f(\sqrt[p]{y^p+y_n^p})|\le\varphi(x), \tag{2.4}$$

$$|(f\left(\sqrt[p]{x^p+(y^p-y_n^p)}\right)+f\left(\sqrt[p]{x^p-(y^p-y_n^p)}\right)-\lambda g(x)f(\sqrt[p]{y^p-y_n^p})|\le\varphi(x), \tag{2.5}$$

for all $x,y,y_n\in\mathbb{R}$. By (2.4) and (2.5), we obtain

$$\begin{aligned}|f&\left(\sqrt[p]{x^p+(y^p+y_n^p)}\right)+f\left(\sqrt[p]{x^p+(y^p-y_n^p)}\right)+f\left(\sqrt[p]{x^p-(y^p-y_n^p)}\right)\\&+f\left(\sqrt[p]{x^p-(y^p+y_n^p)}\right)-\lambda g(x)[f(\sqrt[p]{y^p+y_n^p})+f(\sqrt[p]{y^p-y_n^p})]|\le 2\varphi(x)\end{aligned}$$

for all $x,y,y_n\in\mathbb{R}$.

This implies that

$$\begin{aligned}|&\frac{f\left(\sqrt[p]{(x^p+y^p)+y_n^p}\right)+f\left(\sqrt[p]{(x^p+y^p)-y_n^p}\right)}{\lambda f(y_n)}\\&+\frac{f\left(\sqrt[p]{(x^p-y^p)+y_n^p}\right)+f\left(\sqrt[p]{(x^p-y^p)-y_n^p}\right)}{\lambda f(y_n)}\\&-\lambda g(x)\frac{f(\sqrt[p]{y^p+y_n^p})+f(\sqrt[p]{y^p-y_n^p})}{\lambda f(y_n)}|\le\frac{2\varphi(x)}{\lambda f(y_n)}\end{aligned} \tag{2.6}$$

for all $x,y,y_n\in\mathbb{R}$.

Letting $n\to\infty$ in (2.6), we obtain the desired result (C_r^λ) by applying (2.3).

For the proof of the case (ii), first we show that f (or g) is unbounded if and only if g (or f) is also unbounded. Putting $y=0$ in (2.1) (ii), we obtain

$$|f(x)-\frac{\lambda}{2}g(x)f(0)|\le\frac{\varphi(0)}{2} \tag{2.7}$$

for all $x\in\mathbb{R}$. If g is bounded, then by (2.7), we have

$$|f(x)|=|f(x)-\frac{\lambda}{2}g(x)f(0)+\frac{\lambda}{2}g(x)f(0)|\le\frac{\varphi(0)}{2}+|\frac{\lambda}{2}g(x)f(0)|,$$

which shows that f is also bounded.

On the other hand, if f is bounded, then we choose $y_0\in\mathbb{R}$ such that $f(y_0)\neq 0$, and then by (2.1) we can obtain

$$\begin{aligned}|g(x)|-&\left|\frac{f\left(\sqrt[p]{x^p+y_0^p}\right)+f\left(\sqrt[p]{x^p-y_0^p}\right)}{\lambda f(y_0)}\right|\\&\le\left|\frac{f\left(\sqrt[p]{x^p+y_0^p}\right)+f\left(\sqrt[p]{x^p-y_0^p}\right)}{\lambda f(y_0)}-g(x)\right|\le\frac{\varphi(y_0)}{\lambda|f(y_0)|}\end{aligned} \tag{2.8}$$

and it follows that g is also bounded on $\mathbb{R}$.

That is, if f (or g) is unbounded, then so is g (or f).

Let g be unbounded. Then f is also unbounded. So we can choose sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}$ such that $g(x_n) \neq 0$ and $|g(x_n)| \to \infty$, $f(y_n) \neq 0$ and $|f(y_n)| \to \infty$ as $n \to \infty$.

