The impact of alternative resources and fear on the dynamics of the food chain

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Abstract

In this paper, a three-species food chain model is proposed and studied. It is assumed that there are fear costs in the first two-level due to predation risk and there exists an alternative source of food for a top predator. Holling's disc function is adopted to describe the food transition throughout the chain. All the solution properties are discussed. The conditions of local stability and persistence of the model are established. The Lyapunov function is used to specify the basin of attraction for each equilibrium. Local bifurcation analyses are studied. The global dynamics of the model are investigated numerically. Different bifurcation diagrams and attractors are obtained. It is obtained that the system is rich in dynamics that include chaos.

Keywords: Food chain; Alternative resource; Fear; Stability; Bifurcation; chaos.

1. Introduction

It is well known that the most important biological processes in ecology and population biology are the interactions among organisms and their environment, and the evolution of species. Mathematical modeling is a strong tool for studying the above biological processes, and hence various kinds of mathematical models have been suggested and studied [11]. These mathematical models are classified as deterministic models and can be written using nonlinear ordinary differential equations. Prey-predator interaction is a focal subject in ecology and evolutionary biology. It has been extensively investigated by many researchers throughout the past few decades using mathematical models, which included different biological factors. In fact, these models have played an important role in understanding the effect of various biological factors. Most of the existing prey-predator models are based upon the framework of the classical Lotka–Volterra model. The Lotka-Volterra

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models assumed that the predators can influence the prey population directly by killing them. However, the existence of a predator may significantly change the behavior in addition to the existence of prey [1, 8]. Later on, the Lotka-Volterra model was thereafter modified by including a logistic growth for the prey and a Holling type-II functional response for the predator [7]. Such a model was investigated extensively by Rosenzweig and MacArthur [15] as a more realistic representation of a prey-predator system. Other modifications with different types of functional responses have been studied extensively in the tri-trophic food chains and food webs [6, 3, 12, 18]. Different, types of complex dynamical behavior have been obtained in these models, including periodic, and chaos.

On the other hand, the preys face predation risk by using a variety of anti-predator responses such as changes in the habitat, foraging, vigilance, and different physiological [1, 14]. So, due to fear of predation risk, the prey population can change its feeding area to a safer place and sacrifice their highest intake rate areas, increase their vigilance, regulate their strategies for reproductive, etc. Consequently, the reproduction of the shocking prey decreases. Keeping the above in view, although the known fact is that predators can influence prey population through direct killing only, recent works showed that indirect effect on prey population can be seriously stronger than the direct effect, and hence manipulation of fear is strong enough to influence the population dynamics of ecological systems [21, 17, 19]. Although all the types of functional responses consider the direct killing of the prey no matter the impact of fear, many prey-predator models using different types of functional responses have been proposed and studied [5, 10, 4], and references therein. The first mathematical model that considers into account the prey population reduction due to the impact of fear of the predation risk was proposed by Wang et al [19]. They observed that high levels of fear can stabilize the prey-predator system. However, comparatively low levels of fear can induce multiple limit cycles, leading to a bi-stability phenomenon. On the other hand, they noted that for the prey-predator model with the linear functional response, the cost of fear does not change the dynamic behaviors of the model and a unique positive equilibrium is still globally asymptotically stable. After this work, many authors proposed and studied varieties of prey-predator models by using the impact of fear in prey reproduction.

Wang and Zou [20] proposed a prey-predator model with age structure in prey population and allowing adaptive avoidance of predators. They observed that both strong adaptation of adult prey and the large cost of fear have a destabilizing effect on the dynamics of the system, but when the predator population is large, it has a stabilizing effect. Panday et al [13] proposed a three-species food chain model, where the growth rate of each species decreases due to the fear of predation risk by upper-level species. They obtained that fear can stabilize a chaotic system. Sasmal [16] proposed an eco-epidemiological model with a strong Allee effect and cost of fear in prey reproduction. he showed that fear can stabilize the system at the interior equilibrium, where all the three populations coexist, or it can create the oscillatory coexistence of all the three populations. Zhang et al [22] proposed and investigated the influence of anti-predator behavior due to the fear of predators with a Holling-type-II prey-predator model incorporating a prey refuge. They concluded that the fear effect can not only reduce the population density of predators at the positive equilibrium but also stabilize the system by excluding the existence of periodic solutions. Later on, Fakhry and Naji [2] proposed and studied a prey-predator system with a square root response function and the prey's fear. They showed that the effect of fear reduces the population density of predators at the positive equilibrium and also stabilizes the system. Recently, Liu et al [9] proposed a time-delayed prey-predator model with Holling-type II functional response that incorporates the gestation period and the cost of fear into prey reproduction. They showed that high levels of fear have a stabilizing effect while relatively low levels of fear have a destabilizing effect on the predator-prey interactions which lead to limit-cycle oscillations.

The present paper, a three-species food chain model with the influence of fear of predation in the first two-level that incorporates an alternative resource of food for a top predator is proposed and studied. The Holling's disc (or Holling type II) functions are used to describe the transferring of the food throughout the food chain. The rest of the paper is organized as follows. In the next section, the formulation of the model is carried out. In the following section, the local stability of equilibrium points is studied. In section 4, the persistence conditions of the model are established. Section 5, treats the global stability analysis. However, local bifurcation analysis is discussed in section 6. Section 7 involves the numerical simulation of the model. Finally, section 8 gives the conclusion of this study.

2. Mathematical Model

In the present section, the real-world food chain system is formulated mathematically. It is assumed that the real-world system consisting of a tri-trophic food chain in which the prey at the lower level grows logistically and the top predator at the upper level has alternative resource for food in case of lacking of their preferred prey at the middle level. Let the variables $X(T)$, $Y(T)$, and $Z(T)$ represent at the time T the prey, intermediate predator, and top predator respectively. It is assumed that due to fear of predation process, the growth rates of prey at the lower level and the growth rate of intermediate predator at the middle level are reduced, so that the growth rate of the prey in the lower level and the growth rate of the intermediate predator at the middle level are monotonic decreasing functions of both c_1 with Y and c_2 with Z respectively. Moreover, the food transferees throughout the levels of the food chain according to the Holling's disc functions. Keeping the above assumptions in view, the dynamics of the food-chain system can be describe using the following set of first order differential equations.

$$
\begin{aligned}
\frac{dX}{dT} &= \left(\frac{rX}{1+c_1Y}\right)\left(1-\frac{X}{K}\right) - \frac{a_1XY}{b_1+X} = F_1(X,Y,Z), \\
\frac{dY}{dT} &= \frac{e_1a_1XY}{(1+c_2Z)(b_1+X)} - \frac{a_2AYZ}{b_2+Y} - d_1Y = F_2(X,Y,Z), \\
\frac{dZ}{dT} &= \frac{e_2a_2AYZ}{b_2+Y} + (1-A)BZ - d_2Z = F_3(X,Y,Z),
\end{aligned}
\tag{2.1}
$$

where $X(0) \geq 0$, $Y(0) \geq 0$, and $Z(0) \geq 0$. It is assumed that all the parameters in the model (2.1) are positive and can be described in the following table.

Table 1: Parameters description

Parameter	Description
r	Intrinsic growth rate in the absence of predator
K	Carrying capacity of the prey
c_1 and c_2	Fear coefficients from intermediate predator and top predator respectively
a_1 and a_2	Maximum attack rates for the prey and intermediate predator respectively
b_1 and b_2	Half saturation constants for the intermediate predator and top predator respectively
e_1 and e_2	Conversion rates for the intermediate predator and top predator respectively
d_1 and d_2	Natural death rates for the intermediate predator and top predator respectively
A	Probability of feeding on the intermediate predator
B	Growth rate from alternative resource

Clearly, the functions $g_1(c_1, Y) = \frac{1}{1+c_1Y}$ and $g_2(c_2, Z) = \frac{1}{1+c_2Z}$, which describe the fear factors in the lower and middle levels respectively satisfy that:

$$g_1(0, Y) = g_1(c_1, 0) = 1, \lim_{c_1 \to \infty} g_1(c_1, Y) = \lim_{Y \to \infty} g_1(c_1, Y) = 0 \text{ and } \frac{\partial g_1}{\partial c_1} < 0, \frac{\partial g_1}{\partial Y} < 0$$

$$g_2(0, Z) = g_2(c_2, 0) = 1, \lim_{c_2 \to \infty} g_2(c_2, Z) = \lim_{Z \to \infty} g_2(c_2, Z) = 0 \text{ and } \frac{\partial g_2}{\partial c_2} < 0, \frac{\partial g_2}{\partial Z} < 0$$

Observe that, the interaction functions $F_i; i = 1, 2, 3$ are continuous functions and have a continuous partial derivatives. Therefore, they are Lipschitz functions. Accordingly, the solution of the model (2.1) is exists and is unique.

Now, in order to simplify the analysis of the model (2.1) and generalize the results, the following dimensionless variables and parameters are used:

$$t = rT, \ x = \frac{X}{K}, \ y = \frac{Y}{K}, \ Z = \frac{Z}{K},$$

$$w_1 = c_1K, \ w_2 = \frac{a_1}{r}, \ w_3 = \frac{b_1}{K}, \ w_4 = \frac{e_1a_1}{r}, \ w_5 = c_2K, \ w_6 = \frac{a_2A}{r}, \ w_7 = \frac{b_2}{K}, \ w_8 = \frac{d_1}{r}, \ w_9 = \frac{e_2a_2A}{r},$$

$$w_{10} = \frac{(1-A)B}{r}, \ w_{11} = \frac{d_2}{r}.$$

Then the non-dimensional form of model (2.1) can be written as:

$$\frac{dx}{dt} = x \left[\frac{1-x}{1+w_1y} - \frac{w_2y}{w_3+x} \right] = xf_1(x, y, z),$$

$$\frac{dy}{dt} = y \left[\frac{w_4x}{(1+w_5z)(w_3+x)} - \frac{w_6z}{w_7+y} - w_8 \right] = yf_2(x, y, z), \qquad (2.2)$$

$$\frac{dz}{dt} = z \left[\frac{w_9y}{w_7+y} + w_{10} - w_{11} \right] = zf_3(x, y, z),$$

with $x(0) \geq 0$, $y(0) \geq 0$, and $z(0) \geq 0$. Moreover, in the following theorem the invariant region and positivity of the solution of the model (2.2) is established.

Theorem 2.1. *The model (2.2) with non-negative initial condition in R_+^3 has a positive invariant solution in the region $\Omega = \left\{ (x, y, z) \in R_+^3 : \frac{w_4}{w_2}x(t) + y(t) + \frac{w_6}{w_9}z(t) \leq \frac{w_4}{w_2\mu} \right\}$*

Proof . From the first equation of model (2.2) we have

$$\frac{dx}{dt} \geq -x \left[\frac{x}{1+w_1y} + \frac{w_2y}{w_3+x} \right]$$

By solving the above differential inequality and using given initial condition, it is obtain

$$x(t) \geq x(0)e^{-\int_0^t \left[\frac{x}{1+w_1y} + \frac{w_2y}{w_3+x} \right]dt} \geq 0$$

Using similar arguments on other equations gives

$$y(t) \geq y(0)e^{-\int_0^t \left[\frac{w_6z}{w_7+y} + w_8 \right]dt} \geq 0$$
$$z(t) \geq z(0)e^{-w_{11}t} \geq 0$$

Accordingly, all the solutions of the model (2.2) are feasible. Now to investigate the boundedness of the solutions, define $P(t) = \frac{w_4}{w_2}x(t) + y(t) + \frac{w_6}{w_9}Z(t)$, then the time derivative of $p(t)$ gives

$$\frac{dp}{dt} \leq \frac{w_4}{w_2}x(1-x) - w_8y - \frac{w_6}{w_9}(w_{11} - w_{10})z \leq \frac{w_4}{w_2} - \mu \left(\frac{w_4}{w_2}x(t) + y(t) + \frac{w_6}{w_9}z(t) \right),$$

where $\mu = min\{1, w_8, w_{11} - w_{10}\}$. Therefore, it is obtain that for $t \to \infty$
$$\lim_{t \to \infty} sup \quad p(t) \leq \frac{w_4}{w_2 \mu}. \text{ Thus, the proof is complete. } \square$$

3. Local Behavior of the Model

This section treats the study of the local behavior of the solution of the model (2.2). It is observed that model (2.2) has at most six non-negative equilibrium points. The conditions that guarantee the existence of these points are described as follows.

The vanishing equilibrium point, $s_0 = (0, 0, 0)$ always exists.

The first axial equilibrium point $s_1 = (1, 0, 0)$ always exists.

The second axial equilibrium point $s_2 = (0, 0, \hat{z})$, where $\hat{z} \geq 0$ any real number belongs to Ω, exists under the condition

$$w_{10} = w_{11} \tag{3.1}$$

The top predator-free equilibrium point, $s_3 = (\overline{x}, \overline{y}, 0)$ where

$$\overline{x} = \frac{w_8 w_3}{w_4 - w_8}, \overline{y} = -\frac{w_2}{2w_1 w_2} + \frac{\sqrt{w_2^2 + 4w_1 w_2 w_3 w_4 \frac{[w_4 - w_8(1 + w_3)]}{(w_4 - w_8)^2}}}{2w_1 w_2}, \tag{3.2}$$

exists provided that the following condition holds

$$w_8(1 + w_3) \leq w_4 \tag{3.3}$$

The intermediate predator-free equilibrium point, $s_4 = (1, 0, \hat{z})$, where $\hat{z} \geq 0$ any real number belongs to Ω, exists under the condition (3.1).

