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Some results on fuzzy soft sesquilinear functional

Salim Dawood Mohsen[a], Jaafer Hmood Eidi[a,*], Ayed E. Hashoosh[b]

[a] Mustansiriyah University, College of Education, Baghdad-Iraq
[b] Thi-Qar University, College of Education of Pure Sciences, Iraq

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Abstract

In this paper, we study and discussion new kinds of Sesquilinear functional which is fuzzy soft Sesquilinear functional and given some properties with characterization and also theories related on fuzzy soft Sesquilinear functional have been given. Additionally, we present the relationship between this kind and other kinds.

Keywords: Fuzzy soft set, fuzzy soft Hilbert space, fuzzy soft adjoint operator
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1. Introduction

Functional analysis is a branch of pure mathematics. It was first developed in about century ago. It aims to solve many problems in pure mathematics. Therefore it provides an indispensable tool for solving those problems. It also provides us with techniques for estimating error in the solutions of infinite and finite dimensional problems.

In our everyday life, we often faced with uncertainty that arises from the ambiguity of the phenomenon under study. This type of problems arises in areas like economics, medical science, business and engineering. Our classical mathematical methods often fail to tackle such problems.

The mathematical models related to real-world is too problematical and we cannot usually to find the exact solutions [4]. Then may be interested to use the concept 0f approximate approach to compute their solutions by using some mathematical tools in Hilbert space like, fuzzy, soft, or fuzzy

*Corresponding author
Email addresses: Salim78@uomustansiriyah.edu.iq (Salim Dawood Mohsen), drjaffarmath@uomustansiriyah.edu.iq (Jaafer Hmood Eidi), ayed197991@yahoo.com (Ayed E. Hashoosh)

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soft 0f set theory [7].

Thus, In 1965 , a generalization of set theory was introduced, by Zadeh [14]. The resulting theory was called fuzzy set theory. Fuzzy set theory soon became an excellent tool to deal with problems that associate with uncertainty. In classical set theory, a set X is define with its characteristic function from X to set $\{0,1\}$. On the other hand, in fuzzy set theory, a set is defined with its membership function form X to the closed interval $[0,1]$.

Also In 1999, a yet another generalization was introduced by Molodsov [9] to deals with uncertainty. The resulting theory was called soft set theory. Since then, it was used to solve complicated problems in subjects like computer science, medicine, engineering, etc. A soft set is a parametrized collection of a universal set. The concept soft set was then applied on various mathematical concepts in functional analysis resulting in concepts like soft point [1], Soft normed Spaces [13], Soft Inner Product Spaces [3], Soft Hilbert space [12] and Projection operators on soft inner product spaces[11].

In 2001, Maji et.al [8], was first to introduce the concept of a fuzzy soft set. The concept resulted from combining the concept of a fuzzy and a soft set. The necessity of combining the two concepts was to provide more accurate and general results. The other concepts consequently followed this structure resulting in the introduction fuzzy soft point [10] and fuzzy soft normed spaces [2].

In 2020, the fuzzy soft Hilbert space [4] were introduce by Faried et al. In addition Fuzzy soft linear operators [6]. and finally fuzzy soft self $-$ adjoint operators [5] and studied its properties.

In this paper, Introduce a new kind is said to be fuzzy soft Sesquilinear functional, and given some theorems relating to this functional with properties.

2. BASIC CONCEPTS

Definition 2.1. [14] *If X universe set and $\hat{A}$ is a set characterized by a membership function $\mu_{\widehat{\mathrm{A}}} : \mathcal{X} \to \mathfrak{T}$, such as $\mathfrak{T} = [0,1]$ then $\hat{A}$ is said to be fuzzy set over X, and $\hat{A} = \left\{ \frac{\mu_{\hat{A}}(x)}{x} : x \in \mathcal{X} \right\}$.
And $\mathfrak{T}^x = \{\hat{A} : \hat{A}$ is a function from X into $\mathfrak{T}\}$*

Definition 2.2. [9] *Let $\mathcal{P}(X)$ be the power set of universe set X and E be set of parameters and $A \subseteq E$. The mapping $\mathcal{G} : A \to \mathcal{P}(X)$, where $(\mathcal{G}, A) = \{\mathcal{G}(a) \in \mathcal{P}(X) : a \in A\}$. The pair $(\mathcal{G}, A)$ is said to be soft set.*