For the case $\varphi(y)$ in (2.1) (ii), taking $x = x_n$, we deduce

$$\lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right) + f\left(\sqrt[p]{x_n^p - y^p}\right)}{\lambda g(x_n)} = f(y) \tag{2.9}$$

for all $y \in \mathbb{R}$. Using (2.1) we have

$$\begin{aligned} &|f\left(\sqrt[p]{(x_n^p + x^p) + y^p}\right) + f\left(\sqrt[p]{(x_n^p + x^p) - y^p}\right) - \lambda g(\sqrt[p]{x_n^p + x^p}) f(y) \\ &+ f\left(\sqrt[p]{(x_n^p - x^p) + y^p}\right) + f\left(\sqrt[p]{(x_n^p - x^p) - y^p}\right) - \lambda g(\sqrt[p]{x_n^p - x^p}) f(y)| \leq 2\varphi(y) \end{aligned} \tag{2.10}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Consequently,

$$\begin{aligned} &|\frac{f\left(\sqrt[p]{x_n^p + (x^p + y^p)}\right) + f\left(\sqrt[p]{x_n^p - (x^p + y^p)}\right)}{\lambda g(x_n)} \\ &+ \frac{f\left(\sqrt[p]{x_n^p + (x^p - y^p)}\right) + f\left(\sqrt[p]{x_n^p - (x^p - y^p)}\right)}{\lambda g(x_n)} \\ &- \frac{\lambda g(\sqrt[p]{x_n^p + x^p}) + g(\sqrt[p]{x_n^p - x^p})}{\lambda g(x_n)} f(y)| \leq \frac{2\varphi(y)}{\lambda g(x_n)}, \end{aligned} \tag{2.11}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Take the limit as $n \longrightarrow \infty$ with the use of $|g(x_n)| \to \infty$ in (2.11). Since g satisfies (C_r^λ) by (i), we get that f and g are solutions of (K_r^λ),

Next, replace (x, y) by $(\sqrt[p]{x_n^p + y^p}, x)$ and replace (x, y) by $(\sqrt[p]{x_n^p - y^p}, x)$ for the case $\varphi(y)$ in (2.1) (ii), respectively. Let us follows the same procedure as from (2.10) to (2.11). Then

$$\begin{aligned} &|f\left(\sqrt[p]{(x_n^p + y^p) + x^p}\right) + f\left(\sqrt[p]{(x_n^p + y^p) - x^p}\right) - \lambda g(\sqrt[p]{x_n^p + y^p}) f(x) \\ &+ f\left(\sqrt[p]{(x_n^p - y^p) + x^p}\right) + f\left(\sqrt[p]{(x_n^p - y^p) - x^p}\right) - \lambda g(\sqrt[p]{x_n^p - y^p}) f(x)| \leq 2\varphi(y). \end{aligned}$$

Hence we have

$$\begin{aligned} &|\frac{f\left(\sqrt[p]{x_n^p + (x^p + y^p)}\right) + f\left(\sqrt[p]{x_n^p - (x^p + y^p)}\right)}{\lambda g(x_n)} \\ &+ \frac{f\left(\sqrt[p]{x_n^p + (x^p - y^p)}\right) + f\left(\sqrt[p]{x_n^p - (x^p - y^p)}\right)}{\lambda g(x_n)} \\ &- \frac{\lambda g(\sqrt[p]{x_n^p + y^p}) + g(\sqrt[p]{x_n^p - y^p})}{\lambda g(x_n)} f(x)| \leq \frac{2\varphi(y)}{\lambda g(x_n)}, \end{aligned} \tag{2.12}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Then, by appling (2.9) and (i)'s result, it follows from (2.12) that f and g are solutions of (W_r^λ). □

By a similar process of the proof of Theorem 2.1, we can prove the following theorem.

Theorem 2.5. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)g(y)| \le \begin{cases} (i) \ \ \varphi(y) \\ (ii) \ \ \varphi(x) \ \ and \ \ \varphi(y). \end{cases} \tag{2.13}$$

Then

(i) either f is bounded or g satisfies (C_r^λ),

(ii) either g(or f) is bounded or g satisfies (C_r^λ), and f and g satisfy (K_r^λ) and (W_r^λ).