Finally, the coexistence equilibrium point, $s_5 = (\check{x}, \check{y}, \check{z})$, where

$$\check{x} = \frac{-\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2}}{2}, \quad \check{y} = \frac{w_7(w_{11} - w_{10})}{w_9 - (w_{11} - w_{10})}, \quad \check{z} = \frac{-\delta_2}{2\delta_1} + \frac{1}{2\delta_1}\sqrt{\delta_2^2 + 4\delta_1 \delta_3}, \tag{3.4}$$

where $\gamma_1 = w_3 - 1$, $\gamma_2 = w_3 - w_2 \check{y}(1 + w_1 \check{y})$, $\delta_1 = w_6 w_5(w_3 + \check{x})$, $\delta_2 = (w_3 + \check{x})[w_6 + w_5 w_8(w_7 + \check{y})]$, $\delta_3 = [w_4 \check{x} - w_8(w_3 + \check{x})](w_7 + \check{y})$.

Here the constant solutions $\check{x}$ and $\check{z}$ are the positive roots of the following two quadratic equations respectively.

$$x^2 + (w_3 - 1)x - w_3 + w_2 \check{y}(1 + w_1 \check{y}) = 0 \tag{3.5a}$$

$$w_6 w_5(w_3 + \check{x})z^2 + (w_3 + \check{x})[w_6 + w_5 w_8(w_7 + \check{y})]z$$
$$- [w_4 \check{x} - w_8(w_3 + \check{x})](w_7 + \check{y}) = 0 \tag{3.5b}$$

Obviously, the equations (3.5a) and (3.5b) have unique positive roots provided the following conditions hold.

$$w_2 \check{y}(1 + w_1 \check{y}) < w_3, \tag{3.6a}$$

$$w_8(w_3 + \check{x}) < w_4 \check{x}. \tag{3.6b}$$

While, the constant solution $\check{y}$ is positive under the following condition:

$$0 < (w_{11} - w_{10}) < w_9. \tag{3.6c}$$

Now, to study the local behavior near the above equilibrium points, the Jacobian matrix is computed at general point (x, y, z) by:

$$
J = \begin{pmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{pmatrix},
\tag{3.7}
$$

where $\dfrac{\partial f_1}{\partial x} = \dfrac{-1}{1+w_1 y} + \dfrac{w_2 y}{(w_3+x)^2}$, $\dfrac{\partial f_1}{\partial y} = \dfrac{-w_1(1-x)}{(1+w_1 y)^2} - \dfrac{w_2}{w_3+x}$, $\dfrac{\partial f_1}{\partial z} = 0$,

$\dfrac{\partial f_2}{\partial x} = \dfrac{w_3 w_4}{(w_3+x)^2(1+w_5 z)}$, $\dfrac{\partial f_2}{\partial y} = \dfrac{w_6 z}{(w_7+y)^2}$, $\dfrac{\partial f_2}{\partial z} = \dfrac{-w_6}{w_7+y} - \dfrac{w_4 w_5 x}{(w_3+x)(1+w_5 z)^2}$,

$\dfrac{\partial f_3}{\partial x} = 0$, $\dfrac{\partial f_3}{\partial y} = \dfrac{w_7 w_9}{(w_7+y)^2}$, $\dfrac{\partial f_3}{\partial z} = 0$.

Therefore, the Jacobian matrix at the vanishing equilibrium point, $s_0 = (0,0,0)$ can be written as:

$$
J(s_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -w_8 & 0 \\ 0 & 0 & w_{10} - w_{11} \end{pmatrix}.
\tag{3.8}
$$

Clearly, the eigenvalues are $\lambda_{01} = 1$, $\lambda_{02} = -w_8$ and $\lambda_{03} = w_{10} - w_{11}$. Thus, the point s_0 is a saddle point.

The Jacobian matrix for the first axial equilibrium point $s_1 = (1, 0, 0)$ can be written as

$$
J(s_1) = \begin{pmatrix} -1 & -\dfrac{w_2}{w_3+1} & 0 \\ 0 & \dfrac{w_4}{w_3+1} - w_8 & 0 \\ 0 & 0 & w_{10} - w_{11} \end{pmatrix}.
\tag{3.9}
$$

Here the eigenvalues of $J(s_1)$ are given by $\lambda_{11} = -1$, $\lambda_{12} = \dfrac{w_4}{w_3+1} - w_8$ and $\lambda_{13} = w_{10} - w_{11}$. Therefore, the point s_1 is locally asymptotically stable if and only if the following conditions hold:

$$
\frac{w_4}{w_3+1} < w_8
\tag{3.10a}
$$

$$
w_{10} < w_{11}
\tag{3.10b}
$$

The Jacobian matrix for the second axial equilibrium point is given by:

$$
J(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\dfrac{w_6}{w_7}\hat{z} - w_8 & 0 \\ 0 & \dfrac{w_7 w_9}{w_7^2}\hat{z} & 0 \end{pmatrix}.
\tag{3.11}
$$

Clearly, the eigenvalues are $\lambda_{21} = 1$, $\lambda_{22} = -\dfrac{w_6}{w_7}\hat{z} - w_8$ and $\lambda_{23} = 0$. Thus, the point s_2 is unstable non-hyperbolic point due to existence of zero eigenvalue.

The Jacobian matrix at the top predator-free equilibrium point, $s_3 = (\overline{x}, \overline{y}, 0)$ can be written as

$$
J(s_3) = \begin{pmatrix} \dfrac{-\overline{x}}{1+w_1\overline{y}} + \dfrac{w_2\overline{x}\,\overline{y}}{(w_3+\overline{x})^2} & \dfrac{-w_1\overline{x}(1-\overline{x})}{(1+w_1 y^*)^2} - \dfrac{w_2\overline{x}}{w_3+\overline{x}} & 0 \\ \dfrac{w_3 w_4\overline{y}}{(w_3+\overline{x})^2} & 0 & -\dfrac{w_6\overline{y}}{w_7+\overline{y}} - \dfrac{w_4 w_5\overline{x}\,\overline{y}}{w_3+\overline{x}} \\ 0 & 0 & \dfrac{w_9\overline{y}}{w_7+\overline{y}} + w_{10} - w_{11} \end{pmatrix}.
\tag{3.12}
$$

Therefore, the eigenvalues of $J(s_3)$ are determined as

$$\lambda_{31} = \frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}, \lambda_{32} = \frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}, \lambda_{33} = \frac{w_9\overline{y}}{w_7 + \overline{y}} + w_{10} - w_{11}, \tag{3.13}$$

where $\beta_1 = \dfrac{-\overline{x}}{1 + w_1\overline{y}} + \dfrac{w_2\overline{x}\ \overline{y}}{(w_3 + \overline{x})^2}$, $\beta_2 = \left[\dfrac{w_1\overline{x}(1 - \overline{x})}{(1 + w_1\overline{y})^2} + \dfrac{w_2\overline{x}}{w_3 + \overline{x}}\right]\left[\dfrac{w_3 w_4\overline{y}}{(w_3 + \overline{x})^2}\right]$. Straightforward computation shows that all the eigenvalues in equation (6) have negative real parts, and hence the point s_3 is locally asymptotically stable if and only if the following two conditions are satisfied.

$$\frac{w_2\overline{y}}{(w_3 + \overline{x})^2} < \frac{1}{1 + w_1\overline{y}}, \tag{3.14a}$$

$$\frac{w_9\overline{y}}{w_7\overline{y}} + w_{10} < w_{11}. \tag{3.14b}$$

Now, the Jacobian matrix at the intermediate predator-free equilibrium point, $s_4 = (1, 0, \check{z})$ is determined by:

$$J(s_4) = \begin{pmatrix} -1 & -\dfrac{w_2}{w_3 + 1} & 0 \\ 0 & \dfrac{w_4}{(1 + w_5\hat{z})(w_3 + 1)} - \dfrac{w_6\hat{z}}{w_7} - w_8 & 0 \\ 0 & \dfrac{w_9\hat{z}}{w_7} & 0 \end{pmatrix}. \tag{3.15}$$

Consequently, the eigenvalues of $J(s_4)$ are given by

$$\lambda_{41} = -1, \ \lambda_{42} = \frac{w_4}{(1 + w_5\hat{z})(w_3 + 1)} - \frac{w_6\hat{z}}{w_7} - w_8, \ \lambda_{43} = 0. \tag{3.16}$$

Therefore, the existence of the zero eigenvalues makes the point s_4 non-hyperbolic point, and hence the stability type cannot studied through the linearization method and we may study their stability using Lyapunov method.

Finally, the local stability conditions for the coexistence equilibrium points are established in the following theorem.

Theorem 3.1. *The coexistence equilibrium point of the model* (2.2) *is locally asymptotically stable if and only if the following conditions are satisfied.*

$$w_2\check{x}\check{y}\check{A}_1\check{A}_3^2 + w_6\check{y}\check{z}\check{A}_1\check{A}_2^2 \le \check{x}\check{A}_2^2\check{A}_3^2, \tag{3.17a}$$

$$\left(\frac{w_2 w_6\check{x}\check{y}^2\check{z}\check{A}_1 - w_6\check{x}\check{y}\check{z}\check{A}_2^2}{\check{A}_1\check{A}_2^2\check{A}_3^2}\right) + \left(\frac{w_1 w_3 w_4\check{x}(1 - \check{x})\check{y}\check{A}_2 + w_2 w_4\check{x}\check{y}\check{A}_1^2}{\check{A}_1^2\check{A}_2^3\check{A}_4}\right) > 0, \tag{3.17b}$$

$$\left(\frac{\check{x}A_2^2 A_3^2 - w_2\check{x}\check{y}A_1 A_3^2 - w_6\check{y}\check{z}A_1 A_2^2}{\check{A}_1\check{A}_2^2\check{A}_3^2}\right)\left[\left(\frac{w_2 w_6\check{x}\check{y}^2\check{z}A_1 - w_6\check{x}\check{y}\check{z}A_2^2}{\check{A}_1\check{A}_2^2\check{A}_3^2}\right)\right.$$
$$\left.\left(\frac{w_1 w_3 w_4\check{x}(1 - \check{x})\check{y}A_2 + w_2 w_3 w_4\check{x}\check{y}A_1^2}{\check{A}_1^2\check{A}_2^3\check{A}_4}\right)\right] > \left(\frac{w_6 w_7 w_9\check{y}\check{z}^2}{\check{A}_3^4}\right)\left(\frac{w_6\check{y}A_2 A_4^2 + w_3 w_5\check{x}\check{y}A_3}{\check{A}_2\check{A}_3\check{A}_4^2}\right), \tag{3.17c}$$

where

$$\check{A}_1 = 1 + w_1\check{y}, \ \check{A}_2 = w_3 + \check{x}, \ \check{A}_3 = w_7 + \check{y} \ and \ \check{A}_4 = 1 + w_5\check{z}.$$

Proof . Direct computation shows that the Jacobian matrix at the coexistence equilibrium point $s_5 = (\check{x}, \check{y}, \check{z})$ is determined as:

$$J(s_5) = \begin{pmatrix} \dfrac{-\check{x}}{\check{A}_1} + \dfrac{w_2 \check{x}\check{y}}{\check{A}_2^2} & -\dfrac{w_1 \check{x}(1-\check{x})}{\check{A}_2^2} - \dfrac{w_2 \check{x}}{\check{A}_2} & 0 \\ \dfrac{w_3 w_4 \check{y}}{\check{A}_2^2 \check{A}_4} & \dfrac{w_6 \check{y}\check{z}}{\check{A}_3^2} & -\dfrac{w_6 \check{y}}{\check{A}_3} - \dfrac{w_3 w_5 \check{x}\check{y}}{\check{A}_2 \check{A}_4^2} \\ 0 & \dfrac{w_7 w_9 \check{z}}{\check{A}_3^2} & 0 \end{pmatrix} = (P_{ij}) \qquad (3.18)$$

Therefore, the characteristic equation of $J(s_5)$ can be determined as:

$$\lambda_5^3 + \Theta_1 \lambda_5^2 + \Theta_2 \lambda_5 + \Theta_3 = 0, \qquad (3.19)$$

where $\Theta_1 = -(p_{11} + p_{22})$, $\Theta_2 = p_{11}p_{22} - p_{12}p_{21} - p_{23}p_{32}$, $\Theta_3 = p_{11}p_{23}p_{32}$, while the value of $\Delta = \Theta_1 \Theta_2 - \Theta_3 = -(p_{11} + p_{22})[p_{11}p_{22} - p_{12}p_{21}] + p_{22}p_{23}p_{32}$.

Now, the proof is follows if all the eigenvalues of $J(s_5)$ have negative real parts, which is satisfied if and only if the coefficients of the characteristic equation (3.19) satisfy that $\Theta_1 > 0$, $\Theta_3 > 0$ and $\Delta > 0$.

