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On approximation by Szasz-Mirakyan-Schurer-Kantrovich operators preserving e^{-bx}, $b > 0$

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Abstract

Through this treatise, a study has been submitted about modified of Szasz-Mirakyan-Schurer-Kantrovich operators which that preserving e^{-bx} , $b > 0$ function. We interpret and study the uniform convergence of the modern operators to f. Also, by analyzing the asymptotic conduct of our operator.

Keywords: Szasz-mirakyan-kantorovich operators, Exponential function, linear positive operators.

1. Introductions

Previous research has shown that approximation theory is frequently based on the best mistakes of existing linear operators. The following operator, as shown below had been introduced by Kantorovich [1]:

$$K_n(f)(x) = K_n(f;x) = (n+1)\sum_{k=0}^{\infty}\binom{n}{k}x^k(1-x)^{n-k}\int_{k/n}^{(k+1)/n} f(t)dt$$

Where $f \in L_1[0,1]$ and $x \in [0,\infty)$.

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Many researchers discussed the approximation problems for Kantorovich type operators [3] and regarding to the same subject, it is necessary start with Szasz definition [5]:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f(\frac{k}{n}), \qquad n>0, \text{ for } x\in[0,\infty]$$

And followed by some modifications which called Szasz-Mirakyan-Kantorovich variants [9, 4, 12].

The introduction of Schurer [13], in 1962, is:

$$K_{r,s}(f;x) = (r+s)\sum_{k=0}^{\infty}\frac{e^{-(r+s)^x((r+s)x)^k}}{k!}\int_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} f(t)dt, \quad \text{where } r\in[0,1] \; and \; s\geq 0$$

These operators motivated several authors to investigate Schurer extensions of other linear positive operators, a few of which can still be found in utilize nowadays [6, 7]. In 2010, linear and positive operator was introduced by Aldaz and Render[2] where exponential function is being preserved. Then the discussing preserving exponential functions of operators took a wide area by authors [1, 5, 8]. During our work was created established a unused generalization of Szasz-mirakyan-Schurer-Kantorovich operator

$$K_{r,s}(f;x) = (r+s)\sum_{k=0}^{\infty}\frac{e^{-(r+s)^x((r+s)x)^k}}{k!}\int_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} f(t)dt, \quad \text{such that } r>0 \; and \; s\in[0,\infty].$$

So, the modified form is

$$Z_{r,s}(e^{-2t},x) = (r+s)\sum_{k=0}^{\infty}\frac{e^{-(r+s)T}(r+s)^{(x)}\Big((r+s)T(r+s)^{(x)}\Big)^k}{k!}\int_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} e^{-2t}dt \tag{1.1}$$

We take into account the modification of the szasz-mirakyan-schurer-kantorovich operators preserving e^{-2x}, leading to the function $T_{r,s}$ which satisfying $K_{r.s}\,(f;x) = K_{r.s}\,(e^{-2x};x) = e^{-2x}$ as follows

$$\begin{aligned}
e^{-2x} &= (r+s)\sum_{k=0}^{\infty}\frac{e^{-(r+s)T_{r,s}(x)}\Big((r+s)T_{r,s}(x)\Big)^k}{k!}\int_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} e^{-2t}dt\\
&= (r+s)\sum_{k=0}^{\infty}\frac{e^{-(r+s)T_{r,s}(x)}\Big((r+s)T_{r,s}(x)\Big)^k}{k!}\left[-\frac{1}{2}e^{-2t}\right]_{\frac{k}{r+s}}^{\frac{k+1}{r+s}}\\
&= (r+s)\sum_{k=0}^{\infty}\frac{e^{-(r+s)T_{r,s}(x)}\Big((r+s)T_{r,s}(x)\Big)^k}{k!}\left[\frac{1}{2}e^{\frac{-2k}{r+s}}\left(1-e^{\frac{-2}{r+s}}\right)\right]\\
&= \frac{1}{2}(r+s)\left(1-e^{\frac{-2}{r+s}}\right)\sum_{k=0}^{\infty}\frac{e^{-(r+s)T_{r,s}(x)}\Big((r+s)T_{r,s}(x)e^{\frac{-2k}{r+s}}\Big)^k}{k!}\\
&= \frac{1}{2}(r+s)\left(1-e^{\frac{-2}{r+s}}\right)e^{(r+s)T_{r,s}(x)\left(1-e^{\frac{-2}{r+s}}\right)}\\
&= e^{\ln\left(\frac{1}{2}(r+s)\left(1-e^{\frac{-2}{r+s}}\right)e^{(r+s)T_{r,s}(x)\left(1-e^{\frac{-2}{r+s}}\right)}\right)}
\end{aligned}$$