Proof . The proof follows from that of Theorem 2.4. Let us choose $\{x_n\}$ in $\mathbb{R}$ such that $0 \ne |f(x_n))| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ (with $n \in \mathbb{N}$) in (2.13), dividing both sides by $|\lambda \cdot f(x_n)|$, and passing to the limit as $n \to \infty$, we obtain that

$$g(y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right) + f\left(\sqrt[p]{x_n^p - y^p}\right)}{\lambda f(x_n)} \tag{2.14}$$

for all $y \in \mathbb{R}$.

(i) Replace (x, y) by $(\sqrt[p]{x_n^p + y^p}, x)$ and replace (x, y) by $(\sqrt[p]{x_n^p - y^p}, x)$ in (2.13). Thereafter we go through the same procedure as in (2.4) $\sim$ (2.6) of Theorem 2.4. Then we obtain

$$\begin{aligned} &|\frac{f\left(\sqrt[p]{(x_n^p + y^p) + x^p}\right) + f\left(\sqrt[p]{(x_n^p + y^p) - x^p}\right)}{\lambda f(x_n)} \\ &+ \frac{f\left(\sqrt[p]{(x_n^p - y^p) + x^p}\right) + f\left(\sqrt[p]{(x_n^p - y^p) - x^p}\right)}{\lambda f(x_n)} \\ &- \lambda \frac{f(\sqrt[p]{x_n^p + y^p}) + f(\sqrt[p]{x_n^p - y^p})}{\lambda f(x_n)} g(x)| \le \frac{2\varphi(x)}{\lambda f(x_n)}. \end{aligned} \tag{2.15}$$

Since the right-hand side of the inequality converges to zero as $n \to \infty$ in (2.15), g satisfies (C_r^λ).

(ii) As (2.8), we can see by some calculation that if f is bounded, then g is also bounded.

Assume that g is unbounded. Then f is unbounded and hence g satisfies (C_r^λ).

Let us choose $\{y_n\}$ in $\mathbb{R}$ such that $0 \ne |g(y_n))| \to \infty$ as $n \to \infty$.

As before, for the chosen sequence $\{y_n\}$, we obtain that

$$f(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda g(y_n)} \tag{2.16}$$

for all $x \in \mathbb{R}$

First, replace (x, y) by $(x, \sqrt[p]{y^p + y_n^p})$ and replace (x, y) by $(x, \sqrt[p]{y^p - y_n^p})$ for the case $\varphi(x)$ in (2.13). Thereafter we go through the same procedure as in (2.4) $\sim$ (2.6) of Theorem 2.4. Then we obtain

$$\begin{aligned} &|\frac{f\left(\sqrt[p]{x^p + y^p + y_n^p}\right) + f\left(\sqrt[p]{x^p + y^p - y_n^p}\right)}{\lambda g(y_n)} \\ &+ \frac{f\left(\sqrt[p]{x^p - y^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y^p - y_n^p}\right)}{\lambda g(y_n)} \\ &- \lambda f(x) \frac{g(\sqrt[p]{y^p + y_n^p}) + g(\sqrt[p]{y^p - y_n^p})}{\lambda g(y_n)}| \le \frac{2\varphi(x)}{\lambda g(y_n)}. \end{aligned} \tag{2.17}$$

Since the right-hand side of the inequality converges to zero as $n \to \infty$ in (2.17), by appling (i)'s result and (2.16), (2.17) implies that f and g satisfy (W_r^λ).

Finally, for (K_r^λ), we also apply the same procedures as above.