Simple calculations show that condition (3.17a) guaranties that $p_{11} < 0$, $\Theta_1 > 0$, and $\Theta_3 > 0$, while conditions (3.17b) and (3.17c) guarantee that $\Delta > 0$. Hence, the proof is complete. $\square$

4. The Persistence

It is well known that persistence represents a global property of a dynamic system. It depends upon solution behavior near extinction boundaries rather than the interior solution space. Biologically, the persistence of a system means the survival of all populations of the system in future times. On the other hand, mathematically it means that strictly positive solutions do not have an omega limit set on the boundary of the non-negative cone. Accordingly, if the dynamic system does not persist, then the dynamic system faces extinction.

Now, before examining the persistence of the food chain model using the method of average Lyapunov function, we need to study the global dynamics in the boundary planes. The food chain system represented by the model (2.2), has two subsystems in the boundary planes xy-plane and xz-plane respectively. These two subsystems can be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= x \left[\frac{1-x}{1 + w_1 y} - \frac{w_2 y}{w_3 + x} \right] = N_1(x, y) \\ \frac{dy}{dt} &= y \left[\frac{w_4 x}{w_3 + x} - w_8 \right] = N_2(x, y) \end{aligned} \qquad (4.1a)$$

While the second subsystem is given by:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) = N_3(x, y), \\ \frac{dz}{dt} &= z(w_{10} - w_{11}) = N_4(x, y). \end{aligned} \qquad (4.1b)$$

Furthermore, these subsystems have unique interior coexistence points in the positive quadrant of xy-plane and xz-plane respectively, which represented by a projection of the boundary planar equilibrium

points $(\overline{x}, \overline{y})$ and $(1, \hat{z})$ of the model (2.2).

Now, define the function $B_1(x, y) = \dfrac{1}{xy}$, which is a continuously differential function in the interior of positive quadrant of xy-plane.

$\Delta(x, y) = \dfrac{\partial}{\partial x}(B_1 N_1) + \dfrac{\partial}{\partial y}(B_1 N_2) = \dfrac{-1}{y(1 + w_1 y)} + \dfrac{w_2}{(w_3 + x)^2}.$

Clearly $\Delta(x, y)$ don't identically zero and does not change sign provided that

$$\frac{1}{y(1 + w_1 y)} < \frac{w_2}{(w_3 + x)^2}. \tag{4.2}$$

Therefore, the subsystem (4.1a) has no periodic dynamics in the interior of positive quadrant of xy-plane. In fact the interior point $(\overline{x}, \overline{y})$ is a globally asymptotically stable whenever it exists and locally stable. Similarly, it is observed that the subsystem (4.1b) has no periodic dynamics in the interior of the positive quadrant of the xz-plane.

Theorem 4.1. *Assume that, the boundary planes of the model (2.2) have no periodic dynamics, then model (2.2) is uniformly persistent provided that the following conditions hold.*

$$\left.\begin{array}{c} \dfrac{w_4}{w_3 + 1} > w_8 \\ \\ or \\ \\ w_{10} > w_{11} \end{array}\right\}. \tag{4.3a}$$

$$\frac{w_9 \overline{y}}{w_7 + \overline{y}} + w_{10} > w_{11}. \tag{4.3b}$$

$$\frac{w_4}{(1 + w_5 \hat{z})(w_3 + 1)} > \frac{w_6 \hat{z}}{w_7} + w_8. \tag{4.3c}$$

Proof . Consider the following function $\delta(x, y, z) = x^{d_1} y^{d_2} z^{d_3}$ where $d_i, i = 1, 2, 3$ are positive constants. Obviously, $\delta(x, y, z)$ is a non-negative continuously differentiable function so that $\delta(x, y, z) \to 0$ if any one of the variables x, y, or z approach zero.

Furthermore, it is observed that:

$\Psi(x, y, z) = \dfrac{\delta'(x, y, z)}{\delta(x, y, z)} = d_1 f_1 + d_2 f_2 + d_3 f_3,$

where the functions $f_i, i = 1, 2, 3$, are given in the model (2.2). Accordingly, we have

$$\Psi(x, y, z) = d_1 \left[\frac{1 - x}{1 + w_1 y} - \frac{w_2 y}{w_3 + x} \right] + d_2 \left[\frac{w_4 x}{(1 + w_5 z)(w_3 + x)} - \frac{w_6 z}{w_7 + y} - w_8 \right]$$
$$+ d_3 \left[\frac{w_9 y}{w_7 + y} + w_{10} - w_{11} \right].$$

Thus, it is obtained that

$\Psi(s_0) = d_1[1] = d_2[-w_8] + d_3[w_{10} - w_{11}]$

Clearly by choosing the constant $d_1 > 0$ sufficiently large with respect to the positive constants d_2 and d_3, it is obtain that $\Psi(s_0) > 0$.

$$\Psi(s_1) = d_2 \left[\frac{w_4}{(w_3 + 1)} - w_8 \right] + d_3[w_{10} - w_{11}].$$

Clearly, for suitable chose of constants d_2 and d_3 with the condition (4.3a) it is obtain that $\Psi(s_1) > 0$. Now,

$$\Psi(s_2) = d_1[1] + d_2 \left[-\frac{w_6 \hat{z}}{w_7} - w_8 \right].$$

Similarly, for suitable chose of constants d_1 and d_2, we get $\Psi(s_2) > 0$. Moreover,

$$\Psi(s_3) = d_3 \left[\frac{w_9 \overline{y}}{w_7 + \overline{y}} + w_{10} - w_{11} \right].$$

Clearly, $\Psi(s_3) > 0$ under the condition (4.3). Finally, we have

$$\Psi(s_4) = d_2 \left[\frac{w_4}{(1 + w_5 \hat{z})(w_3 + 1)} - \frac{w_6 \hat{z}}{w_7} - w_8 \right].$$

Again, condition (4.3c) guarantees that $\Psi(s_4) > 0$. Therefore, due to the average Lyapunov method, model (2.2) is uniformly persistent. $\square$

5. Global Dynamics

In this section, the global dynamics of all the equilibrium points of the model (2.2), which are locally asymptotically stable is investigated using Lyapunov function.

Theorem 5.1. *Assume that, the first axial equilibrium point, $s_1 = (1, 0, 0)$, of the model (2.2) is locally asymptotically stable in R_+^3 then it is a globally asymptotically stable provided that the following conditions hold:*

$$\frac{w_2}{w_3} < w_8. \tag{5.1a}$$

$$w_{10} < w_{11} \tag{5.1b}$$

Proof . Consider the positive definite function:

$$M_1 = (x - 1 - \ln x) + y + x.$$

Straightforward computation gives:

$$\frac{dM_1}{dt} \leq -\frac{(x-1)^2}{1 + w_1 y} - \frac{xy}{w_3 + x}[w_2 - w_4] - y \left[w_8 - \frac{w_2}{w_3} \right]$$
$$- \frac{yz}{w_7 + y}[w_6 - w_9] - z[w_{11} - w_{10}].$$

Therefore, due to the biological meaning of the parameters, it is obtain:

$$\frac{dM_1}{dt} \leq -\frac{(x-1)^2}{1 + w_1 y} - y \left[w_8 - \frac{w_2}{w_3} \right] - z[w_{11} - w_{10}].$$

Therefore, the function $\dfrac{dM_1}{dt}$ is negative definite due to conditions (5.1a) and (5.1b). This, M_1 is a strong Lyapunove function that is readily unbounded. Hence, s_1 is a globally asymptotically stable. $\square$

Theorem 5.2. *Assume that, the top predator-free equilibrium point, $s_3 = (\overline{x}, \overline{y}, 0)$, of the model (2.2) is locally asymptotically stable, then the basin of attraction of s_3 is a sunset of Ω that satisfies the following sufficient conditions.*

$$\frac{w_2\overline{y}}{A_2\overline{A}_2} < \frac{1}{A_1}, \tag{5.2a}$$

$$\left[\frac{w_1(1-\overline{x})}{A_1\overline{A}_1} + \frac{w_2}{A_2} - \frac{w_3w_4}{A_2\overline{A}_2A_4}\right]^2 < 4\left[\frac{1}{A_1} - \frac{w_2\overline{y}}{A_2\overline{A}_2}\right], \tag{5.2b}$$

$$w_{10} + \frac{w_4w_5\overline{x}\ \overline{y}}{\overline{A}_2} + \frac{w_6}{w_7}\overline{y} < w_{11} \tag{5.2c}$$

$$(y - \overline{y})^2 < \left[w_{11} - w_{10} - \frac{w_4w_5\overline{x}\ \overline{y}}{\overline{A}_2} - \frac{w_6}{w_7}\overline{y}\right]y, \tag{5.2d}$$

where

$$A_1 = 1 + w_1y, \ \overline{A}_1 = 1 + w_1\overline{y}, \ A_2 = w_3 + x, \ \overline{A}_2 = w_3 + \overline{x}, \ A_3 = w_7 + y, \ and \ A_4 = 1 + w_5z.$$

Proof . Consider the positive definite function:

$$M_2 = \left(x - \overline{x} - \overline{x}\ln\left(\frac{x}{\overline{x}}\right)\right) + \left(y - \overline{y} - \overline{y}\ln\left(\frac{y}{\overline{y}}\right)\right) + z.$$

Straightforward computation gives:

$$\begin{aligned}
\frac{dM_2}{dt} \leq &-\left[\frac{1}{A_1} - \frac{w_2\overline{y}}{A_2\overline{A}_2}\right](x - \overline{x})^2 - (y - \overline{y})^2 \\
&-\left[\frac{w_1(1-\overline{x})}{A_1\overline{A}_1} + \frac{w_2}{A_2} - \frac{w_3w_4}{A_2\overline{A}_2A_4}\right](x - \overline{x})(y - \overline{y}) \\
&-\left[w_{11} - w_{10} - \frac{w_4w_5\overline{x}\ \overline{y}}{\overline{A}_2} - \frac{w_6}{w_7}\overline{y}\right]y + (y - \overline{y})^2.
\end{aligned}$$

Now, by using the conditions (5.2a)-(5.2b), it is obtain that

$$\begin{aligned}
\frac{dM_2}{dt} \leq &-\left[\sqrt{\frac{1}{A_1} - \frac{w_2\overline{y}}{A_2\overline{A}_2}}(x - \overline{x}) + (y - \overline{y})\right]^2 \\
&-\left[w_{11} - w_{10} - \frac{w_4w_5\overline{x}\ \overline{y}}{\overline{A}_2} - \frac{w_6}{w_7}\overline{y}\right]y + (y - \overline{y})^2.
\end{aligned}$$

Therefore, using the conditions (5.2c)-(5.2d), it is obtain that, the derivative $\dfrac{dM_2}{dt}$ is negative definite in the interior of sub region of Ω that satisfies the given sufficient conditions. Hence, this sub region represents a basin of attraction for s_3, and the proof is complete $\square$

Theorem 5.3. *Assume that, the intermediate predator-free equilibrium point, $s_4 = (1, 0, \hat{z})$, of the model (2.2)) exists, then it is a globally asymptotically stable in R_+^3 provided that the condition (5.1a) holds.*

Proof . Consider the positive definite function:

$$M_3 = (x - 1 - \ln x) + y + \left(z - \hat{z} - \hat{z} \ln \left(\frac{z}{\hat{z}} \right) \right).$$

Straightforward computation gives:

$$\frac{dM_3}{dt} \leq -\frac{(x-1)^2}{1 + w_1 y} - \frac{xy}{w_3 + x}[w_2 - w_4] - \left[w_8 - \frac{w_2}{w_3} \right] y$$
$$- \frac{yz}{w_7 + y}[w_6 - w_9] - \frac{w_9 y \hat{z}}{w_7 + y}.$$

Therefore, due to the biological meaning of the parameters, it is obtain:

$$\frac{dM_3}{dt} \leq -\frac{(x-1)^2}{1 + w_1 y} - \left[w_8 - \frac{w_2}{w_3} \right] y$$

Clearly, the derivative $\dfrac{dM_3}{dt}$, is negative semi definite under the condition (5.1a). Hence, so by the Lyapunov-Lasalle's invariance principle [Lasalle 1976], the intermediate predator-free equilibrium point is a globally asymptotically stable. □

Theorem 5.4. *Assume that, the coexistence equilibrium point $s_5 = (\check{x}, \check{y}, \check{z})$ of the model (2.2) is locally asymptotically stable, then the basin of attraction of s_5 is a subset of Ω that satisfies the following sufficient conditions.*

$$\frac{w_2 \check{y}}{A_2 \check{A}_2} < \frac{1}{A_1}, \tag{5.3a}$$

$$\left[\frac{w_1(1 - \check{x})}{A_1 \check{A}_1} + \frac{w_2}{A_2} - \frac{w_3 w_4}{A_2 \check{A}_2 A_4} \right]^2 < 2 \left[\frac{1}{A_1} - \frac{w_2 \check{y}}{A_2 \check{A}_2} \right] \left[\frac{w_6 \check{z}}{w_7 \check{A}_3} \right], \tag{5.3b}$$

$$\left[\frac{w_4 w_5 \check{x}}{\check{A}_2 A_4 \check{A}_4} + \frac{w_6}{A_3} - \frac{w_7 w_9}{A_3 \check{A}_3} \right]^2 < 2 \left[\frac{w_6 \check{z}}{w_7 \check{A}_3} \right], \tag{5.3c}$$

$$2 \frac{w_6 \check{z}}{w_7 \check{A}_3}(y - \check{y})^2 + \frac{w_6 \check{z}}{w_7 \check{A}_3}(z - \check{z})^2 < \left[\sqrt{\frac{w_6 \check{z}}{2 w_7 \check{A}_3}}(y - \check{y}) + \sqrt{\frac{w_6 \check{z}}{w_7 \check{A}_3}}(z - \check{z}) \right]^2. \tag{5.3d}$$

