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Turàn type inequalities for a class of Polynomials with constraints

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Abstract

Let $p(z)=\sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n. Let $p'(z)$ and $D_\alpha p(z)=np(z)+(\alpha-z)p'(z)$ be the ordinary and polar derivative of $p(z)$ respectively. In this paper some sharp lower bound estimates for the maximal modulus of $p'(z)$ and $D_\alpha p(z)$ are established in terms of their degrees, coefficients and maximal modulus of $p(z)$ over unit disk under the assumption that all the zeros of $p(z)$ lie in $|z|\le k$, $k\ge 1$.

Keywords: Lower bound, Maximum modulus, Polar derivative, Polynomial.
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1. Introduction

Let $\mathscr{P}$ be the linear space of all polynomials $p(z)=\sum_{v=0}^{n} a_v z^v$ over $\mathbb{C}$ of degree n and $p'(z)$ be the derivative of $p(z)$. Concerning the maximum of $|p'(z)|$ in terms of maximum of $|p(z)|$ on $|z|=1$. Turàn [15] showed that if $p\in\mathscr{P}$ has all its zeros in $|z|\le 1$, then

$$\max_{|z|=1}|p'(z)|\ge \frac{n}{2}\max_{|z|=1}|p(z)| \tag{1.1}$$

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Equality holds in (1.1) for the polynomials $p \in \mathscr{P}$ which have all their zeros in $|z| = 1$. As a generalization of (1.1), Govil [8] proved that if $p \in \mathscr{P}$ has all its zeros in $|z| \leq k$, $k \geq 1$ then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)| \tag{1.2}$$

There exist several extensions and generalizations of (1.1) and (1.2) in the literature (see [2], [4], [13], [3], [1]). In an intriguing refinement of (1.1), Dubinin [6] used the boundary Schwarz lemma due to Osserman [11] and proved that if all the zeros of $p \in \mathscr{P}$ lie in $|z| \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right) \max_{|z|=1} |p(z)| \tag{1.3}$$

For $p \in \mathscr{P}$, the polar derivative $D_\alpha p(z)$ of $p(z)$ with respect to $\alpha \in \mathbb{C}$ is defined as

$$\begin{aligned} D_\alpha p(z) &:= -\left[\frac{p(z)}{(z-\alpha)^n}\right]' (z-\alpha)^{n+1} \\ &= np(z) + (\alpha - z)p'(z) \end{aligned}$$

Note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$. It generalizes the ordinary derivative of $p(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z) \tag{1.4}$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Aziz [3], Aziz and Rather ([2], [4]) established several sharp estimates for maximum modulus of $D_\alpha p(z)$ on unit circle and among other things they extended inequality (1.2) to the polar derivative of a polynomial by proving that if $p \in \mathscr{P}$ has all its zeros in $|z| \leq k$, $k \geq 1$ then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n(|\alpha| - k)}{1+k^n} \max_{|z|=1} |p(z)| \tag{1.5}$$

Very recently Nisar and Ishfaq [14] markedly proved the refinements and generalizations of inequalities (1.1), (1.2), (1.3), and (1.5) by using the boundary Schwarz lemma due to Osserman [11]. In fact, they proved the following results.

Theorem A: If $p \in \mathscr{P}$ and $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$ then

$$\max_{|z|=1} |p'(z)| \geq \frac{1}{1+k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=1} |p(z)|$$

The result is sharp and equality holds for $p(z) = z^n + k^n$.

Theorem B: If all the zeros of $p \in \mathscr{P}$ lie in $|z| \leq k$, $k \geq 1$, then for $0 \leq l < 1$,

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n}\left(\max_{|z|=1} |p(z)| + lm\right) + \frac{1}{k^n(1+k^n)}\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\{k^n \max_{|z|=1} |p(z)| - lm\}$$

where $m = \min\limits_{|z|=k} |p(z)|$. The result is sharp and equality holds for $p(z) = z^n + k^n$.

Theorem C: If $p \in \mathscr{P}$ has all its zeros in $|z| \leq k$, $k \geq 1$ then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{|\alpha| - k}{1 + k^n} \left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|} \right) \max_{|z|=1} |p(z)|$$

In view of (1.4), the result is sharp in limiting case when $|\alpha| \to \infty$ as shown by polynomial $p(z) = z^n + k^n$.