Definition 2.3. [8] *The soft set (G, A) is called fuzzy soft set ($\mathcal{FS} -$ set) over X, whenever $\mathcal{G} : A \to \mathfrak{T}^x$,and $\{\mathcal{G}(a) \in \mathfrak{T}^x : a \in A\}$.
The collection of every $\mathcal{FS}-$ sets , denoted by $\mathcal{FSS}(\widetilde{\mathcal{X}})$*

Definition 2.4. [10] *If $(\mathcal{G}, A) \in \mathcal{FSS}(\tilde{X})$ is called $\mathcal{FS}$ -point over X, symbolized by $\tilde{a}_{\mu_{\mathcal{G}(e)}}$, if $e \in A$ and $a \in X$,*

$$\mu_{\mathcal{G}(e)}(x) = \begin{cases} \delta & , \text{if } a = a_o \in X \text{ and } e = e_0 \in A \\ 0 & , \text{if } a \in X - \{a_0\} \text{ or } e \in A - \{e_0\} \end{cases} , \text{ where } \delta \in (0,1]$$

Remark 2.5. [10] *The collection of every $\mathcal{FS}-$ Complex numbers denoted by $\tilde{\mathcal{C}}(A)$ and the collection of every $\mathcal{FS}-$ Real numbers denoted by $\tilde{\mathcal{R}}(A)$.*

Definition 2.6. [2] *The mapping* $\|\tilde{.}\| : \widetilde{X} \to \mathcal{R}(A)$ *where* $\widetilde{X}$ *is* $\mathcal{FS}-$ *vector space is said to be* $\mathcal{FS}-$ *norm on* $\widetilde{\mathcal{X}}$ *if* $\|\tilde{.}\|$ *satisfies:*

(1) $\left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\| \geq \tilde{0}$, *for all* $\tilde{a}_{\mu_{G(e)}} \in \widetilde{X}$, *and* $\left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\| = \tilde{0}$ *if and only if* $\tilde{a}_{\mu_{G(e)}} = \tilde{\theta}$

(2) $\left\|\tilde{r}\tilde{\tilde{a}}_{\mu_{\mathcal{G}(e)}}\right\| = |\tilde{r}| \left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\|$, *for all* $\tilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}, \tilde{r} \in \mathcal{C}(A)$

(3) $\left\|\tilde{a}_{\mu_{\mathcal{G}(e)}} \tilde{+} \tilde{b}_{\mu_2 g(e_2)}\right\| \leq \left\|\widetilde{\tilde{a}_{\mu_{G(e)}}}\right\| + \left\|\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right\|, \forall \tilde{a}_{\mu_{\mathcal{G}(e)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$

Then $(\widetilde{\chi}, \|\tilde{.}\|)$ is called $\boldsymbol{\mathcal{FS}}-$ normed vector space ($\mathcal{FSN}-$ space)

Definition 2.7. [4] *The mapping* $\widetilde{\langle ., . \rangle} : \widetilde{X} \times \widetilde{X} \to (\mathcal{C}(A)$ *or* $\mathcal{R}(A))$ *where* $\widetilde{X}$ *is* $\mathcal{FSV}-$ *space is called* $\mathcal{FS}-$ *inner product on* $\widetilde{X}(\mathcal{FS}\mathfrak{T})$ *if* $\widetilde{(\ldots)}$ *satisfies:*

(1) $\left\langle \widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}} \tilde{a}_{\mu_{\mathcal{G}(e)}}} \right\rangle \geq \tilde{0}$, *for all* $\tilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\chi}$ *and* $\left\langle \tilde{a}_{\mu_{\mathcal{G}(e)}, \tilde{a}_{\mu_{\mathcal{G}(e)}}} \right\rangle = \tilde{0}$ *if and only if* $\tilde{a}_{\mu_{\mathcal{G}(e)}} = \tilde{\theta}$

(2) $\left\langle \widetilde{\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}} \right\rangle = \overline{\left\langle \widetilde{\tilde{b}_{\mu_2\zeta(e_2)}, \tilde{a}_{\mu_1\mathcal{G}(e_1)}} \right\rangle}$, *for all* $\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \tilde{\chi}$,

(3) $\left\langle \widetilde{\tilde{\alpha}\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}} \right\rangle = \tilde{\alpha} \left\langle \widetilde{\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}} \right\rangle$, *for all* $\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\chi}$, *for all* $\tilde{\alpha} \in \mathcal{C}(A)$.