Proof . Consider the positive definite function:

$$M_4 = \left(x - \check{x} - \check{x} \ln \left(\frac{x}{\check{x}} \right) \right) + \left(y - \check{y} - \check{y} \ln \left(\frac{y}{\check{y}} \right) \right) + \left(z - \check{z} - \check{z} \ln \left(\frac{z}{\check{z}} \right) \right).$$

Accordingly, the derivative of M_4 with respect to time can be written as:

$$\frac{dM_4}{dt} \leq - \left[\frac{1}{A_1} - \frac{w_2 \check{y}}{A_2 \check{A}_2} \right] (x - \check{x})^2 - \left[\frac{w_1(1 - \check{x})}{A_1 \check{A}_1} + \frac{w_2}{A_2} - \frac{w_3 w_4}{A_2 \check{A}_2 A_4} \right] (x - \check{x})(y - \check{y})$$
$$- \left[\frac{w_4 w_5 \check{x}}{\check{A}_2 A_4 \check{A}_4} + \frac{w_6}{A_3} - \frac{w_7 w_9}{A_3 \check{A}_3} \right] (y - \check{y})(z - \check{z}) - \frac{w_6 \check{z}}{w_7 \check{A}_3}(y - \check{y})^2$$
$$+ 2 \frac{w_6 \check{z}}{w_7 \check{A}_3}(y - \check{y})^2 - \frac{w_6 \check{z}}{w_7 \check{A}_3}(z - \check{z})^2 + \frac{w_6 \check{z}}{w_7 \check{A}_3}(z - \check{z})^2.$$

Therefore, using the conditions (5.3a)-(5.3c) gives that:

$$
\begin{aligned}
\frac{dM_4}{dt} \leq &- \left[\sqrt{\frac{1}{A_1} - \frac{w_2 \check{y}}{A_2 \check{A}_2}}(x - \check{x}) + \sqrt{\frac{w_6 \check{z}}{2 w_7 \check{A}_3}}(y - \check{y}) \right]^2 \\
&- \left[\sqrt{\frac{w_6 \check{z}}{2 w_7 \check{A}_3}}(y - \check{y}) + \sqrt{\frac{w_6 \check{z}}{w_7 \check{A}_3}}(z - \check{z}) \right]^2 \\
&+ 2 \frac{w_6 \check{z}}{w_7 \check{A}_3}(y - \check{y})^2 + \frac{w_6 \check{z}}{w_7 \check{A}_3}(z - \check{z})^2.
\end{aligned}
$$

Therefore, using the condition (5.3d), it is clear that, the derivative $\dfrac{dM_4}{dt}$ is negative definite in the interior of sub region of Ω that satisfies the given sufficient conditions. Hence, this sub region represents a basin of attraction for s_5, and the proof is complete. $\square$

6. Bifurcation Analysis

This section interests how the equilibrium configurations of model (2.2) depend on the parameters that characterize the model. Indeed, it may occur that as one of the parameters crosses a specific value, the solution will tend toward another equilibrium position. The aim of this section is to study the bifurcation of the model (2.2) that may occur with the varying of the values of parameters. Rewrite the model (2.2) in the form:

$$
\frac{dX}{dt} = F(X), \text{ where } X = (x, y.z)^T, \text{ and } F(x) = (x f_1, y f_2, z f_3)^T
$$

Therefore, the second derivative of $F(X)$, with respect to the vector X can be written as

$$
D^2 F(X) = (\mho, \mho) = [c_{i1}]_{3 \times 1}, \tag{6.1}
$$

where:

$$
c_{11} = \left[-\frac{2}{(1 + w_1 y)} + \frac{2 w_2 w_3 y}{(w_3 + x)^3} \right] v_1^2 + \left[-\frac{2 w_1 (1 - 2x)}{(1 + w_1 y)^2} \right] v_1 v_2 + \left[\frac{2 w_1^2 x (1 - x)}{(1 + w_1) y)^3} \right] v_2^2.
$$

$$
\begin{aligned}
c_{21} = &\left[-\frac{2 w_2 w_4 y}{(x + w_3)^3 (1 + w_5 z)} \right] v_1^2 + \left[\frac{2 w_3 w_4}{(x + w_3)^2 (1 + w_5 z)} \right] v_1 v_2 \\
&+ \left[-\frac{2 w_3 w_4 w_5 y}{(x + w_3)^2 (1 + w_5 z)^2} \right] v_1 v_3 + \left[\frac{w_6 z (w_7 - y)}{(y + w_7)^3} + \frac{w_6 z}{(y + w_7)^2} \right] v_2^2 \\
&+ \left[-\frac{2 w_4 w_5 x}{(w_3 + x)(1 + w_5 z)^2} - \frac{2 w_6 w_7}{(y + w_7)^2} \right] v_2 v_3 + \left[\frac{2 w_4 w_5^2 x y}{(x + w_3)(1 + w_5 z)^3} \right] v_3^2.
\end{aligned}
$$

$$
c_{31} = \left[-\frac{2 w_7 w_9 z}{(y + w_7)^3} \right] v_2^2 + \left[\frac{2 w_7 w_9}{(y + w_7)^2} \right] v_2 v_3.
$$

With $\mho = (v_1, v_2, v_3)$ be a non-zero real vector.

Theorem 6.1. *Assume that condition (3.10b) holds, then the model (2.2) at the first axial equilibrium point $s_1 = (1, 0, 0)$ possesses a transcritical bifurcation when the parameter w_8 satisfies that*
$$
w_8 = \frac{w_4}{w_3 + 1}(\equiv w_8^*)
$$

Proof . It is easy to verify that the Jacobian matrix of the (2.2) at the first axial equilibrium point with $w_8 = w_8^*$ can be written in the form:

$$J(s_1, w_8^*) = \begin{pmatrix} -1 & -\dfrac{w_2}{w_3+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w_{10} - w_{11} \end{pmatrix}$$

Obviously, due to condition (3.10b) the eigenvalues of $J(s_1, w_8^*)$ are given by $\lambda_{11}^* = -1 < 0$, $\lambda_{12}^* = 0$, and $\lambda_{13}^* = w_{10} - w_{11} < 0$.

Let $\mathcal{L}_1 = (\ell_{11}, \ell_{12}, \ell_{13})^T$ represents the eigenvector corresponding to $\lambda_{12}^* = 0$. Then direct computation gives that $\mathcal{L}_1 = \left(-\dfrac{w_2}{w_3+1}\ell_{12}, \ell_{12}, 0 \right)^T$, where $\ell_{12} \neq 0$ any real number.

Let $\mathcal{H}_1 = (h_{11}, h_{12}, h_{13})^T$ represents the eigenvector corresponding to $\lambda_{13}^* = 0$ for the $[J(s_1, w_8^*)]^T$. Then, direct computation gives that $\mathcal{H}_1 = (0, h_{12}, 0)^T$, where $h_{12} \neq 0$ any real number. Accordingly, the following is obtained:

$$\frac{\partial F}{\partial w_8} = F_{w_8} = (0, -y, 0)^T \Rightarrow \mathcal{H}_1^T F_{w_8}(s_1, w_8^*) = 0.$$

$$DF_{w_8}(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{H}_1^T DF_{w_8}(s_1, w_8^*)\mathcal{L}_1 = -\ell_{12}h_{12} \neq 0.$$

$$D^2 F(s_1, w_8^*)(\mathcal{L}_1, \mathcal{L}_1) = \begin{pmatrix} -2\dfrac{w_2}{w_3+1}\left(\dfrac{w_2}{w_3+1} + w_1 \right)\ell_{12}^2 \\ -\dfrac{2w_2 w_3 w_4}{(1+w_3)^3}\ell_{12}^2 \\ 0 \end{pmatrix}.$$

This gives that:

$$\mathcal{H}_1^T D^2 F(s_1, w_8^*)(\mathcal{L}_1, \mathcal{L}_1) = -\frac{2w_2 w_3 w_4}{(1+w_3)^3}\ell_{12}^2 h_{12} \neq 0.$$

Therefore , in the sense of Sotomayor's theorem, the model possesses a transcritical bifurcation and the proof is complete. □

Theorem 6.2. *Assume that condition (3.14a) holds, then the model (2.2) at the top predator-free equilibrium point, $s_3 = (\overline{x}, \overline{y}, 0)$ possesses a transcritical bifurcation when the parameter w_{11} satisfies that $w_{11} = \dfrac{w_9 \overline{y}}{w_7 + \overline{y}} + w_{10} (\equiv w_{11}^*).$*

Proof . Note that the Jacobian matrix of the model (2.2) at (s_3, w_{11}^*) can be written in the form:

$$J(s_3, w_{11}^*) = \begin{pmatrix} \dfrac{-\overline{x}}{1+w_1\overline{y}} + \dfrac{w_2\overline{x}\,\overline{y}}{(w_3+\overline{x})^2} & \dfrac{-w_1\overline{x}(1-\overline{x})}{(1+w_1\overline{y})^2} - \dfrac{w_2\overline{x}}{w_3+\overline{x}} & 0 \\ \dfrac{w_3 w_4 \overline{y}}{(w_3+\overline{x})^2} & 0 & -\dfrac{w_6\overline{y}}{w_7+\overline{y}} - \dfrac{w_4 w_5 \overline{x}\,\overline{y}}{w_3+\overline{x}} \\ 0 & 0 & 0 \end{pmatrix} = (\sigma_{ij}).$$

Therefore, the eigenvalues of $J(s_3, w_{11}^*)$ are determined as

$$\lambda_{31}^* = \frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}, \quad \lambda_{32}^* = \frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}, \quad \lambda_{33}^* = 0,$$

where β_1 and β_2 are given in equation (6). Clearly, due to condition (3.14a) the eigenvalues λ_{31}^* and λ_{32}^* have negative real parts, while the third one is zero. Thus the top predator free equilibrium point becomes non-hyperbolic point.

Define $\mathcal{L}_2(\ell_{21}, \ell_{22}, \ell_{23})^T$ represents the eigenvector corresponding to $\lambda_{33}^* = 0$. Then direct computation gives that $\mathcal{L}_2 = \left(-\dfrac{\sigma_{23}}{\sigma_{21}} \ell_{23}, \dfrac{\sigma_{11}\sigma_{23}}{\sigma_{12}\sigma_{21}} \ell_{23}, \ell_{23} \right)^T$, where $\ell_{23} \neq 0$ any real number.

Let $\mathcal{H}_2 = (h_{21}, h_{22}, h_{23})^T$ represents the eigenvector corresponding to $\lambda_{33} = 0$ for the $[J(s_3, w_{11}^*)]^T$. Then, direct computation gives that $\mathcal{H}_2 = (0, 0, h_{23})^T$, where $h_{23} \neq 0$ any real number.

Accordingly, the following is obtained:

$$\frac{\partial F}{\partial w_{11}} = F_{w_{11}} = (0, 0, z)^T \Rightarrow \mathcal{H}_2^T F_{w_{11}}(s_3, w_{11}^*) = 0.$$

$$DF_{w_{11}}(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathcal{H}_2^T DF_{w_{11}}(s_3, w_{11}^*)\mathcal{L}_2 = \ell_{23}h_{23} \neq 0.$$

$$D^2 F(s_3, w_{11}^*)(\mathcal{L}_2, \mathcal{L}_2) = \begin{pmatrix} \overline{c}_{11} \\ \overline{c}_{21} \\ \overline{c}_{31} \end{pmatrix},$$

where

$$\overline{c}_{11} = 2\left(\left[-\frac{1}{(1+w_1\overline{y})} + \frac{w_2 w_3 \overline{y}}{(w_3 + \overline{x})^3} \right] + \left[\frac{w_1(1 - 2\overline{x})}{(1+w_1\overline{y})^2} \right] \left(\frac{\sigma_{11}}{\sigma_{12}} \right) + \left[\frac{w_1^2 x(1-\overline{x})}{(1+w_1\overline{y})^3} \right] \left(\frac{\sigma_{11}}{\sigma_{12}} \right)^2 \right) \left(\frac{\sigma_{23}}{\sigma_{21}} \right)^2 \ell_{23}^2,$$

$$\overline{c}_{21} = 2\left(\left[-\frac{w_2 w_4 \overline{y}}{(\overline{x} + w_3)^3} \right] \left(\frac{\sigma_{23}}{\sigma_{21}} \right)^2 - \left[\frac{w_3 w_4}{(\overline{x} + w_3)^2} \right] \left(\frac{\sigma_{23}}{\sigma_{21}} \right)^2 \left(\frac{\sigma_{11}}{\sigma_{12}} \right) + \left[\frac{w_3 w_4 w_5 \overline{y}}{(\overline{x} + w_3)^2} \right] \left(\frac{\sigma_{23}}{\sigma_{21}} \right) \right.$$
$$\left. - \left[\frac{w_4 w_5 \overline{x}}{(w_3 + \overline{x})} + \frac{w_6 w_7}{(\overline{y} + w_7)^2} \right] \left(\frac{\sigma_{11}\sigma_{23}}{\sigma_{12}\sigma_{21}} \right) + \left[\frac{w_4 w_5^2 \overline{x}\, \overline{y}}{(\overline{x} + w_3)} \right] \right) \ell_{23}^2,$$

$$\overline{c}_{31} = \left[\frac{2w_7 w_9}{(\overline{y} + w_7)^2} \right] \left(\frac{\sigma_{11}\sigma_{23}}{\sigma_{12}\sigma_{21}} \right) \ell_{23}^2$$