Theorem D: If all the zeros of $p \in \mathscr{P}$ lie in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, $0 \leq l < 1$,

$$\begin{aligned} \max_{|z|=1} |D_\alpha p(z)| \geq & \frac{n}{1 + k^n} \{ (|\alpha| - k) \max_{|z|=1} |p(z)| + (|\alpha| + 1/k^{n-1}) lm \} \\ & + \frac{(|\alpha| - k)}{k^n(1 + k^n)} \left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|} \right) \{ k^n \max_{|z|=1} |p(z)| - lm \} \end{aligned}$$

where $m = \min\limits_{|z|=k} |p(z)|$. In view of (1.4), the result is sharp in limiting case when $|\alpha| \to \infty$ as shown by polynomial $p(z) = z^n + k^n$.

2. Main Results

In above results the bound depends on the zero with largest modulus and on the coefficients a_n, a_0. In this paper we will refine all the above results such that the bound depends on the zero of largest modulus and on the coefficients a_n, a_0, a_1, and a_2. In fact, we prove the following results.

Theorem 1: If $p \in \mathscr{P}$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\begin{aligned} \max_{|z|=1} |p'(z)| \geq & \frac{1}{1 + k^n} \left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|} \right) \max_{|z|=1} |p(z)| + \frac{2}{k^{n-1}} \left(\frac{k^{n-1} - 1}{n + 1} \right) |a_1| \\ & + \frac{2}{k^{n-1}} \left(\frac{k^{n-1} - 1}{n - 1} - \frac{k^{n-3} - 1}{n - 3} \right) |a_2|, \qquad \textit{if} \quad n > 3 \end{aligned} \tag{2.1}$$

and

$$\max_{|z|=1} |p'(z)| \geq \frac{1}{1 + k^3} \left(3 + \frac{k^3|a_3| - |a_0|}{k^3|a_3| + |a_0|} \right) \max_{|z|=1} |p(z)| + \frac{(k-1)}{2k^2} [(k+1)|a_1| + 2(k-1)|a_2|], \qquad \textit{if} \quad n = 3 \tag{2.2}$$

The result is best possible and equality in (2.1) and (2.2) holds for $p(z) = z^n + k^n$.

Theorem 2: If $p \in \mathscr{P}$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \geq 1$, then for $\gamma \in \mathbb{C}$ with $0 \leq |\gamma| < 1$

$$\begin{aligned} \max_{|z|=1} |p'(z)| \geq & \frac{n}{1 + k^n} \left(\max_{|z|=1} |p(z)| + |\gamma| m \right) + \frac{1}{k^n(1 + k^n)} \left(\frac{k^n|a_n| - |\gamma|m - |a_0|}{k^n|a_n| - |\gamma|m + |a_0|} \right) \{ k^n \max_{|z|=1} |p(z)| - |\gamma| m \} \\ & + \frac{2}{k^{n-1}} \left(\frac{k^{n-1} - 1}{n + 1} \right) |a_1| + \frac{2}{k^{n-1}} \left(\frac{k^{n-1} - 1}{n - 1} - \frac{k^{n-3} - 1}{n - 3} \right) |a_2|, \quad \textit{if} \quad n > 3 \end{aligned} \tag{2.3}$$

and

$$\max_{|z|=1}|p'(z)| \geq \frac{3}{1+k^3}\left(\max_{|z|=1}|p(z)|+|\gamma|m\right)+\frac{1}{k^3(1+k^3)}\left(\frac{k^3|a_3|-|\gamma|m-|a_0|}{k^3|a_3|-|\gamma|m+|a_0|}\right)\{k^3\max_{|z|=1}|p(z)|-|\gamma|m\}$$
$$+\frac{(k-1)}{2k^2}[(k+1)|a_1|+2(k-1)|a_2|], \quad if \quad n=3 \tag{2.4}$$

where $m=\min\limits_{|z|=k}|p(z)|$. The result is best possible and equality in (2.3) and (2.4) holds for $p(z)=z^n+k^n$.

Theorem 3: If $p\in\mathscr{P}$ is a polynomial of degree $n\geq 3$ having all its zeros in $|z|\leq k$, $k\geq 1$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{1+k^n}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|p(z)|+\frac{2|\alpha a_1+na_0|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)\\ +\frac{|2\alpha a_2+(n-1)a_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right), \quad if \quad n>3\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{1+k^3}\left(3+\frac{k^3|a_3|-|a_0|}{k^3|a_3|+|a_0|}\right)\max_{|z|=1}|p(z)|\\ +\frac{(k-1)}{2k^2}[(k+1)|\alpha a_1+3a_0|+(k-1)|2\alpha a_2+2a_1|], \quad if \quad n=3\end{aligned} \tag{2.6}$$

The result is sharp. In view of (1.4), equality in (2.5) and (2.6) holds in limiting case when $|\alpha|\to\infty$ as shown by polynomial $p(z)=z^n+k^n$.