(4) $\left\langle \widetilde{\tilde{a}_{\mu_1\mathcal{G}(e_1)} + \tilde{b}_{\mu_2\mathcal{G}(e_2)}, \tilde{c}_{\mu_{3\mathcal{G}(e_3)}}} \right\rangle = \left\langle \widetilde{\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{c}_{\mu_{3\mathcal{G}(e_3)}}} \right\rangle + \left\langle \widetilde{\tilde{b}_{\mu_2\mathcal{G}(e_2)}, \tilde{c}_{\mu_{3\mathcal{G}(e_3)}}} \right\rangle$

For all $\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \tilde{c}_{\mu_{3\mathcal{G}(e_3)}} \in \widetilde{\mathcal{X}}$
Then $(\widetilde{\chi}, \widetilde{\langle ., . \rangle})$ *is called* $\mathcal{FS}-$ *inner product space* ($\mathcal{FS}\mathfrak{T}-$ *space*)

Definition 2.8. [7] *The* $\mathcal{FSN}-$ *space* $(\tilde{X}, \widetilde{\|\cdot\|})$ *is called* $\boldsymbol{\mathcal{FS}}$ *-complete if all* $\mathcal{FS}$ *- Cauchy sequence in* X *is* $\mathcal{FS}$ *- convergence in* X.

Definition 2.9. [4] *The* $\mathcal{FS}-$ *complete inner product space* $(\widetilde{\mathcal{X}}, \widetilde{\langle ., . \rangle})$ *is called* $\mathcal{FS}$ *-Hilbert space* ($\mathcal{FSH}-$ *space*), *and symbolized by* $(\widetilde{\mathcal{H}}, \widetilde{\langle , \rangle})$.

Definition 2.10. *If* $\widetilde{\mathcal{H}}$ *be* $\mathcal{FSH}-$ *space and* $\widetilde{\mathfrak{S}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *be* $\mathcal{FS}-$ *operator .Then* $\widetilde{\mathfrak{S}}$ *is called* $\mathcal{FS}-$ *linear operator* ($\mathcal{FSL}-$ *operator*) *if:*
$\widetilde{\mathfrak{S}}\left(\tilde{\alpha}\tilde{a}_{\mu_1\mathcal{G}(e_1)} + \tilde{\beta}\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right) = \tilde{\alpha}\widetilde{\mathfrak{S}}\left(\tilde{a}_{\mu_1\mathcal{G}(e_1)}\right) + \tilde{\beta}\left(\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right)$ *for all* $\tilde{a}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_2\mathcal{G}(e_2)} \in \widetilde{\mathcal{H}}$ *and* $\tilde{\alpha}, \tilde{\beta} \in \mathcal{C}(A)$

Definition 2.11. *If* $\widetilde{\mathcal{H}}$ *be* $\mathcal{FSH}-$ *space and* $\widetilde{\mathfrak{S}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *be* $\mathcal{FS}$ *- operator is called* $\mathcal{FS}-$ *bounded operator, if* $\exists \tilde{m} \in \mathcal{R}(A)$ *such that* $\left\|\widetilde{\mathfrak{S}\left(\tilde{a}_{\mu_1\mathcal{G}(e_1)}\right)}\right\| \leq \tilde{m} \left\|\widetilde{\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}}\right\|$, *for all* $\tilde{a}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}$.

Now, the family of all $\mathcal{FS}-$ bounded linear operators denoted by $\tilde{B}(\widetilde{\mathcal{H}})$.

Example 2.12. [6] *The* $\mathcal{FS}-$ *operator* $\tilde{I} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *defined by* $\tilde{I}\left(\tilde{a}_{\mu_1\mathcal{G}(e_1)}\right) = \tilde{a}_{\mu_1\mathcal{G}(e_1)}, \forall \tilde{a}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}$, *it is called* $\mathcal{FS}$ *-identity operator.*

Definition 2.13. [6] *If $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-$ space and $\tilde{\mathfrak{S}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ be $\mathcal{FSB}-$ operator, then The $\mathcal{FS}-$ adjoint operator $\tilde{\mathfrak{S}}^*$ is defined by*
$\left\langle \widetilde{\tilde{\mathfrak{S}}\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2G(e_2)}}} \right\rangle = \left\langle \tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \widetilde{\overline{\tilde{\mathfrak{S}}^*}\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}} \right\rangle$