Therefore, it is obtained that:

$$\mathcal{H}_2^T D^2 F(s_3, w_{11}^*)(\mathcal{L}_2, \mathcal{L}_2) = \left[\frac{2w_7 w_9}{(\overline{y} + w_7)^2} \right] \left(\frac{\sigma_{11}\sigma_{23}}{\sigma_{12}\sigma_{21}} \right) \ell_{23}^2 h_{23} \neq 0.$$

Thus, in the sense of Sotomayor's theorem, the model possesses a transcritical bifurcation and the proof is complete. $\square$

Theorem 6.3. *The model* (2.2) *undergoes a saddle node bifurcation near the coexistence equilibrium point as the parameter w_2 passes through the value $w_2^* = \dfrac{\breve{A}_2^2}{\breve{y}\breve{A}_1}$.*

Proof . From the Jacobian matrix $J(s_5)$ of the model (2.2) at $s_5 = (\breve{x}, \breve{y}, \breve{z})$ that given in equation (3.18), the determinant of it is determined by $\Theta_3 = p_{11}p_{23}p_{32}$, which given in equation (3.19). Clearly, $p_{11}(w_2^*) = 0$, hence $\Theta_3 = 0$. Therefore, the characteristic equation of $J(s_5)$ has a zero eigenvalue, and hence the coexistence equilibrium point becomes a non-hyperbolic point. Furthermore, the Jacobian matrix at (s_5, w_2^*) is written as:

$$J^* = J(s_5, w_2^*) = [p_{ij}]_{3\times 3},$$

where p_{ij}, for all $i, j = 1, 2, 3$ are the same as given in equation (3.18) except $p_{11}(w_2^*) = 0$. Now define $\mathcal{L}_3 = (\ell_{13}, \ell_{32}, \ell_{33})^T$, where $\mathcal{L}_3$ represents the eigenvector corresponding to λ_{51}^*. Then direct computation gives that $\mathcal{L}_3 = \left(\ell_{31}, 0, -\dfrac{p_{21}}{p_{23}} \ell_{31} \right)^T$, where $\ell_{31} \neq 0$ any real number.

Let $\mathcal{H}_3 = (h_{31}, h_{32}, h_{33})^T$ be the eigenvector corresponding to $\lambda_{51}^* = 0$ for J^{*T}. Then direct computation gives that $\mathcal{H}_2 = \left(h_{31}, 0, -\dfrac{p_{12}}{p_{32}} h_{31} \right)^T$, where $h_{31} \neq 0$ any real number.

Accordingly, the following is obtained:

$$\frac{\partial F}{\partial w_2} = F_{w_2} = \left(-\frac{xy}{A_2}, 0, 0 \right)^T \Rightarrow \mathcal{H}_2^T F_{w_{11}}(s_5, w_2^*) = -\frac{\check{x}\check{y}}{\check{A}_2} h_{31} \neq 0.$$

Using the eigenvector $\mathcal{L}_3$ and (s_5, w_2^*) in (6.1) gives that:

$$D^2 F(s_5, w_2^*)(\mathcal{L}_3, \mathcal{L}_3) = [c_{i1}(s_5, w_2^*)]_{3 \times 1},$$

where:

$$c_{11}(s_5, w_2^*) = \frac{2}{\check{A}_1} \left[-1 + \frac{w_3}{\check{A}_2} \right] \ell_{31}^2.$$

$$c_{21}(s_5, w_2^*) = -\left[\frac{2w_2^* w_4 \check{y}}{\check{A}_2^3 \check{A}_4} \right] \ell_{31}^2 - \left[\frac{2w_3 w_4 w_5 \check{y}}{\check{A}_2^2 \check{A}_4^2} \right] \left(-\frac{p_{21}}{p_{23}} \right) \ell_{31}^2$$
$$+ \left[\frac{2w_4 w_5^2 \check{x}\check{y}}{\check{A}_2 \check{A}_4^3} \right] \left(-\frac{p_{21}}{p_{23}} \right)^2 \ell_{31}^2.$$

$$c_{31}(s_5, w_2^*) = 0.$$

Hence, it is obtain that:

$$\mathcal{H}_2^T D^2 F(s_5, w_2^*)(\mathcal{L}_3, \mathcal{L}_3) = \frac{2}{\check{A}_1} \left[-1 + \frac{w_3}{\check{A}_2} \right] \ell_{31}^2 h_{31}.$$

Since, $\dfrac{w_3}{\check{A}_2} < 1$, then $\mathcal{H}_2^T D^2 F(s_5, w_2^*)(\mathcal{L}_3, \mathcal{L}_3) \neq 0$. Hence the model (2.2) undergoes a saddle node bifurcation in the sense of Sotomayor. $\square$

7. Numerical Simulation

In this section, the global dynamics of model (2.2) are studied numerically. Model (2.2) is solved numerically for different sets of parameters and different sets of initial conditions. The objective is to understand the global dynamic behavior of model (2.2) when the parameter values are varying. It is observed that, for the following set of hypothetical parameter values, the trajectory of the model (2.2) approaches asymptotically to the chaotic attractor, as shown in figure (1) and figure (2).

$$\begin{aligned} &w_1 = 0.2, \ w_2 = 0.5, \ w_3 = 0.2, w_4 = 0.4, \ w_5 = 0.1, \ w_6 = 0.4, \\ &w_7 = 0.5, \ w_8 = 0.15, \ w_9 = 0.2, \ w_{10} = 0.1, \ w_{11} = 0.15. \end{aligned} \tag{7.1}$$

Clearly, figures (1) and figure (2) show the existence of a strange attractor in the model (2.2) with their projection on the boundary planes and their time series for the data given in equation
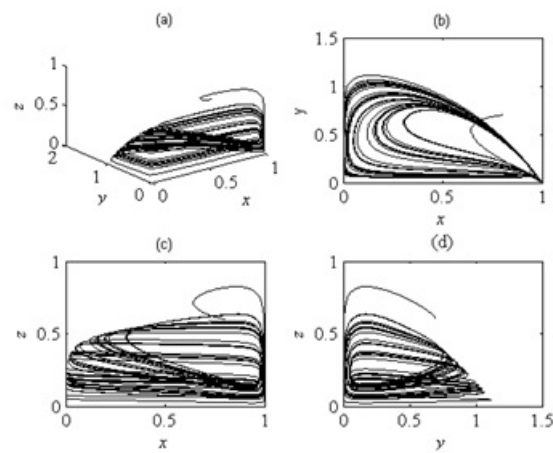
Figure 1: *The trajectory of the model (2.2) approaches asymptotically to a chaotic attractor for data (7.1). (a) 3D strange attractor. (b) Projection of the attractor in xy-plane. (c) Projection of the attractor in xz-plane. (d) Projection of the attractor in yz-plane.*
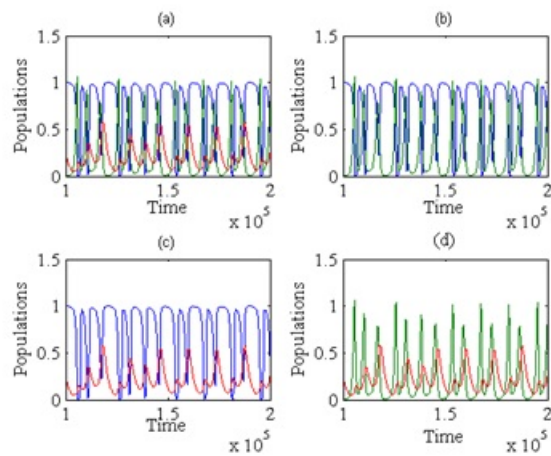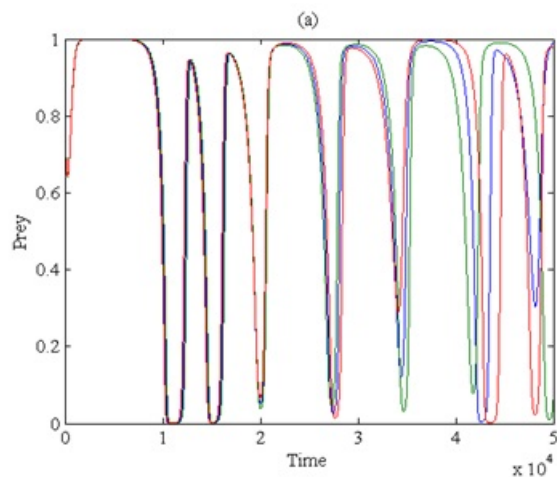

Figure 2: *Time series of the chaotic attractor at data (7.1). (a) Time series of the strange attractor. (b) Time series in xy-plane. (c) Time series in xz-plane. (d) Time series in yz-plane.*
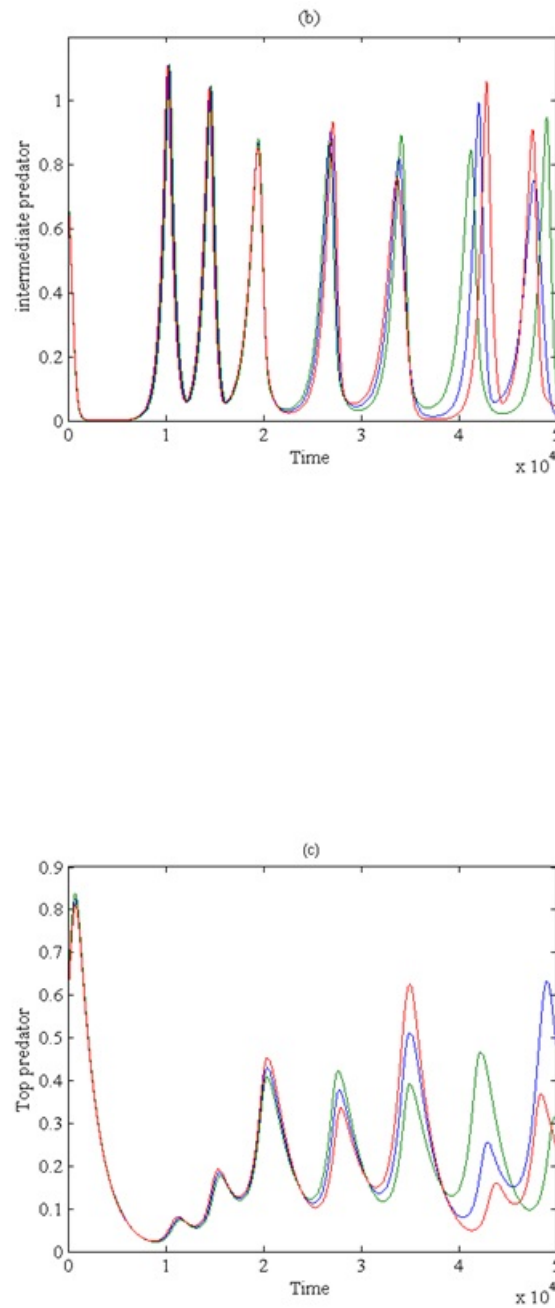
Figure 3: *Sensitivity to initial points for data . (a) Trajectories of the prey: blue starts at 0.8, green starts at 0.81, red starts at 0.79. (b) Trajectories of the intermediate predator: blue starts at 0.7, green starts at 0.71, red starts at 0.69. (c) Trajectories of the top predator: blue starts at 0.6, green starts at 0.61, red starts at 0.59.*

(7.1).   It is well known that the most important characteristic property for chaotic dynamics is their sensitivity to slight changes in the initial points. Therefore, figure (3) explains the sensitivity of the changing in initial point, so that changing the initial point by (0.01) leads to a huge change in the trajectories of the model (2.2), which means the existence of chaos.