Theorem 4: Let $p\in\mathscr{P}$ be a polynomial of degree $n\geq 3$ having all its zeros in $|z|\leq k$, $k\geq 1$, then for every $\alpha,\gamma\in\mathbb{C}$, with $|\alpha|\geq k$ and $0\leq|\gamma|<1$

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n}{1+k^n}\{(|\alpha|-k)\max_{|z|=1}|p(z)|+(|\alpha|+1/k^{n-1})|\gamma|m\}\\ +\frac{(|\alpha|-k)}{k^n(1+k^n)}\left(\frac{k^n|a_n|-|\gamma|m-|a_0|}{k^n|a_n|-|\gamma|m+|a_0|}\right)\{k^n\max_{|z|=1}|p(z)|-|\gamma|m\}\\ +\frac{2|\alpha a_1+na_0|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)+\frac{|2\alpha a_2+(n-1)a_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right), \quad if \quad n>3\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{3}{1+k^3}\{(|\alpha|-k)\max_{|z|=1}|p(z)|+(|\alpha|+1/k^2)|\gamma|m\}\\ +\frac{(|\alpha|-k)}{k^3(1+k^3)}\left(\frac{k^3|a_3|-|\gamma|m-|a_0|}{k^3|a_3|-|\gamma|m+|a_0|}\right)\{k^3\max_{|z|=1}|p(z)|-|\gamma|m\}\\ +\frac{(k-1)}{2k^2}[(k+1)|\alpha a_1+3a_0|+(k-1)|2\alpha a_2+2a_1|], \quad if \quad n=3\end{aligned} \tag{2.8}$$

where $m = \min_{|z|=k} |p(z)|$. The result is sharp and in view of (1.4), equality in (2.7) and (2.8) holds in limiting case when $|\alpha| \to \infty$ as shown by polynomial $p(z) = z^n + k^n$.

Remark 1: The bound obtained by Theorem 1 is sharper than the bound obtained from Theorem A. In order to show the sharpness of the bounds, here we produce the following example.

Example 1: Consider $p(z) = (z+1)^2(z+2)(z+3)$. Here we take $k = 3$ then we find that all the zeros of $p(z)$ lie in $|z| \leq 3$. For this polynomial, the bound for $\max_{|z|=1} |p'(z)|$ by Theorem A comes out to be 2.85, and by inequality (2.1) of Theorem 1 it comes out to be 17.80, which is a significant improvement over the bound obtained from Theorem A.

Now consider the case for $n = 3$. Let $p(z) = (z+1)(z+2)(z+3)$, and take $k = 3$ then we can see that all the zeros of $p(z)$ lie in $|z| \leq 3$. For this polynomial the bound for $\max_{|z|=1} |p'(z)|$ by Theorem A comes out to be 3.11, and by inequality (2.2) of Theorem 1 it comes out to be 10.66, which is also considerable improvement over the bound obtained from Theorem A.

Remark 2: Theorem 2 is generalization of Theorem 1. For $\gamma = 0$ Theorem 2 reduces to Theorem 1.

Remark 3: Theorem 3 is an extension of Theorem 1 to the polar derivative and in this direction Theorem 3 presents a refinement of inequality (1.5). In view of (1.4), it generalizes Theorem 1, when we divide Theorem 3 both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, we get Theorem 1.

Remark 4: Theorem 4 is an extension of Theorem 2 to the polar derivative and in view of (1.4) it generalizes Theorem 2 as when we divide Theorem 4 both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, we get Theorem 2.

3. Lemmas

For the proof of our theorems, we require the following lemmas.

Lemma 1: If $p \in \mathscr{P}$, then for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|$$

Lemma 1 is a simple deduction from Maximum Modulus Principle (MMP) (see [12], [10]).