Replace (x, y) by $(y, \sqrt[p]{x^p + y_n^p})$ and replace (x, y) by $(y, \sqrt[p]{x^p - y_n^p})$ for the case $\varphi(y)$ in (2.13). As above, let us go through the same procedure as in (2.16) $\sim$ (2.17), then we obtain

$$\begin{aligned} &|\frac{f\left(\sqrt[p]{x^p + y^p + y_n^p}\right) + f\left(\sqrt[p]{x^p + y^p - y_n^p}\right)}{\lambda g(y_n)} \\ &+ \frac{f\left(\sqrt[p]{x^p - y^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y^p - y_n^p}\right)}{\lambda g(y_n)} \\ &- \lambda \frac{g(\sqrt[p]{x^p + y_n^p}) + g(\sqrt[p]{x^p - y_n^p})}{\lambda g(y_n)} f(y)| \le \frac{2\varphi(y)}{\lambda g(y_n)}. \end{aligned} \tag{2.18}$$

Taking the limit as $n \to \infty$ in (2.18), applying (i)'s result and (2.16) and (2.18), we obtain the required result that f and g satisfy (K_r^λ). $\square$

Notice that, in Theorems 2.4 and 2.5, the second term $\varphi(x)$ and $\varphi(y)$ of (ii) in (2.1) and (2.13) can be replaced by the fact that g satisfies (C_r^λ), respectively.

The following corollaries follow immediate from Theorems 2.4 and 2.5.

Corollary 2.6. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x) f(y)| \le \varepsilon.$$

Then

(i) either f is bounded or g satisfies (C_r^λ),

(ii) either g(or f) is bounded or g satisfies (C_r^λ), also f and g satisfy (K_r^λ) and (W_r^λ).

Corollary 2.7. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x) g(y)| \le \varepsilon.$$

Then

(i) either f is bounded or g satisfies (C_r^λ),

(ii) either g(or f) is bounded or g satisfies (C_r^λ), also f and g satisfy (K_r^λ) and (W_r^λ).

Corollary 2.8. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x) f(y)| \le \begin{cases} (i)\ \varphi(x), \\ (ii)\ \varphi(y), \\ (iii)\ \varepsilon. \end{cases}$$

Then either f is bounded or f satisfies (C_r^λ),

Remark 2.9. *In results, letting $p = 1$ or $\lambda = 2$, one can obtain* (C), (W), (K), (1.1), (1.2), (1.3) (C_r), (W_r), (K_r). *Hence they can be applied to stability results of cosine, Wilson, Kim, trigonometric functional eqautions, etc. See Badora [4], Badora and Ger [5], Baker [6], Fassi, et al.[11], Kannappan and Kim [17], Kim [18, 19, 20, 21, 22], and Almahalebi, et al.[2]. Letting $p = 2, 3, 4$ and $\lambda = 1, 2$, we can obtain the other functional equations. If the obtained results can be extend to them, then it will be applied similarly to stability results.*

3. Applications of the case $\widetilde{f}(x) := f(x)f(0)^{-1}$ in (W_r^λ) and (K_r^λ)

Let $\widetilde{f}(x) := f(x)f(0)^{-1}$. The following lemmas show that similar arguments hold without assuming the continuity. To make it easy to write, we continue using this notation $\widetilde{f}$ and note that it is legal only when $f(0) \neq 0$.

The following lemmas can be easy to check.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function satisfying*

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda f(x)f(y)$$

for all $x, y \in \mathbb{R}$. If f is an even function such that $f(0) \neq 0$, then $\widetilde{f}$ satisfies (C_r).

Lemma 3.2. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda f(x)g(y)$$

for all $x, y \in \mathbb{R}$. If f is an even function such that $f(0) \neq 0$, then $\widetilde{f}$ satisfies (C_r).

Lemma 3.3. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = \lambda g(x)f(y)$$

for all $x, y \in \mathbb{R}$. Then, for $f(0) \neq 0$, $\widetilde{f}$ satisfies (C_r).

Theorem 3.4. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)f(y)\right| \leq \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \ \text{ and } \ \varphi(x). \end{cases} \tag{3.1}$$

(i) If f is unbounded, then $\widetilde{g}$ satisfies (C_r).
(ii) If g(or f) is unbounded, then $\widetilde{f}$ and $\widetilde{g}$ satisfy (C_r).

Proof . (i) It follows trivially from Theorem 2.4 (i) and Lamma 3.3.