According to the above, the effects of the changing of the parameters of the model (2.2) are investigated using the bifurcation diagram, which shows the attractors approached asymptotically (fixed points, periodic attractors, or strange attractors) of a model as a function of a bifurcation parameter in the model. Therefore, in the following figures   According to the above bifurcation diagram
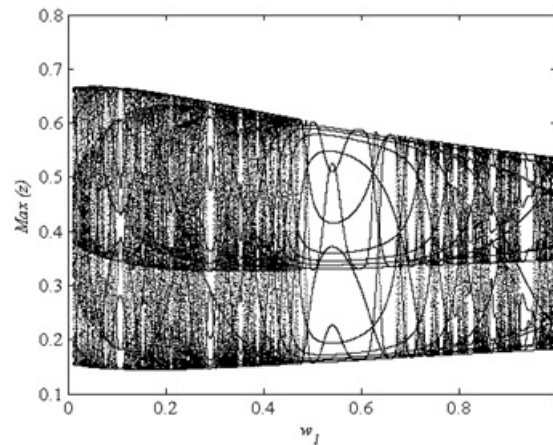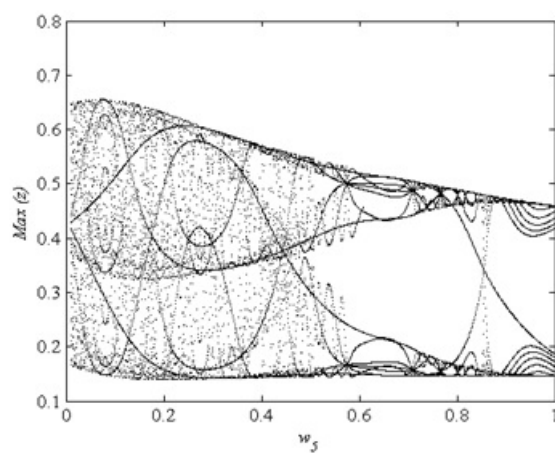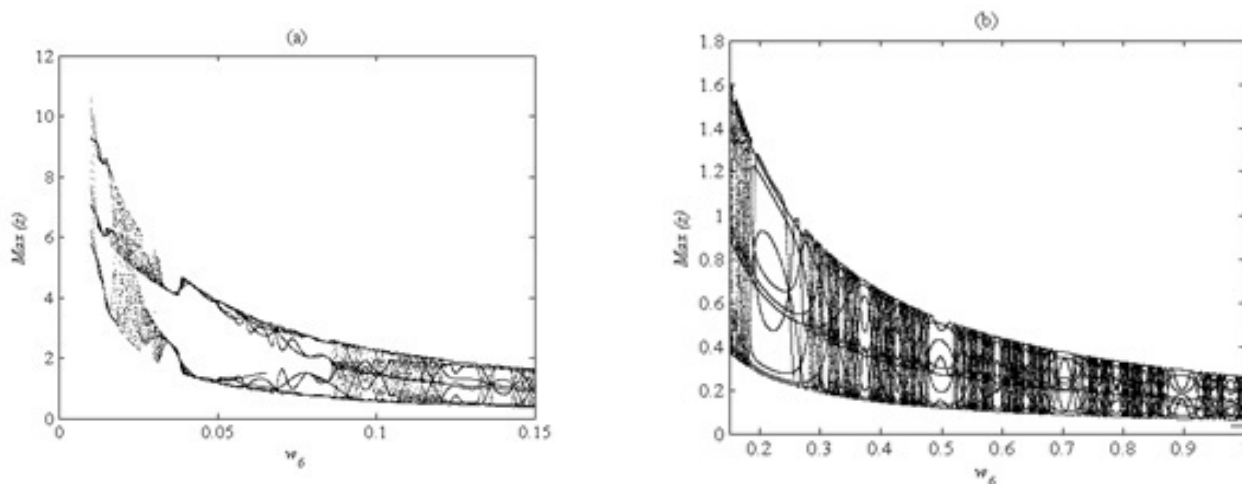

Figure 4: *Bifurcation diagram as a function of w_1 in the range $0.01 < w_1 < 1$.*
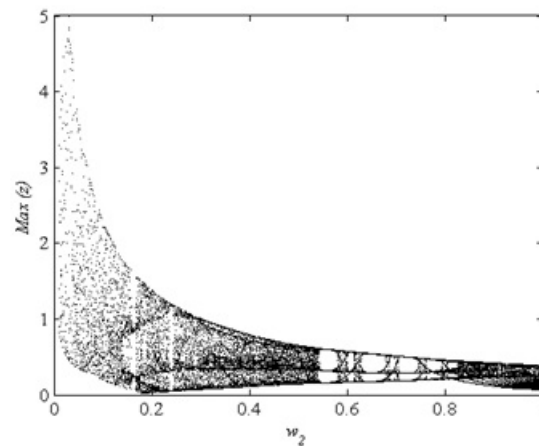
Figure 5: *Bifurcation diagram as a function of w_2 in the range $0.01 < w_2 < 1$.*

figures (4)-(14), It is clear that model (2.2) is rich in their dynamics and there are ranges of chaotic dynamics in between there are windows of periodic dynamics that leading to chaos. Regarding the fear rates parameters, it is observed from the bifurcation diagrams given in figures (4), (8) and their extended for values above the upper limit in these figures that there is a gradual reduction to the chaotic region and the solution of model (2.2) stabilize at a periodic dynamics as shown in the figure (15) for typical values of fear rates. Regarding the bifurcation diagrams given by figures (5) and (9),
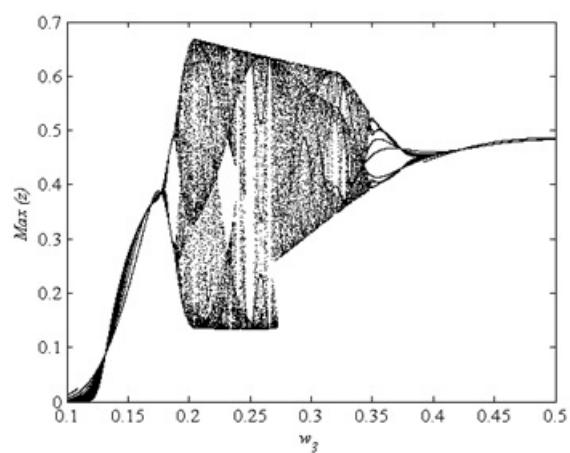
Figure 6: *Bifurcation diagram as a function of w_3 in the range $0.1 < w_3 < 0.5$.*
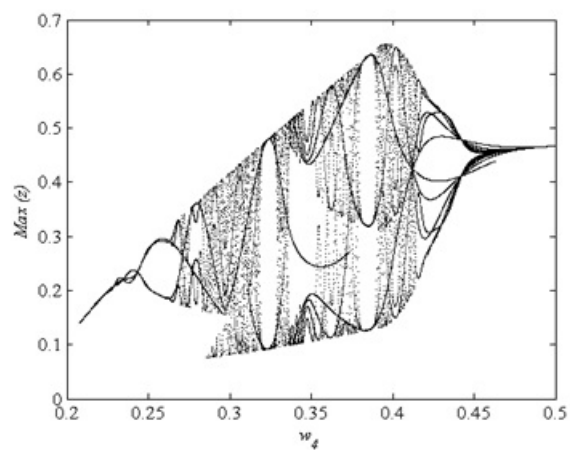
Figure 7: *Bifurcation diagram as a function of w_4 in the range $0.2 < w_4 < 0.5$.*

Figure 8: *Bifurcation diagram as a function of w_5 in the range $0.01 < w_5 < 1$.*

Figure 9: *Bifurcation diagram as a function of w_6 in the ranges $0.01 < w_6 < 0.15$ and $0.15 \leq w_6 < 1$.*
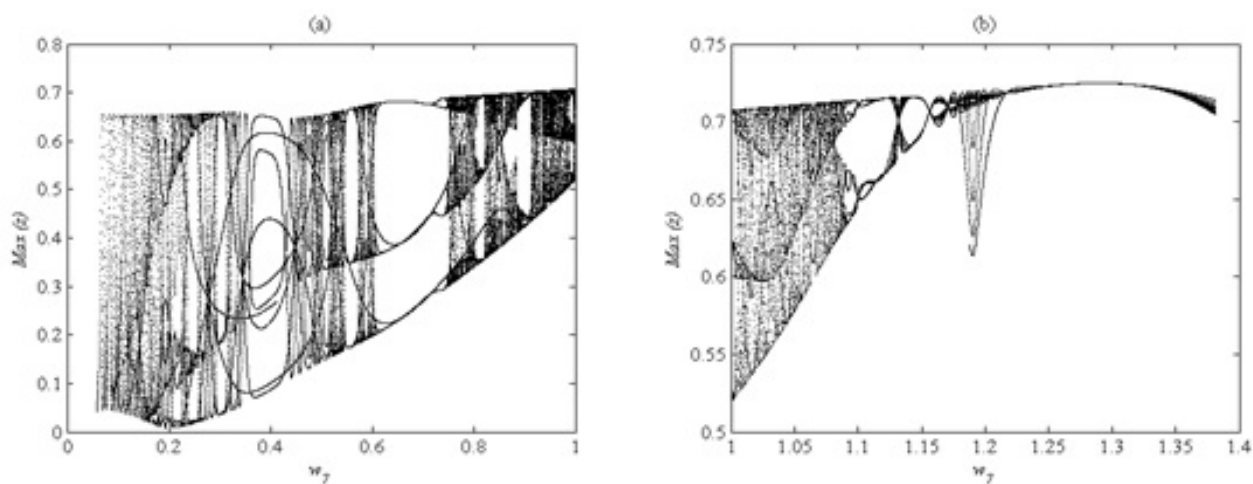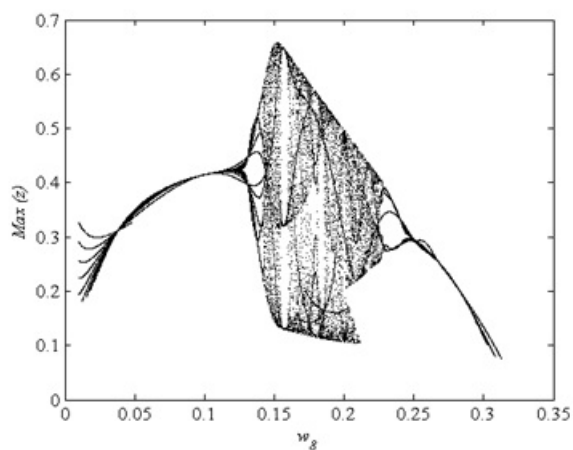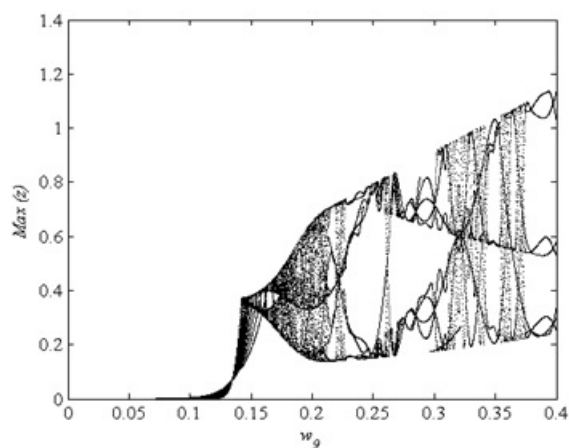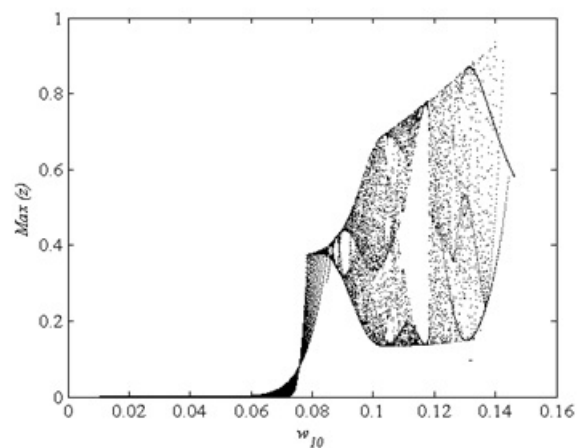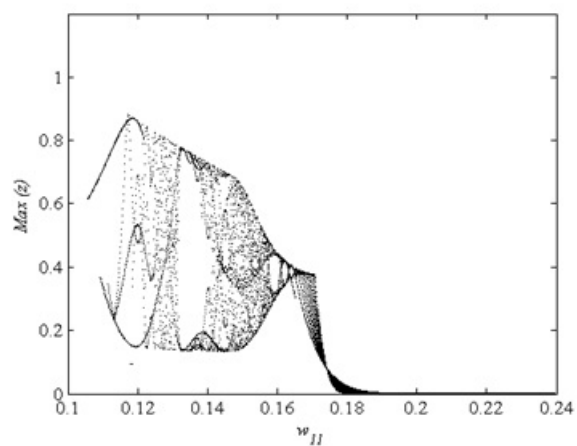

Figure 10: *Bifurcation diagram as a function of w_7 in the ranges $0.01 < w_7 < 1$ and $1 \leq w_7 < 1.4$.*

Figure 11: *Bifurcation diagram as a function of w_8 in the ranges $0.01 < w_8 < 0.35$.*

Figure 12: *Bifurcation diagram as a function of w_9 in the ranges $0.01 < w_9 < 0.4$.*

Figure 13: *Bifurcation diagram as a function of w_{10} in the ranges $0.01 < w_{10} < 0.16$.*