Lemma 2: If $p \in \mathscr{P}$, then

$$\max_{|z|=R>1} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - \frac{2(R^n-1)}{n+2}|a_0| - |a_1| \left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right], \quad if \quad n > 2 \tag{3.1}$$

and

$$\max_{|z|=R>1} |p(z)| \leq R^2 \max_{|z|=1} |p(z)| - \frac{(R-1)}{2}[(R+1)|a_0| + (R-1)|a_1|], \quad if \quad n = 2 \tag{3.2}$$

The above lemma is due to Dewan et.al [7]

Lemma 3: If $p \in \mathscr{P}$ has all its zeros in $|z| \leq k$, $k \geq 1$ then for $|z| = 1$

$$\max_{|z|=k} |p(z)| \geq \frac{2k^n}{1+k^n} \max_{|z|=1} |p(z)|$$

The above lemma is due to Aziz [3].

The next lemma can be deduced from Lemma 3.

Lemma 4: If $p \in \mathscr{P}$ has all its zeros in $|z| \leq k$, $k \geq 1$ then for $\lambda \in \mathbb{C}$ with $0 \leq \lambda < 1$

$$\max_{|z|=k} |p(z)| \geq \frac{2k^n}{1+k^n} \max_{|z|=1} |p(z)| + \left(\frac{k^n-1}{k^n+1}\right) |\lambda| m$$

where $m = \min\limits_{|z|=k} |p(z)|$.

The next lemma is due to R. Osserman [11] known as boundary Schwarz lemma.

Lemma 5: If a function $T(z)$ is such that

(i) $T(z)$ is analytic in $|z| < 1$.

(ii) $|T(z)| < 1$ for $|z| < 1$.

(iii) $T(0) = 0$.

(iv) For some b with $|b| = 1$ extends continuous to b, $|T(b)| = 1$ and $T'(b)$ exists.

Then

$$|T'(b)| \geq \frac{2}{1 + |T'(0)|}$$

Lemma 6: If $p \in \mathscr{P}$ has all its zeros in $|z| \leq 1$, then for $|z| = 1$,

$$|q'(z)| \leq |p'(z)|$$

Where $q(z) = z^n \overline{p(1/\overline{z})}$.

The above lemma is special case of a result due to Aziz and Rather [5, 2].

Lemma 7: *If all the zeros of $p \in \mathscr{P}$ lie in a circular region C and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C.*

The above lemma is the famous Laguerre Theorem [9].

Lemma 8: If $p \in \mathscr{P}$ has all its zeros in $|z| \leq 1$, then for some $k \geq 1$

$$\max_{|z|=k} |p'(z)| \geq \frac{1}{2k} \left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=k} |p(z)|$$

The above lemma can be simply deduced from inequality (1.3).

4. Proof of the Theorems

Proof of Theorem 1. Let $n > 3$. Consider $v(z) = p(kz)$, then it can be seen that $v \in \mathscr{P}$ and $v(z)$ has all its zeros in $|z| \leq 1$. By applying inequality (1.3) to $v(z)$, we have

$$\max_{|z|=1} |v'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right) \max_{|z|=1} |v(z)|$$

Replacing $v(z)$ by $p(kz)$, we get for $|z| = 1$,

$$k \max_{|z|=1} |p'(kz)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=1} |p(kz)|$$

or equivalently

$$\max_{|z|=k} |p'(z)| \geq \frac{1}{2k}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=k} |p(z)|$$

Since $p(z)$ is a polynomial of degree $n > 3$, it follows that polynomial $p'(z)$ is of degree $n \geq 3$, hence applying inequality (3.1) of Lemma 2 to $p'(z)$ with $R = k \geq 1$, we have

$$\begin{aligned} k^{n-1} \max_{|z|=1} |p'(z)| - 2\left(\frac{k^{n-1} - 1}{n+1}\right)|a_1| - 2\left(\frac{k^{n-1} - 1}{n-1} - \frac{k^{n-3} - 1}{n-3}\right)|a_2| &\geq \max_{|z|=k} |p'(z)| \\ &\geq \frac{1}{2k}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=k} |p(z)| \end{aligned}$$

which implies on using Lemma 3,

$$\begin{aligned} \max_{|z|=1} |p'(z)| \geq \frac{1}{1+k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=1} |p(z)| + \frac{2}{k^{n-1}}\left(\frac{k^{n-1} - 1}{n+1}\right)|a_1| \\ + \frac{2}{k^{n-1}}\left(\frac{k^{n-1} - 1}{n-1} - \frac{k^{n-3} - 1}{n-3}\right)|a_2| \end{aligned}$$

This completes the proof of Theorem 1 for $n > 3$.
Now instead of using inequality (3.1) of Lemma 2 we use inequality (3.2) of the same Lemma the proof of Theorem 1 for $n = 3$ follows on the same lines as that of $n > 3$.