Then we have

$$T_{r,s}(x) = \frac{-x - \ln\left((r+s)\left(1 - e^{\frac{-2}{r+s}}\right)\right)}{\frac{1}{2}(r+s)\left(e^{\frac{-2}{r+s}} - 1\right)} \tag{1.2}$$

2. Basic outcomes

Lemma 2.1. *Where $B > 0$ and $s > 0$, $r \geq 0$, for $T_{r,s}$be given by (1.1), then we arrive to*

$$K_{r,s}(e^{Bt}; x) = 2(r+s)e^{(r+s)T_{r,s}(x)\left(-1+e^{\frac{-2B}{r+s}}\right)}\left(\frac{-1+e^{\frac{-2B}{r+s}}}{B}\right) \tag{2.1}$$

Proof . *we've got*

$$\begin{aligned} Z_{r,s}(e^{Bt}, x) &= (r+s)\sum_{k=0}^{\infty} \frac{e^{-(r+s)T}(r+s)^{(x)}\left((r+s)T(r+s)^{(x)}\right)^k}{k!} \int_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} e^{Bt} dt \\ &= (r+s)\sum_{k=0}^{\infty} \frac{e^{-(r+s)T_{r,s}(x)}\left((r+s)T_{r,s}(x)\right)^k}{k!}\left[\frac{-2e^{Bt}}{B}\right]_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} \\ &= \frac{(r+s)}{B}\sum_{k=0}^{\infty} \frac{e^{-(r+s)T_{r,s}(x)}\left((r+s)T_{r,s}(x)\right)^k}{k!}\left[2e^{\frac{-2Bk}{r+s}}\left(1 - e^{\frac{-2B}{r+s}}\right)\right] \\ &= \frac{-2}{B}(r+s)\left(1 - e^{\frac{-2B}{r+s}}\right)e^{(r+s)T_{r,s}(x)}\left(-1 + e^{\frac{-2B}{r+s}}\right) \end{aligned}$$

□

Lemma 2.2. *For $j = 0, 1, 2, \ldots$ $e_j(t) = t^j$, we get the moments as*

$Z_{r.s}(e_0.x) = 1$
$Z_{r.s}(e_1.x) = T_{r,s}(x) + \frac{1}{2n}$
$Z_{r.s}(e_2.x) = (T_{r,s}(x))^2 - \frac{2}{n}T_{r,s}(x) + \frac{1}{3n^2}$

Lemma 2.3. *From Lemma 2.2 and let $Z_{r.s}(\varnothing_x^m(t).x) = Z_{r.s}((t ¸ x)^m ¸ x)$, $m = 0, 1, 2, \ldots$. then*

(i) $Z_{r.s}(\varnothing_x^0(t).x) = 1$
(ii) $Z_{r.s}(\varnothing_x^1(t).x) = T_{r,s}(x) + \frac{1}{2n} - x$
(iii) $Z_{r¸s}(\varnothing_x^2(t)¸ x) = (T_{r¸s}(x))^2 - 2xT_{r¸s}(x) + x^2 + \frac{2}{n}T_{r¸s}(x) - \frac{x}{n} + \frac{1}{3n^2}$

In addition to that, from (1.2)

$$\lim_{r\to\infty}\left[T_{r,s}(x) + \frac{1}{2n} - x\right] = \frac{1}{2}x \tag{2.2}$$

$$\lim_{r\to\infty}\left[(T_{r,s}(x))^2 - 2xT_{r,s}(x) + x^2 + \frac{2}{n}T_{r,s}(x) - \frac{x}{n} + \frac{1}{3n^2}\right] = x \tag{2.3}$$