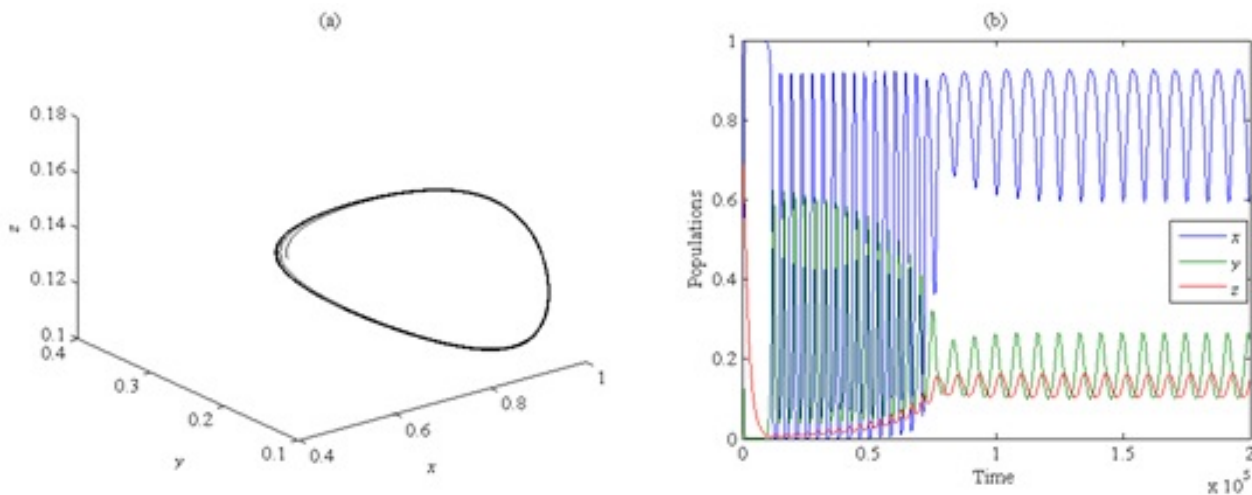Figure 14: *Bifurcation diagram as a function of w_{11} in the ranges $0.1 < w_{11} < 0.25$.*

Figure 15: *The trajectory of the model* (2.2) *approaches asymptotically to a periodic attractor for data* (7.1) *with* $w_1 = 6$ *and* $w_5 = 3$. *(a)* $3D$ *periodic attractor. (b) Time series of the attractor in (a).*

it is observed that model (2.2) approaches to chaotic attractor as the parameters w_2 and w_6 varying in the ranges $w_2 \in (0, 1)$ and $w_6 \in (0, 1)$ respectively through a cascade of periodic doubling as shown in the figure (16) for a selected value of w_2. However, increasing these parameters causes extinction in top predators and the trajectory approaches periodic dynamics in the xy-plane as shown in figure (17). Regarding the bifurcation diagrams given by figures (6) and (10), it is observed that model (2.2) is rich in their dynamics in the ranges $w_3 \in (0, 0.5)$ and $w_7 \in (0, 1.4)$ respectively including the chaotic region in between there are periodic regions. However, increasing these parameters leads to stabilizing the system first at a simple periodic attractor and then at a coexistence equilibrium point as shown in figure (18) for typical values of these parameters. Obviously, according to the figures (18 c) and (18 d) model (2.2) approaches to coexistence equilibrium point given by $(0.85, 0.43, 0.25)$ as the parameters w_3 and w_7 increase above specific values. Regarding the bifurcation diagrams given by figures (7) and (12), it is observed that model (2.2) is rich in their dynamics in the ranges $w_4 \in (0.1, 0.5)$ and $w_9 \in (0, 0.4)$ respectively including the chaotic region in between there are periodic regions. However, decreasing these parameters leads to stabilizing the system first at a simple periodic attractor in $3D$ space, periodic in the interior of xy-plane, and then at top predator-free equilibrium point as shown in figure (19) for typical values of these parameters. However, increasing these parameters above specific values leads to reduce the chaotic region and the model approaches periodic dynamics in the interior of $3D$ space. From the bifurcation diagrams given by figures (11) and (14), it is observed that model (2.2) is rich in their dynamics in the ranges $w_8 \in (0, 0.35)$ and $w_{11} \in (0, 0.24)$ respectively including the chaotic region in between there are periodic regions. However, increasing these parameters leads to extinction in intermediate or top predator and the trajectory approach asymptotically either to periodic or to the boundary equilibrium points.
On the other hand, from the bifurcation diagram given by figure (13) in the range $0.01 < w_{10} < 0.16$, Clearly, figure (20 b) shows the existence of intermediate predator-free equilibrium point when the condition (3.1) is satisfied.
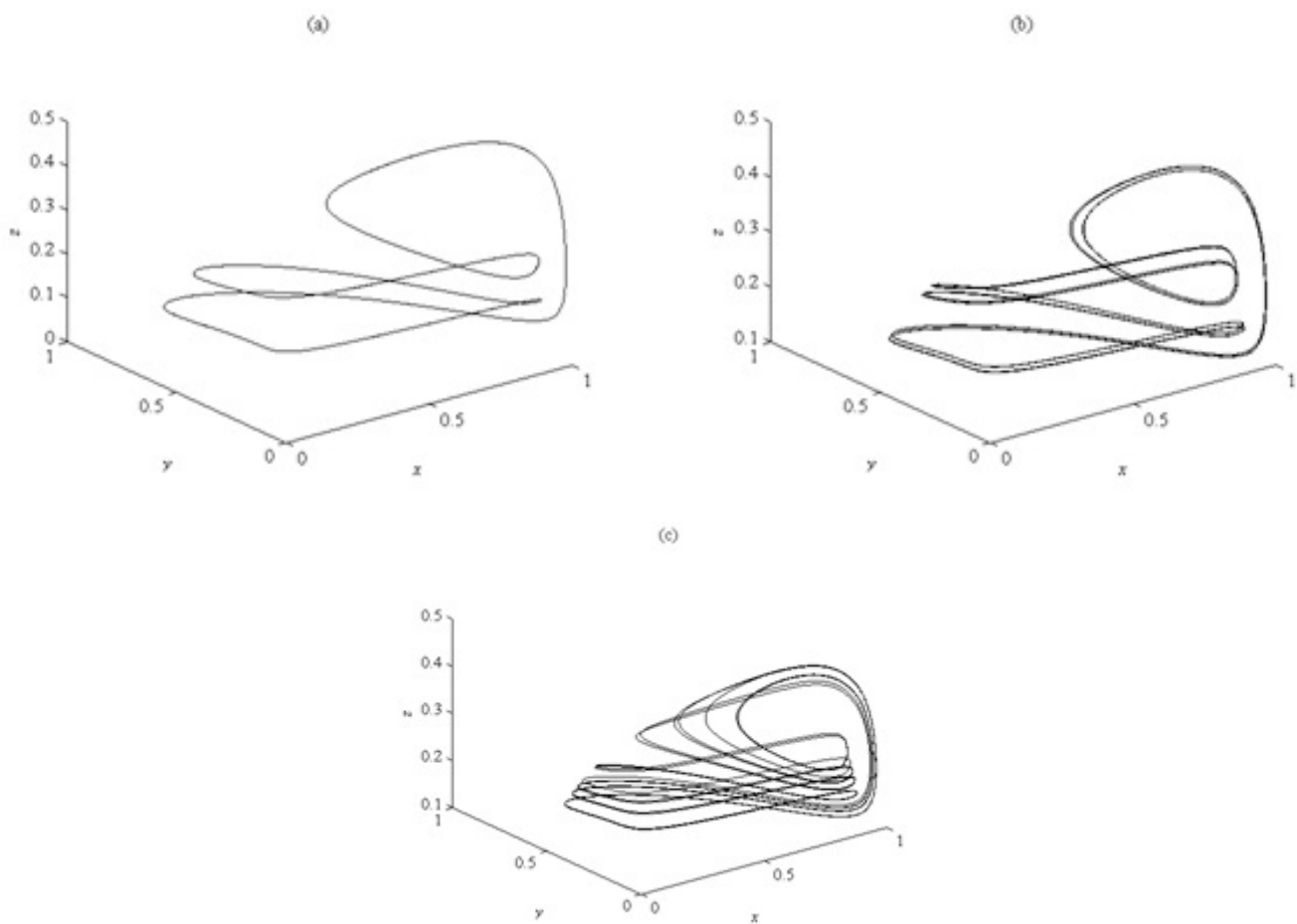
Figure 16: *The route to chaos for the trajectory of the model* (2.2) *using data* (7.1) *with different values of* w_2. *(a) period*-3 *when* $w_2 = 0.73$. *(b) long periodic when* $w_2 = 0.8$. *(c) Strange attractor when* $w_2 = 0.85$.
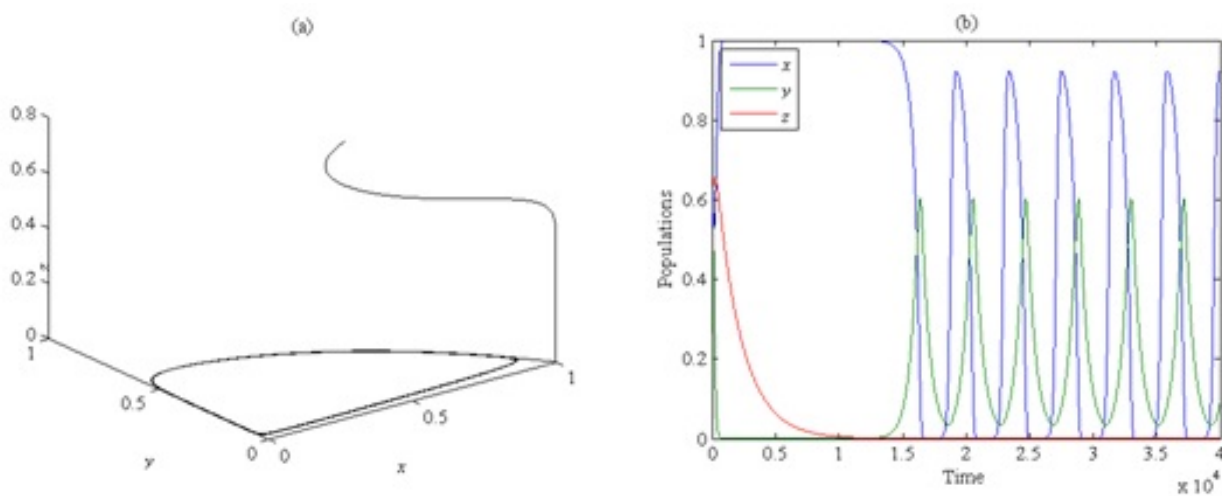