Proof of Theorem 2. Since $p \in \mathscr{P}$ has all its zeros in $|z| \leq k$, $k \geq 1$. If $p(z)$ has a zero on $|z| = k$ then $m = 0$ and the result follows from Theorem 1. Assume that $p(z)$ becomes zero in $|z| < k$, $k \geq 1$ so that $m > 0$.
Now if $f(z) = p(kz)$, then $f(z)$ has all its zeros in $|z| < 1$ and $m \leq |f(z)|$ for $|z| = 1$ which implies for every $\gamma \in \mathbb{C}$ with $|\gamma| < 1$

$$|m\gamma z^n| < |f(z)| \qquad \text{for} \qquad |z| = 1$$

It follows by Rouches theorem that all the zeros of $h(z) = f(z) - \gamma m z^n$ has all its zeros in $|z| < 1$. By applying similar arguments to $h(z)$ as in the proof of Theorem 1, we obtain

$$|h'(z)| \geq \frac{1}{2}\left(n + \frac{|k^n a_n + \gamma m| - |a_0|}{|k^n a_n + \gamma m| + |a_0|}\right)|h(z)| \qquad \text{for} \qquad |z| = 1$$

The function $s(x) = \frac{x-|a|}{x+|a|}, a \in \mathbb{R}$ is an increasing function of x and $|k^n a_n + \gamma m| \geq k^n|a_n| - |\gamma m|$, we get for $|z| = 1$

$$|h'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|h(z)| \tag{4.1}$$

This implies for $|z| = 1$ and $|\gamma| < 1$,

$$|f'(z) - nm\gamma z^{n-1}| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)(|f(z)| - m|\gamma|)$$

Choosing an argument of γ in the left hand side of above inequality such that

$$|f'(z) - nm\gamma z^{n-1}| = |f'(z)| - nm|\gamma| \qquad \text{for} \qquad |z| = 1$$

we get,

$$|f'(z)| - nm|\gamma| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)(|f(z)| - m|\gamma|)$$

i.e

$$|f'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|f(z)| + \frac{1}{2}\left(n - \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|\gamma|m$$

Replace $f(z)$ by $p(kz)$, we get

$$\max_{|z|=k}|p'(z)| \geq \frac{1}{2k}\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)\max_{|z|=k}|p(z)| + \frac{1}{2k}\left(n - \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|\gamma|m$$

For $n > 3$, the polynomial $p'(z)$ is of degree $n \geq 3$, hence applying inequality (3.1) of Lemma 2 to $p'(z)$ with $R = k \geq 1$, we have

$$\begin{aligned} k^{n-1}\max_{|z|=1}|p'(z)| \geq{}& \frac{1}{2k}\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)\max_{|z|=k}|p(z)| + \frac{1}{2k}\left(n - \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|\gamma|m \\ &+ 2\left(\frac{k^{n-1}-1}{n+1}\right)|a_1| + 2\left(\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right)|a_2| \end{aligned}$$

Using Lemma 4 and taking $\lambda = \gamma$, we obtain the following

$$\begin{aligned} \max_{|z|=1}|p'(z)| \geq{}& \frac{n}{1+k^n}\left(\max_{|z|=1}|p(z)| + |\gamma|m\right) + \frac{1}{k^n(1+k^n)}\left(\frac{k^n|a_n| - |\gamma|m - |a_0|}{k^n|a_n| - |\gamma|m + |a_0|}\right)\{k^n\max_{|z|=1}|p(z)| - |\gamma|m\} \\ &+ \frac{2}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)|a_1| + \frac{2}{k^{n-1}}\left(\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right)|a_2| \end{aligned}$$

This completes the proof of Theorem 2 for $n > 3$.
Now using inequality (3.2) of Lemma 2, the proof of Theorem 2 follows for $n = 3$.