3. Main results

Theorem 3.1. *[10] For an arrangement of a operators for a linear positive* $L_r : C^*[\ 0\ ¸\ \infty)\ \to C^*[\ 0\ ¸\ \infty)$ *satisfy attached equations.*

$$\begin{aligned} \|L_r\left(e_0\right)-1\|_{[0,\infty)} &= B_r \\ \left\|L_r\left(e^{-t}\right)-e^{-x}\right\|_{[0,\infty)} &= \gamma_r \\ \left\|L_r\left(e^{-2t}\right)-e^{-2x}\right\|_{[0,\infty)} &= \delta_r, \qquad \textit{then,} \end{aligned}$$

$$\|L_r f-f\|_{[0,\infty)} \leq B_r\,\|f\|_{[0,\infty)} + (2+B_r)\,\omega^*(f,\sqrt{B_r+\gamma_r+\delta_r}), \tag{3.1}$$

for every function of $f \in C^*[0,\infty)$.

And the modules of continuity can also be defined in this theorem as

$$\omega^*\left(f,\mu\right) = \sup_{|e^{-x}-e^{-t}|\leq \mu x, t>0} |f(t)-f(x)|.$$

Here B_r, γ_r and δ_r tend to zero as $r \to \infty$.

Theorem 3.2. *let a function* $\in C^*\,[0¸\infty)$ *, we have*
$\left\|Z_{r¸s} - \right\|_{[0,\infty)} \leq 2\omega^*(Z_{r¸s}, \sqrt{\delta_{r¸s}})$*; Were* $\delta_{r¸s}$ *tends to zero as* $r \to \infty$ *and*
$\{\ Z_{r¸s}$ *converges uniformly to the above function.*

Proof . *the szasz-mirakyan-schurer-kantrovich operators* $Z_{r¸s}$ *preserve stability in addition to* $e^{-2bx}¸$ $b>0$ *and upon it,*

$$B_{r¸s} = \|Z_r\left(e_0\right)-1\|_{[0,\infty)} = 0 \ \ \textit{and} \ \ \gamma_{r¸s} = \left\|Z_r\left(e^{-t}\right)-e^{-x}\right\|_{[0,\infty)} = 0.$$

Now we only have to evaluate$\delta_{r,s}$.
From lemma 2.1 we have.

$$\begin{aligned} Z_{r,s}(e^{-2t},x) &= (r+s)\sum_{k=0}^{\infty} \frac{e^{-(r+s)T_{(r+s)}(x)}\left((r+s)T(r+s)^{(x)}\right)^k}{k!} \int_{\frac{k}{r+s}}^{\frac{k+1}{r+s}} e^{-2t}dt \\ &= \frac{(r+s)\left(1-e^{\frac{-2}{r+s}}\right)}{2} e^{(r+s)T_{(r+s)}(x)\left[e^{\frac{-2}{r+s}}-1\right]} \end{aligned}$$

Where $T_{(r+s)}(x)$ *is find by (2) as*

$$T_{r,s}\left(x\right) = \left(2\left(r+s\right)\left(e^{\frac{-1}{r+s}} \ \ -1\right)\right)^{-1} \ \left(-x-\ln\left[\left(r+s\right)\left(1-e^{\frac{-1}{r+s}}\right)\right]\right)$$

To find the right hand part of above equality using the software Mathematica, we get

$$\begin{aligned} Z_{r,s}(e^{-2t},x) &= (r+s)\left(1-e^{\frac{-2}{r+s}}\right) e^{(r+s)T_{(r+s)}(x)\left[e^{\frac{-2}{r+s}}-1\right]} \\ &= (r+s)\left(1-e^{\frac{-2}{r+s}}\right) e^{\left(1-e^{\frac{1}{r+s}}\right)\left(-x-\ln\left[(r+s)\left(1-e^{\frac{-1}{r+s}}\right)\right]\right)} \end{aligned}$$