Theorem 2.14. [6] *If $\widetilde{\mathfrak{S}}, \tilde{\varphi} \in \tilde{B}(\widetilde{\mathcal{H}})$,where $\widetilde{\mathcal{H}}$ is $\mathcal{FSH}-$ space and $\tilde{\beta} \in \mathcal{C}(A)$, then $\widetilde{\mathfrak{S}}^{*^*} = \widetilde{\mathfrak{S}}, (\tilde{\beta}\widetilde{\mathfrak{S}})^* = \overline{\tilde{\beta}}\widetilde{\mathfrak{S}}^*, (\widetilde{\mathfrak{S}} + \tilde{\varphi})^* = \widetilde{\mathfrak{S}}^* + \tilde{R}^*$ and $(\widetilde{\mathfrak{S}}\tilde{\varphi})^* = \tilde{\varphi}^*\widetilde{\mathfrak{S}}^*$*

Definition 2.15. [5] *The $\mathcal{FS}$ - operator $\widetilde{\mathfrak{S}}$ of $\mathcal{FSH}-$ space $\widetilde{\mathcal{H}}$ is called $\mathcal{FS}-$ self adjoint operator if $\widetilde{\mathfrak{S}} = \widetilde{\mathfrak{S}}^*$.*

3. MAIN RESULTS

Definition 3.1. *Let $\widetilde{\mathcal{H}}$ be $\mathcal{FSH}-$ space . A mapping $\widetilde{\tilde{a}(.,.)} : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \to \mathcal{C}(A)$ is called a fuzzy soft Sesquilinear functional ($\mathcal{FS}-$ sesquilinear functional) if the following conditions are satisfied:*

1. $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \tilde{y}_{\mu_{2\mathcal{G}(e_2)}}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right) = \tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right) + \tilde{a}\left(\widetilde{\tilde{x}_{\mu_1\zeta(e_1)} + \tilde{y}_{\mu_{2\mathcal{G}(e_2)}}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right)$
2. $\tilde{a}\left(\widetilde{\tilde{\beta}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) = \tilde{\beta}\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right)$
3. $\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_{2\mathcal{G}(e_2)}} + \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) = \tilde{a}\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}\overline{\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) + \tilde{a}\left(\tilde{x}_{\mu_1\zeta(e_1)'}\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)$
4. $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\beta}\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) = \overline{\tilde{\beta}}\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right)$

Remark 3.2.
1. *The $\mathcal{FS}-$ sesquilinear functional is $\mathcal{FS}$ -Linearity in the first variable, but not in the second variable. $\mathcal{FS}-$ sesquilinear functional Which is also $\mathcal{FS}$ -linear in second variable is said to be $\mathcal{FS}$ -bilinear form or a $\mathcal{FS}-$ bilinear functional. Thus, $\mathcal{FS}-$ bilinear form $\widetilde{\tilde{a}(.,.)}$ is a mapping defined on $\widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \to \mathcal{C}(A)$ which satisfies condition (1) through (3) of Definition 3.1 and (4) $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\beta}\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) = \tilde{\beta}\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right)$*
2. *If $\widetilde{\mathcal{H}}$ is a real $\mathcal{FSH}-$ space, then the concepts of $\mathcal{FS}-$ sesquilinear functional and $\mathcal{FS}-$ bilinear forms coincide.*
3. *If $\widetilde{\tilde{a}(.,.)}$ is $\mathcal{FS}-$ sesquilinear function, so $\widetilde{g}(x,y) = \widetilde{\tilde{a}(.,.)}$ is $\mathcal{FS}-$ sesquilinear functional*

Definition 3.3. *suppose that $\widetilde{\tilde{a}(.,.)}$ is $\mathcal{FS}-$ bilinear forms. Then*