(ii) Assume that g(or f) is unbounded. Then f is unbounded. By (i), $\widetilde{g}$ satisfies (C_r). From Theorem 2.4 (ii), f and g satisfy (K_r^λ) and (W_r^λ). By Lemma 3.3, $\widetilde{f}$ satisfies (C_r). □

Theorem 3.5. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)g(y)\right| \leq \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \ \text{ and } \ \varphi(y). \end{cases}$$

(i) If f is unbounded, then $\widetilde{g}$ satisfies (C_r).
(ii) If g(or f) is unbounded, then $\widetilde{f}$ and $\widetilde{g}$satisfy (C_r).

Proof . (i) It follows trivially from Theorem 2.5 (ii) and Lemma 3.2.

(ii) Assume that g(or f) is unbounded. Then f is unbounded. By (i), $\widetilde{g}$ satisfies (C_r). From Theorem 2.5 (ii), f and g satisfy (K_r^λ) and (W_r^λ). By Lemma 3.3, $\widetilde{f}$ satisfies (C_r). □

Corollary 3.6. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)|\leq\varepsilon,$$

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)|\leq\varepsilon.$$

(i) If f is unbounded, then $\widetilde{g}$ satisfies (C_r).

(ii) If g(or f) is unbounded, then $\widetilde{f}$ and $\widetilde{g}$satisfy (C_r).

Corollary 3.7. *Assume that $f : \mathbb{R} \to \mathbb{C}$ satisfies the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)f(y)|\leq\begin{cases}(i)\ \ \varphi(x)\\(ii)\ \ \varphi(y)\\(iii)\ \ \varepsilon\end{cases}$$

If f is unbounded, then $\widetilde{f}$ satisfies (C_r).

Remark 3.8. *As Remark 2.9, letting $p = 1$ or $\lambda = 2$, we obtain some results, which are the results given in Fassi[11].*

4. Extension to Banach algebras

In this section, we will extend our main results to Banach algebras.

Theorem 4.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)\|\leq\begin{cases}(i)\ \ \varphi(x)\\(ii)\ \ \varphi(y)\ \ and\ \ \varphi(x).\end{cases}\tag{4.1}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies (C_r^λ).

(ii) If $z^* \circ g$ (or $z^* \circ f$) is unbounded, then g satisfies (C_r^λ), and f and g satisfy (K_r^λ) and (W_r^λ).

Proof . Assume that (4.1) holds and let $z^* \in E^*$ be a linear multiplicative functional. Since $\|z^*\| = 1$, for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\varphi(x)&\geq\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)\|\\&=\sup_{\|w^*\|=1}\left|w^*\big(f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)\big)\right|\\&\geq\left|z^*\big(f\left(\sqrt[p]{x^p+y^p}\right)\big)+z^*\big(f\left(\sqrt[p]{x^p-y^p}\right)\big)-\lambda\cdot z^*\big(g(x)\big)\cdot z^*\big(f(y)\big)\right|,\end{aligned}$$

which states that the superpositions $z^* \circ f$ and $z^* \circ g$ yield solutions of the inequality (2.1) in Theorem 2.4.

Hence we can apply to Theorem 2.4 (i).

(i) Since, by assumption, the superposition $z^* \circ f$ is unbounded, an appeal to Theorem 2.4 shows that the superposition $z^* \circ g$ is a solution of (C_r^λ), that is,

$$(z^*\circ g)\big(\sqrt[p]{x^p+y^p}\big)+(z^*\circ g)\big(\sqrt[p]{x^p-y^p}\big)=\lambda(z^*\circ g)(x)(z^*\circ g)(y).$$

Since z^* is a linear multiplicative functional, we get

$$z^*\big(g\big(\sqrt[p]{x^p+y^p}\big)+g\big(\sqrt[p]{x^p-y^p}\big)-\lambda g(x)g(y)\big)=0.$$

Hence an unrestricted choice of z^* implies that

$$g\big(\sqrt[p]{x^p+y^p}\big)+g\big(\sqrt[p]{x^p-y^p}\big)-\lambda g(x)g(y)\in\bigcap\{\ker z^*:z^*\in E^*\}.$$

Since E is a semisimple Banach algebra, $\bigcap\{\ker z^*:z^*\in E^*\}=0$, which means that g satisfies the claimed equation (C_r^λ).