Hence

$$Z_{r,s}\left(e^{-2t},x\right)-e^{-2x}=\frac{x}{e^{2x}\left(r+s\right)}+\frac{6\left(x^{2}-x-\frac{5}{6}\right)}{e^{2x}\left(r+s\right)^{2}}+O\left(r+s\right)^{-3}$$

Since $\sup_{x\in[0,\infty)} xe^{-2x}=\frac{1}{2}e^{-1}, \quad \sup_{x\in[0,\infty)} x^{2}\, e^{-2x}=\frac{1}{4}e^{-1}$

$$\begin{aligned}\delta_{r,S}&=\left\|Z_{r,s}\left(e^{-2t}\right)-e^{-2x}\right\|_{[0,\infty)}\\&=\sup_{x\in[0,\infty)}\left|Z_{r,s}\left(e^{-2t}\right)-e^{-2x}\right|\\&\le\frac{1}{2}\left(\left(r+s\right)e\right)^{-1}+\left(r+s\right)^{-2}\left(\frac{9}{2}e^{-1}+5\right)+\left(r+s\right)^{-3}\\&\le O\left(r+s\right)^{-1}\end{aligned}$$

Here $\delta_{r,S}$ *tend to zero as* $r\to\infty$.
From that the proof of this theorem is complete. □

Theorem 3.3. *let a function* $f,\acute{f}\in C^{*}[0,\infty)$, *the following inequality holds.*

$$\begin{aligned}&\left|(r+s)[Z_{r,s}(f;x)-f(x)]-\frac{x}{2}\left(\acute{f}(x)-\acute{\acute{f}}\right)\right|\le|p_{r,s}(x)|\left|\acute{f}(x)\right|+|q_{r,s}|\left|\acute{\acute{f}}(x)\right|\\&+2\left(2q_{r,s}(x)+x+l_{r,s}(x)\right)\omega^{*}\left(\acute{\acute{f}},\frac{1}{\sqrt{r+s}}\right)\end{aligned}$$

where

$$\begin{aligned}&p_{r,s}(x)=\frac{2(r+s)k_{r}(\psi_{1/x}(t),x)-x}{2},\quad q_{r,s}(x)=\frac{(r+s)k_{r}(\psi_{2/x}(t),x)-x}{2}\\&l_{r,s}=n^{2}\sqrt{k_{r}\left((e^{-x}-e^{-t})^{4},x\right)k_{r}(\psi_{x}^{4}(t),x)},\end{aligned}$$

Proof . *from "Taylor's expansion of* f *at the point* x*" we've got*

$$f(t)=f(x)+\acute{f}(x)(t-x)+\frac{\acute{\acute{f}}(x)}{2}(t-x)^{2}+h(t,x)(t-x)\tag{3.2}$$

Where $h(t,x)=\frac{\acute{\acute{f}}(\lambda)-\acute{\acute{f}}(x)}{2}$, λ *may be a number that falls within the period specified by* x *and* t.
Whenever apply Z_{u} *on all side of Taylor's expansion* (3.2) *we can get*

$$Z_{r,s}(f,x)-f(x)-\acute{f}(x)Z_{r,s}(e_{1}^{x}(t),x)-\frac{\acute{\acute{f}}(x)Z_{r,s}(e_{2}^{x}(t),x)}{2}=Z_{r,s}(h(t,x)e_{2}^{x}(t),x)\tag{3.3}$$

using lemma 2.3 we have

$$\begin{aligned}&\left|(r+s)[Z_{r,s}(f;x)-f(x)]-\frac{x}{2}\left(\acute{f}(x)-\acute{\acute{f}}\right)\right|\le\left|\frac{f(x)}{2}\right||2(r+s)Z_{r,s}(e_{1}^{x}(t),x)+x|\\&\left|\frac{\acute{\acute{f}}(x)}{2}\right||(r+s)Z_{r,s}(e_{2}^{x}(t),x)-x|+|(r+s)Z_{r,s}\left(h(t,x)(t-x)^{2},x\right)|.\end{aligned}$$

We define $p_{r,s}(x)$, $q_{r,s}(x)$ *by the following equations,*

$$p_{r,s}(x) = \frac{1}{2}[2(r+s)Z_r(e_1^x(t),x) - x], \qquad q_{r,s}(x) = \frac{(r+s)Z_r(e_2^x(t),x) - x}{2}$$