Figure 17: *Trajectory of model* (2.2) *using data* (7.1) *with* $w_2 = 1$ *and* $w_1 = 1$. *(a) Periodic attractor in the xy-plane. (b) Time series of the trajectory in (a).*
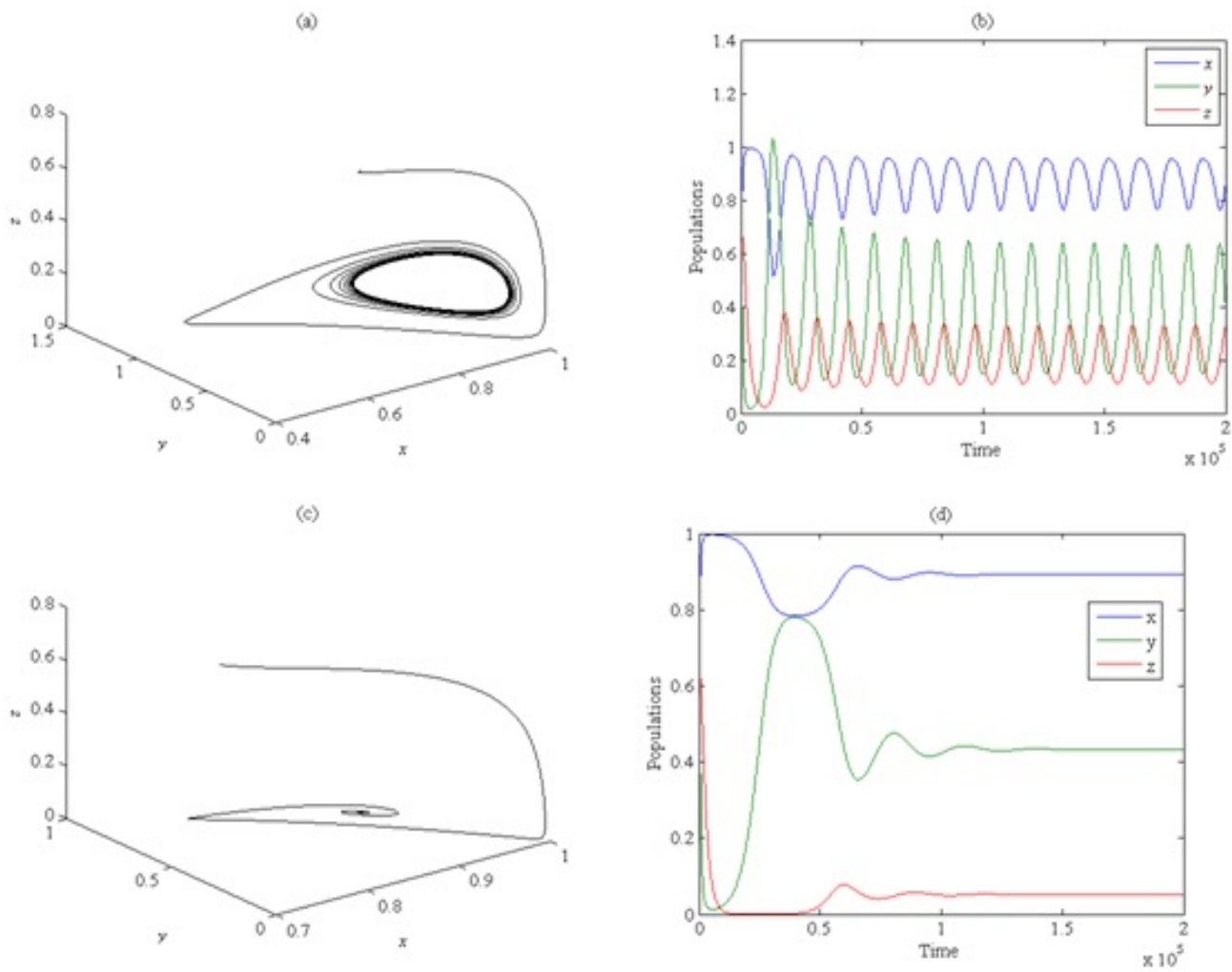
Figure 18: *The trajectory of model* (2.2) *using data* (7.1) *with different values of* w_3 *and* w_7. *(a) Periodic attractor in* 3D *when* $w_3 = 0.75$ *and* $w_7 = 1$. *(b) Time series of the periodic attractor. (c) Asymptotically stable coexistence equilibrium point when* $w_3 = 0.75$ *and* $w_7 = 1.3$. *(d) Time series of the coexistence point attractor.*
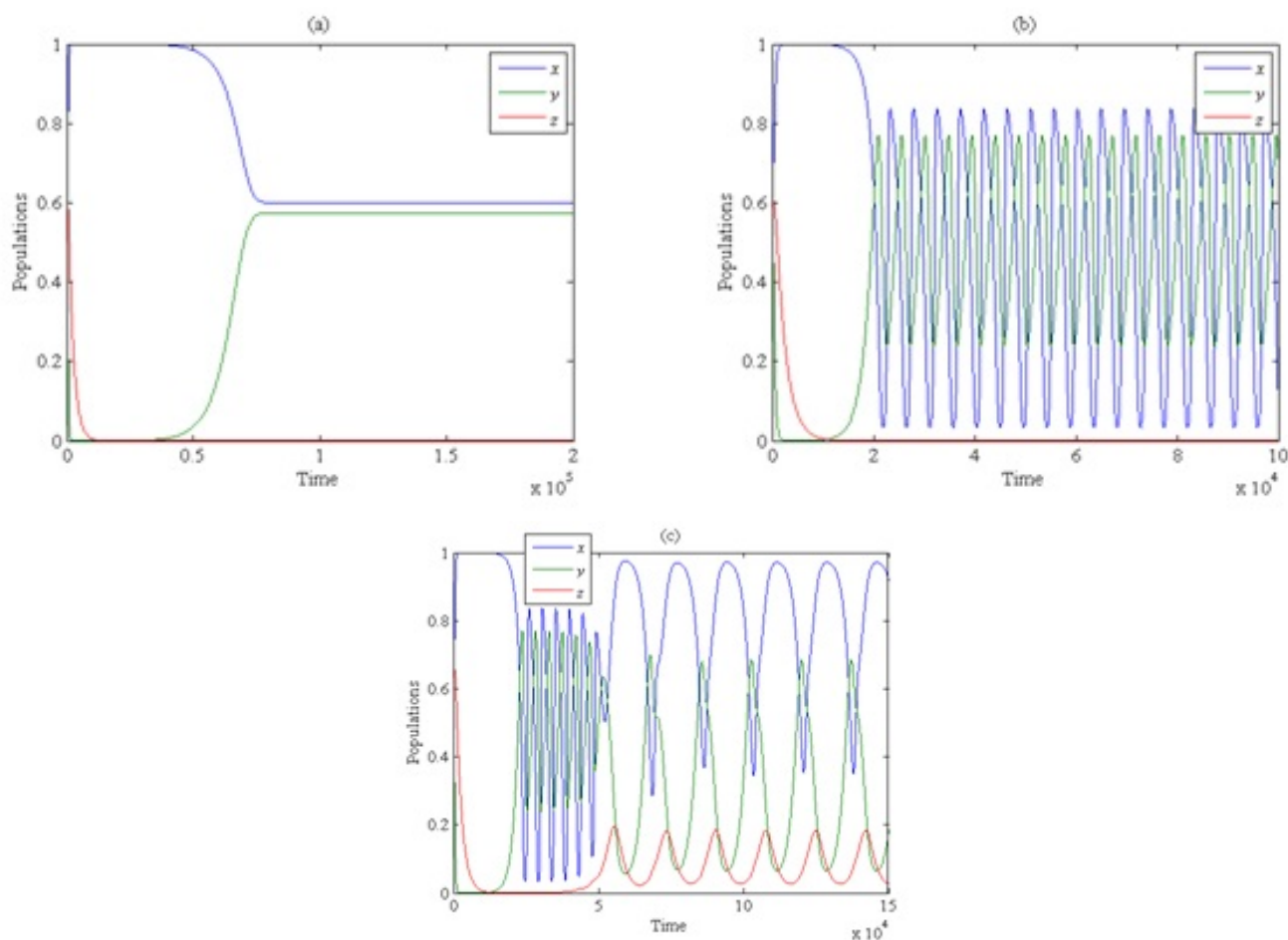
Figure 19: *The trajectory of the model (2.2) using data (7.1) with different values of w_4 and w_9. (a) Approaches to top predator-free stable equilibrium point when $w_4 = 0.2$ and $w_9 = 0.1$. (b) Approaches to periodic attractor in xy-plane when $w_4 = 0.25$ and $w_9 = 0.1.$. (c) Approaches to periodic attractor in 3D space when $w_4 = 0.25$ and $w_9 = 0.15$.*
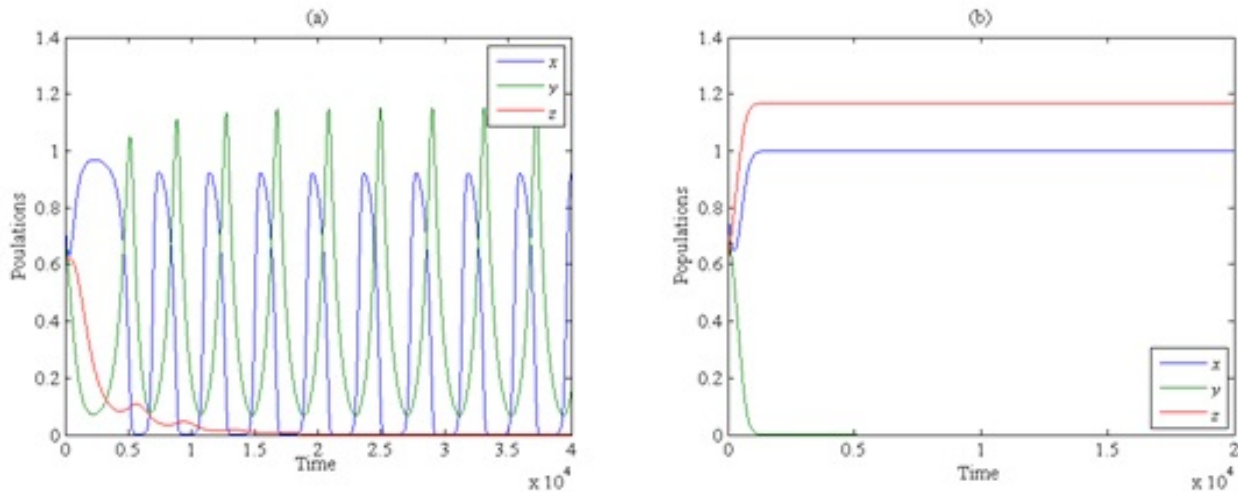
Figure 20: *The trajectory of the model* (2.2) *using data* (7.1) *with different values of* w_{10}. *(a) Approaches to periodic attractor in xy-plane when* $w_{10} = 0.05$. *(c) Approaches to intermediate predator-free equilibrium point in the xz-plane when* $w_{10} = 0.15$.

8. Conclusion and Discussion

In this work, a food chain model with fear in the first and second levels is proposed and analyzed. The effect of alternative food resources for the top predator is also included. The transition of food throughout the food chain is depending on Holling's disc functional response. The properties of the solution of the model are discussed. The equilibrium points of the model are found and then their local stability is performed using the Linearization technique. The persistence conditions are determined. The stability regions for each equilibrium point are determined with the help of the Lyapunov method. Local bifurcation of the model is also studied using the Sotomayer theorem. Finally, the model is numerically simulated in order to understand the global dynamics of the model. Different bifurcation diagrams are drawn to study the effects of varying these parameters on the dynamical behavior of the model. It is observed that the model has different types of dynamics including chaos. Moreover, it is observed that increasing the fear rates in a model works as a reduction factor of the chaotic regions and the solution of model (2.2) stabilizes periodic dynamics in the $3D$ space. On the other hand, increasing the maximum attach rates of predators in the model above a specific level causes loss of persistence and the trajectory approaches periodic dynamics in the xy-plane. However, increasing the half-saturation constants for the intermediate predator and top predator controls the chaotic regions by approaching the periodic attractor first and then the trajectory approaches a coexistence equilibrium point. Also, decreasing the conversion rates of the model reduces the chaos and the solution transfer to periodic first, then the model loses their persistence and the solution approaches to periodic in the interior of xy-plane. Finally, the solution approaches the top predator-free equilibrium point. However, increasing these parameters above specific values leads to periodic dynamics in the interior of $3D$ space. Finally, it is observed that decreasing the alternative resource coefficient leads to extinction in top predator and the trajectory approaches periodic dynamics in the xy-plane. While increasing this coefficient leads to extinction in intermediate predator and the solution approaches to the intermediate predator-free equilibrium point.

References

[1] W. Cresswell,*Predation in bird populations*, J. Ornithol. 152 (2010) 251—263.

[2] N.H. Fakhry and R.K. Naji, *The dynamics of a square root prey-predator model with fear*, Iraqi J. Sci. 61 (2020) 139–146.

[3] S. Gakkhar and R.K. Naji, *On a food web consisting of a specialist and a generalist predator*, J. Biol. Syst. 11 (2003a) 365–376.

[4] K. Garain, U. Kumar and P.S. Mandal, *Global Dynamics in a Beddington–DeAngelis Prey–Predator Model with Density Dependent Death Rate of Predator*, Differ. Equ. Dyn. Syst. 29 (2021) 265–283.

[5] M. Haque, *Ratio-Dependent Predator-Prey Models of Interacting Populations*, Bull. Math. Biol. 71 (2008)430-–452.

[6] A. Hastings and T. Powell, *Chaos in the three-species food chain*, Ecology 72 (1991) 896–903.

[7] C.S. Holling, *The components of predation as revealed by a study of small-mammal predation of the European pine sawfly*, Can. Entomol. 91 (1959) 293–320.

[8] F. Hua, K. E. Sieving, R. J. Fletcher and C. A. Wright, *Increased perception of predation risk to adults and offspring alters avian reproductive strategy and performance*, Behav. Ecol. 25 (2014) 509-–519.

[9] J. Liu, P. Lv, B. Liu and T. Zhang, *Dynamics of a Predator-Prey Model with Fear Effect and Time Delay*, Complex. 2021 (2021) 16 pages.

[10] X. Liu and Y. Lou, *Global dynamics of a predator–prey model*, J. MATH. ANAL. APPL. 371 (2010) 323–340.

[11] J. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications, 3rd edition*, Springer-Verlag, NY, 2003.

[12] R.K. Naji and A.T. Balasim, *Dynamical behavior of a three-species food chain model with Beddington–DeAngelis functional response*, Chaos Solitons Fract. 32 (2007) 1853–1866.

[13] P. Panday, N, Pal, S. Samanta and J. Chattopadhyay, *Stability and bifurcation analysis of a three-species food chain model with fear*, IJBC. 28 (2018) 1850009-1-1850009-20.

[14] S.D. Peacor, B.L. Peckarsky, G.C. Trussell and J.R. Vonesh, *Costs of predator-induced phenotypic plasticity: a graphical model for predicting the contribution of nonconsumptive and consumptive effects of predators on prey*, Oecologia 171 (2012) 1–10.

[15] M.L. Rosenzweig and R.H. MacArthur, *Graphical representation and stability conditions of predator-prey interactions*, Am. Nat. 97 (1963) 209–223.

[16] S.K. Sasmal, *Population dynamics with multiple Allee effects induced by fear factors – A mathematical study on prey-predator interactions*, Appl. Math. Model. 64 (2018) 1-–14.

[17] J.P. Suraci, M. Clinchy, L.M. Dill, D. Roberts and L.Y. Zanette, *Fear of large carnivores causes a trophic cascade*, Nature Commun. 7 (2016) 1–7.

[18] R.K. Upadhyay and R.K. Naji, *Dynamics of a three-species food chain model with Crowley–Martin type functional response*, Chaos Solitons Fract. 42, (2009) 1337-—1346.

[19] X. Wang, L. Zanette and X. Zou, *Modelling the fear effect in predator-prey interactions*, J. Math. Biol. 73 (2016) 1179–1204.

[20] X. Wang, and X. Zou, *Modeling the Fear Effect in Predator-Prey Interactions with Adaptive Avoidance of Predators*, Bull. Math. Biol. 79 (2017) 1325-—1359.

[21] L.Y. Zanette, A. F. White, M. C. Allen and M. Clinchy, *Perceived predation risk reduces the 609 number of offspring songbirds produce per year*, Science, 334 (2011) 1398–1401.

[22] H. Zhang, Y. Cai, S. Fu and W. Wang, *Impact of the fear effect in a prey-predator model incorporating a prey refuge*, Appl. Math. Comput. 356 (2019) 328-–337.