(ii) By assumption, the superposition $z^*\circ g$ is unbounded, an appeal to Theorem 2.4 shows that the results hold.

From a similar process as in (2.8) of Theorem 2.4, we can show that the unboundedness of the superposition $z^*\circ g$ implies the unboundedness of the superposition $z^*\circ f$.

First, it follows from the above result (i) that g satisfies the claimed equation (C_r^λ).

Next, an appeal to Theorem 2.4 shows that $z^*\circ f$ and $z^*\circ g$ are solutions of the equations (K_r^λ) and (W_r^λ), that is,

$$(z^*\circ f)\big(\sqrt[p]{x^p+y^p}\big)+(z^*\circ f)\big(\sqrt[p]{x^p-y^p}\big)=\lambda(z^*\circ g)(x)(z^*\circ f)(y),$$
$$(z^*\circ f)\big(\sqrt[p]{x^p+y^p}\big)+(z^*\circ f)\big(\sqrt[p]{x^p-y^p}\big)=\lambda(z^*\circ f)(x)(z^*\circ g)(y).$$

This means by a linear multiplicativity of z^* that the differences

$$\mathcal{D}K^\lambda(x,y):=f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y),$$
$$\mathcal{D}W^\lambda(x,y):=f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)$$

fall into the kernel of z^*. That is, $z^*\left(\mathcal{D}K^\lambda(z,w)\right)=0$ and $z^*\left(\mathcal{D}W^\lambda(z,w)\right)=0$.

Hence an unrestricted choice of z^* implies that

$$\mathcal{D}K^\lambda(x,y),\ \mathcal{D}W^\lambda(x,y)\in\bigcap\{\ker z^*:z^*\in E^*\}.$$

Since the algebra E is semisimple, $\bigcap\{\ker z^*:z^*\in E^*\}=0$, which means that f and g satisfy the claimed equations (K_r^λ) and (W_r^λ). □

By a similar procedure, we can prove the next theorem as an extension of Theorem 2.5. So we will skip the proof.

Theorem 4.2. *Let $(E,\|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f,g:\mathbb{R}\to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)\|\le\begin{cases}(i)\ \ \varphi(y)\\(ii)\ \ \varphi(x)\ \ and\ \ \varphi(y).\end{cases}\tag{4.2}$$

Let $z^*\in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^*\circ f$ is unbounded, then g satisfies (C_r^λ).

(ii) If $z^*\circ g$ (or $z^*\circ f$) is unbounded, then g satisfies (C_r^λ), and f and g satisfy (K_r^λ) and (W_r^λ).

Corollary 4.3. *Let $(E,\|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f,g:\mathbb{R}\to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)\|\le\varepsilon.$$

Let $z^*\in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^*\circ f$ is unbounded, then g satisfies (C_r^λ).

(ii) If $z^*\circ g$ (or $z^*\circ f$) is unbounded, then g satisfies (C_r^λ), and f and g satisfy (K_r^λ) and (W_r^λ).

Corollary 4.4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)g(y)\| \leq \varepsilon.$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies (C_r^λ).

(ii) If $z^* \circ g$ (or $z^* \circ f$) is unbounded, then g satisfies (C_r^λ), and f and g satisfy (K_r^λ) and (W_r^λ).

Corollary 4.5. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)f(y)\| \leq \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \\ (iii) & \varepsilon. \end{cases}$$

Then either the superposition $z^* \circ f$ is bounded for each linear multiplicative functional $z^* \in E^*$ or f satisfies (C_r^λ).

Remark 4.6. *(1) Letting $p = 1$ or $\lambda = 2$, we can get* (C), (W), (K), (C_r), (1.1), (1.2), (1.3). *Hence they can be applied to stability results of cosine, Wilson, Kim, trigonometric functional equations. See [2, 4, 5, 17, 18, 19, 20, 21].*

(2) The results of Section 3 also can be extended to Banach algebras. By applying $p = 1$ or $\lambda = 2$, some results can be derived.

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