Proof of Theorem 3. Let $n > 3$. If $h(z) = p(kz)$ then $h \in \mathscr{P}$ and $h(z)$ has all its zeros in $|z| \leq 1$. Consider the polynomial $q(z) = z^n\overline{h(1/\overline{z})}$, then

$$|q'(z)| = |nh(z) - zh'(z)| \qquad \text{for} \qquad |z| = 1$$

Combining above equation with Lemma 6, we obtain

$$|h'(z)| \geq |nh(z) - zh'(z)| \qquad \text{for} \qquad |z| = 1$$

Now for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have

$$|D_{\alpha/k}h(z)| \geq |\alpha/k||h'(z)| - |nh(z) - zh'(z)|$$

i.e

$$|D_{\alpha/k}h(z)| \geq \left(\frac{|\alpha| - k}{k}\right)|h'(z)| \tag{4.2}$$

Accordingly,

$$\max_{|z|=k}|D_\alpha p(z)| \geq (|\alpha| - k)\max_{|z|=k}|p'(z)|$$

Since $p(z)$ is a polynomial of degree $n > 3$, it follows that polynomial $D_\alpha p(z)$ is of degree $n \geq 3$, hence applying inequality (3.1) of Lemma 2 to $D_\alpha p(z)$ with $R = k \geq 1$, we have

$$\begin{aligned} k^{n-1}\max_{|z|=1}|D_\alpha p(z)| &\geq \max_{|z|=k}|D_\alpha p(z)| + 2\left(\frac{k^{n-1}-1}{n+1}\right)|D_\alpha p(0)| + \left(\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right)|D_\alpha p'(0)| \\ &\geq (|\alpha| - k)\max_{|z|=k}|p'(z)| + 2\left(\frac{k^{n-1}-1}{n+1}\right)|D_\alpha p(0)| + \left(\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right)|D_\alpha p'(0)| \end{aligned}$$

Using Lemma 8 and then Lemma 3, we get for $|z| = 1$

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha| - k}{1 + k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=1}|p(z)| + \frac{2|\alpha a_1 + na_0|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right) \\ + \frac{|2\alpha a_2 + na_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right) \end{aligned}$$

This completes the proof of Theorem 3 for $n > 3$.

Next let $n = 3$, Since $p(z)$ is a polynomial of degree $n = 3$, it follows that polynomial $D_\alpha p(z)$ is of degree $n = 2$, hence applying inequality (3.2) of Lemma 2 to $D_\alpha p(z)$ with $R = k \geq 1$, we have

$$\begin{aligned} k^2\max_{|z|=1}|D_\alpha p(z)| &\geq \max_{|z|=k}|D_\alpha p(z)| + \frac{(k-1)}{2}[(k+1)|D_\alpha p(0)| + (k-1)|D_\alpha p'(0)|] \\ &\geq (|\alpha| - k)\max_{|z|=k}|p'(z)| + \frac{(k-1)}{2}[(k+1)|D_\alpha p(0)| + (k-1)|D_\alpha p'(0)|] \end{aligned}$$

Using Lemma 8 and then Lemma 3 for $n = 3$, we get for $|z| = 1$

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha| - k}{1 + k^3}\left(3 + \frac{k^3|a_3| - |a_0|}{k^3|a_3| + |a_0|}\right)\max_{|z|=1}|p(z)| \\ + \frac{(k-1)}{2k^2}[(k+1)|\alpha a_1 + 3a_0| + (k-1)|2\alpha a_2 + 3a_1|] \end{aligned}$$

Proof of Theorem 4. By hypothesis $p \in \mathscr{P}$ and $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, therefore applying similar arguments as in the proof of Theorem 2, it follows that the polynomial $h(z) = f(z) - \gamma m z^n$ has all its zeros in $|z| < 1$ where $f(z) = p(kz), m = \min\limits_{|z|=k} |p(z)| = \min\limits_{|z|=1} |f(z)|$ and $|\gamma| < 1$. Now applying inequality (4.2) to the polynomial $h(z)$, it follows for $|z| = 1$ and $|\alpha| \geq k$,

$$|D_{\alpha/k}h(z)| \geq \left(\frac{|\alpha| - k}{k}\right) |h'(z)|$$

Using inequality (4.1), we get

$$|D_{\alpha/k}h(z)| \geq \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|h(z)|$$

which is equivalent to

$$|D_{\alpha/k}f(z) - \frac{nm\alpha\gamma}{k}z^{n-1}| \geq \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)(|f(z)| - |\gamma| m) \tag{4.3}$$

Since all the zeros of $h(z)$ lie in $|z| < 1$ and $|\frac{\alpha}{k}| \geq 1$, it follows by Lemma 7 that all the zeros of

$$D_{\alpha/k}(f(z) - \gamma m z^n) = D_{\alpha/k}f(z) - \frac{nm\alpha\gamma}{k}z^{n-1}$$

lie in $|z| < 1$. This implies that

$$|D_{\alpha/k}f(z)| \geq \frac{nm|\alpha|}{k}|z|^{n-1} \qquad \text{for} \qquad |z| \geq 1$$