Those

$$\left|(r+s)[Z_{r,s}(f;x) - f(x)] - \frac{x}{2}\left(\acute{f}(x) - \acute{\acute{f}}\right)\right| \leq |\acute{f}(x)||p_{r,s}(x)| + |\acute{\acute{f}}(x)||q_{r,s}(x)| + \\ |(r+s)Z_{r,s}\left(h(t,x)(t-x)^2, x\right)|. \tag{3.4}$$

By using (2.2) *and* (2.3), *it is possible noted that if we it is expected* $r \to \infty$, $p_{r,s}(x)$ *and* $q_{r,s}(x)$ *approaches zero when* x *is anywhere.*
To reach the end of the proof, must be counted the term $|(r+s)Z_{r,s}\left(h(t,x)(t-x)^2, x\right)|$.
From the property

$$|f(t) - f(x)| \leq \left(1 - \left(\frac{e^{-x} - e^{-t}}{\delta}\right)^2\right)\omega^*(\acute{\acute{f}}, \delta),$$

Here for $|e^{-x} - e^{-t}| \leq \delta$ *and* $\delta > 0$ *than* $|h(t,x)| \leq 2\omega^*\left(\acute{\acute{f}}, \delta\right)$,
or if $|e^{-x} - e^{-t}| > \delta$ *than* $|h(t \text{ ¸ } x)| \leq 2\left(\frac{e^{-x}-e^{-t}}{\delta}\right)^2 \omega^*(\acute{\acute{f}}, \delta)$ *so we get*

$$|h(t,x)| \leq 2 + 2\left(\frac{e^{-x} - e^{-t}}{\delta}\right)^2 \omega^*(\acute{\acute{f}}, \delta)$$

By using (3.4) *we have*

$$(r+s)Z_{r,s}\left(h(t,x)(t-x)^2, x\right) \leq 2(r+s)\omega^*(\acute{\acute{f}}, \delta)Z_{r,s}(e_2^x(t), x) + \\ 2(r+s)\delta^{-2}\omega^*(\acute{\acute{f}}, \delta)\sqrt{Z_{r,s}\left((e^{-x} - e^{-t})^2 e_2^x(t), x\right)}$$

By "applying Cauchy-Schwarz inequality", we obtain

$$(r+s)Z_{r,s}\left(h(t,x)(t-x)^2, x\right) \leq 2(r+s)\omega^*(\acute{\acute{f}}, \delta)Z_{r,s}(e_2^x(t), x) + \\ 2(r+s)\delta^{-2}\omega^*(\acute{\acute{f}}, \delta)\sqrt{Z_{r,s}\left((e^{-x} - e^{-t})^4, x\right) Z_{r,s}\left(e_2^x(t), x\right)}$$

Through some mathematical operations we can find

$$\begin{aligned} Z_{r,s}\left(e_x^4(t), x\right) = e^{\frac{-1}{r+s}\left(4+4(r+s)x+(r+s)^2 T_{r,s}\right)} \Bigg[& e^{(r+s)T_{r,s} + \frac{4}{r+s}} - \frac{1}{4}(r+s)e^{4x+(r+s)T_{r,s}e^{\frac{-4}{r+s}}} \\ & + \frac{1}{4}(r+s)e^{4x+(r+s)T_{r,s}e^{\frac{-4}{r+s}} + \frac{4}{r+s}} + \frac{4}{3}(r+s)e^{3x+(r+s)T_{r,s}e^{\frac{-3}{r+s}} + \frac{1}{r+s}} \\ & - \frac{4}{3}(r+s)e^{3x+(r+s)T_{r,s}e^{\frac{-3}{r+s}} + \frac{4}{r+s}} - 3(r+s)e^{2x+(r+s)T_{r,s}e^{\frac{-2}{r+s}} + \frac{2}{r+s}} \\ & + 3(r+s)e^{2x+(r+s)T_{r,s}e^{\frac{-2}{r+s}} + \frac{4}{r+s}} + 4(r+s)e^{x+(r+s)T_{r,s}e^{\frac{-1}{r+s}} + \frac{3}{r+s}} \\ & - 4(r+s)e^{x+(r+s)T_{r,s}e^{\frac{-1}{r+s}} + \frac{4}{r+s}} \Bigg] \end{aligned} \tag{3.5}$$