Now choose an argument of γ in the left hand side of inequality (4.3), we get for $|z| = 1$

$$|D_{\alpha/k}f(z)| - \frac{nm|\alpha||\gamma|}{k} \geq \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)(|f(z)| - |\gamma| m)$$

i.e

$$\begin{aligned}|D_{\alpha/k}f(z)| \geq{}& \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|f(z)| + \frac{n}{2}\left(\frac{|\alpha| + k}{k}\right)|\gamma| m \\ &- \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(\frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|\gamma| m\end{aligned}$$

This gives

$$\begin{aligned}\max_{|z|=k}|D_\alpha p(z)| \geq{}& \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)\max_{|z|=k}|p(z)| + \frac{n}{2}\left(\frac{|\alpha| + k}{k}\right)|\gamma| m \\ &- \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(\frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|\gamma| m\end{aligned}$$

Since $p(z)$ is a polynomial of degree $n > 3$, it follows that polynomial $D_\alpha p(z)$ is of degree $n \geq 3$, hence applying inequality (3.1) of Lemma 2 to $D_\alpha p(z)$ with $R = k \geq 1$, we have

$$\begin{aligned}k^{n-1}\max_{|z|=1}|D_\alpha p(z)| \geq{}& \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)\max_{|z|=k}|p(z)| + \frac{n}{2}\left(\frac{|\alpha| + k}{k}\right)|\gamma| m \\ &- \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(\frac{k^n|a_n| - |\gamma m| - |a_0|}{k^n|a_n| - |\gamma m| + |a_0|}\right)|\gamma| m + 2\left(\frac{k^{n-1} - 1}{n + 1}\right)|D_\alpha p(0)| \\ &+ \left(\frac{k^{n-1} - 1}{n - 1} - \frac{k^{n-3} - 1}{n - 3}\right)|D_\alpha p'(0)|\end{aligned}$$

As earlier, Using Lemma 4 and choosing $\lambda=\gamma$, we have for $|z|=1$ and $|\alpha|\geq k$

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq & \frac{n}{1+k^n}\{(|\alpha|-k)\max_{|z|=1}|p(z)|+(|\alpha|+1/k^{n-1})|\gamma|m\}\\ &+\frac{(|\alpha|-k)}{k^n(1+k^n)}\left(\frac{k^n|a_n|-|\gamma|m-|a_0|}{k^n|a_n|-|\gamma|m+|a_0|}\right)\{k^n\max_{|z|=1}|p(z)|-|\gamma|m\}\\ &+\frac{2|\alpha a_1+na_0|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)+\frac{|2\alpha a_2+na_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right)\end{aligned}$$

Here in this way we complete the proof of Theorem 4 for $n>3$.
Now using inequality (3.2) of Lemma 2 instead of inequality (3.1) of the same Lemma. Furthermore using Lemma 4 for $n=3$ the proof of Theorem 4 follows for $n=3$.

5. Conclusion

Inequality (1.1) is known as Turàn's inequality which is best possible for the class of polynomials having all zeros in unit closed disk and equality in (1.1) holds for the polynomials $p\in\mathscr{P}$ which have all their zeros in $|z|=1$. In 1973 Govil established (1.2), the more general result for the class of polynomials whose zeros are distributed in closed disk of radius k, where $k\geq 1$. As an application of boundary Schwarz lemma due to Osserman Dubinin proved inequality (1.3) which presents the interesting refinement of Turàn's inequality. The authors in Theorems A and B established the refinements of inequalities (1.2) and (1.3) and extended them to the polar derivative of a polynomial which can be seen in Theorems C and D under the hypothesis that all the zeros of a polynomial are distributed in closed disk of radius k, where $k\geq 1$. In fact the Theorems C and D provides the refinements of inequality (1.5). The main results ensconced in the present paper for the class of polynomials of degree n, where $n\geq 3$ and whose all zeros lie $|z|\leq k$, $k\geq k$ provides the refinements of all the Theorems mentioned in section 1. Our results improve upon the lower bound estimates corresponding to the Theorems presented in section 1, subject to the Theorem A the bound is significantly improved in Theorem 1 as shown by example 1.

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