And

$$Z_{r,s}\left((e^{-x}-e^{-t})^4,x\right)=\frac{1}{4}\left[1-5x(r+s)+10x^2(r+s)^2-10x^3(r+s)^3+5x^4(r+s)^4\right.$$
$$+30(r+s)T_{r,s}-70x(r+s)^2T_{r,s}+60x^2(r+s)^3T_{r,s}-20x^3(r+s)^4T_{r,s} \qquad (3.6)$$
$$+75(r+s)^2T^2_{r,s}(x)-90x(r+s)^3T^2_{r,s}(x)+30x^2(r+s)^4T^2_{r,s}(x)$$
$$\left.+40(r+s)^3T^3_{r,s}(x)-20x(r+s)^4T^3_{r,s}(x)+5(r+s)^4T^4_{r,s}(x)\right]$$

Choosing $\delta=\frac{1}{\sqrt{r+s}}$ *and setting*

$$l_{r,s}(x)=(r+s)^2\sqrt{Z_{r,s}\left((e^{-x}-e^{-t})^4,x\right)Z_{r,s}\left((t-x)^4,x\right)}$$

From this we can reach to the desired result. □

Theorem 3.4. *Let* $f,\acute{\acute{f}}\in C^*[0,\infty)$ *then for* $x\in[0,\infty)$ *the following statement is valid*

$$\lim_{r\to\infty}(r+s)[Z_{r,s}(f,x)-f(x)]=\frac{x}{2}\left(\acute{f}(x)-\acute{\acute{f}}(x)\right).$$

Proof . *from "Taylor's expansion of f" can wright*

$$f(t)=f(x)+\acute{f}(x)(t-x)+\frac{\acute{\acute{f}}(x)}{2}(t-x)^2+\varphi(t,x)(t-x)^2$$

Where

$$\varphi(t,x)=f(t)(t-x)^{-2}-\acute{f}(t)(t-x)-\frac{x}{2}\acute{\acute{f}}(t)$$

Since $\varphi(\ x\ ¸\ x\ )=0$ *and this function* $\varphi\left(.,x\right)\in C^*\left[0,\infty\right)$.
By lemma 2.3 we can say that

$$\lim_{r\to\infty}(r+s)[Z_{r,s}(f,x)-f(x)]=\left(T_{r,s}(x)+\frac{1}{2n}-x\right)\left(\acute{f}(x)-\acute{\acute{f}}(x)\right)+(r+s)Z_{r,s}(\varphi(t,x)(t-x)^2;x)$$

We can count on "Cauchy- Schwarz inequality" we deduce that

$$Z_{r,s}(\phi^2_x(t)\varphi(t,x);x)\le\sqrt{Z_{r,s}(\varphi^2(t,x);x)}\sqrt{Z_{r,s}((t-x)^4;x)}$$

We can also calculate that

$$\lim_{r\to\infty}Z_{r,s}\left(\varphi^2\left(t,x\right);x\right)=\varphi^2\left(x,x\right)=0$$

This leads to

$$\lim_{r\to\infty}(r+s)Z_{r,s}\left(\varnothing^2_x(t)\varphi\left(t,x\right);x\right)=\varphi^2\left(x,x\right)=0 \qquad (3.7)$$

Consequently,

$$\lim_{r\to\infty}(r+s)[Z_{r,s}(f,x)-f(x)]=\lim_{r\to\infty}\left(T_{r,s}(x)+\frac{1}{2n}-x\right)\left(\acute{f}(x)-\acute{\acute{f}}(x)\right)+(r+s)Z_{r,s}(\varphi(t,x)(t-x)^2;x)$$

From the facts above and $\lim_{r\to\infty}\left[T_{r¸s}\left(x\right)+\frac{1}{2n}-x\right]=\frac{1}{2}x$, *the required results can be obtained.* □

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