1. *$\widetilde{\tilde{a}(.,.)}$ is said to be $\mathcal{FS}-$ symmetric if $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) = \tilde{a}\left(\widetilde{\tilde{y}_{\mu_{2\mathcal{G}(e_2)}}, \tilde{x}_{\mu_1\mathcal{G}(e_1)}}\right)$ $\forall \left(\tilde{x}_{\mu_1G(e_1)}, \tilde{y}_{\mu_{2gG(e_2)}}\right) \in \widetilde{\mathcal{H}} \times \tilde{\mathcal{H}}$*
2. *$\widetilde{\tilde{a}(.,.)}$ is called $\mathcal{FS}-$ positive if $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{x}_{\mu_1\mathcal{G}(e_1)}}\right) \geq \tilde{0}$ $\forall \tilde{x}_{\mu_{1\mathcal{G}(e_1)}} \in \widetilde{\mathcal{H}}$.*
3. *$\widetilde{\tilde{a}(.,.)}$ is called $\mathcal{FS}-$ positive definite if $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{x}_{\mu_1\mathcal{G}(e_1)}}\right) \geq \tilde{0}$ $\forall \tilde{x}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}$ and $\tilde{a}\left(\widetilde{\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{x}_{\mu_1\mathcal{G}(e_1)}}\right) = \tilde{0}$ implies that $\tilde{x}_{\mu_{1\mathcal{G}(e_1)}} = \tilde{0}$*

4. $\widetilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)=\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)}$ *is called* $\mathcal{FS}$ *- quadratic form.*

5. $\tilde{a}\widetilde{(.,.)}$ *is called* $\mathcal{FS}-$ *bounded or* $\mathcal{FS}-$ *continuous if there exists* $\widetilde{M}\in\mathcal{R}(A)$ *such that* $\left|\tilde{a}\widetilde{\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)}\right|\leq\widetilde{M}\left\|\widetilde{\tilde{x}_{\mu_{1g(e_1)}}}\right\|\left\|\widetilde{\tilde{y}_{\mu_{2g(e_2)}}}\right\|$

6. $\tilde{a}\widetilde{(.,.)}$ *is said to be* $\mathcal{FS}-$ *coercive* ($\widetilde{\mathcal{H}}-$ *coercive*) *if there exists* $\tilde{\lambda}\in\mathcal{R}(A)$ *such that* $\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)}\geq\tilde{\lambda}\left\|\widetilde{\tilde{x}_{\mu_{1g(e_1)}}}\right\|^2\forall\tilde{x}_{\mu_{1g(e_1)}}\in\widetilde{\mathcal{H}}$

7. *7. A* $\mathcal{FS}-$ *quadratic form* $\widetilde{\mathfrak{F}}$ *is called* $\mathcal{FS}-$ *real if* $\widetilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1G(e_1)}\right)$ *is* $\mathcal{FS}-$ *real for all* $\tilde{x}_{\mu_{1g(e_1)}}$ $\widetilde{\mathcal{H}}$

Remark 3.4.

1. *If* $\tilde{a}\widetilde{(.,.)}:\widetilde{\mathcal{H}}\times\widetilde{\mathcal{H}}\to\mathcal{R}(A)$, *then the* $\mathcal{FS}-$ *bilinear form* $\tilde{a}\widetilde{(.,.)}$ *is* $\mathcal{FS}-$ *symmetric if* $\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)=\tilde{a}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)$

2.
$$\begin{aligned}\widetilde{\|\tilde{a}\|}&=\sup_{\tilde{x}_{\mu_1\mathcal{G}(e_1)}}=\widetilde{0},\tilde{y}_{\mu_2\mathcal{G}(e_2)}=\widetilde{0}\frac{\left|\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)}\right|}{\left\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right\|\left\|\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right\|}\\&=\sup_{\tilde{x}_{\mu_1\mathcal{G}(e_1)}=\widetilde{0},\tilde{y}_{\mu_2\mathcal{G}(e_2)}=\widetilde{0}}\left|\tilde{a}\widetilde{\left(\frac{\tilde{x}_{\mu_1\mathcal{G}(e_1)}}{\left\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right\|},\frac{\tilde{y}_{\mu_{2\mathcal{G}(e_2)}}}{\left\|\tilde{y}_{\mu_{2\mathcal{G}(e_2)}}\right\|}\right)}\right|\\&=\sup_{\left\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right\|=\widetilde{1},\left\|\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right\|=\widetilde{1}}\left|\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\right|\end{aligned}$$

It is clear that $\left|\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)}\right|\leq\|\widetilde{\tilde{a}}\|\left\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right\|\left\|\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right\|$

3. $\|\widetilde{\mathfrak{F}}\|=\sup_{\tilde{x}_{\mu_1\mathcal{G}(e_1)}=\widetilde{1}}\left|\widetilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)\right|$

4. *If* $\tilde{a}\widetilde{(.,.)}$ *is any fixed* $\mathcal{FS}-$ *sesquilinear form and* $\widetilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)$ *be associated* $\mathcal{FS}-$ *quadratic form on* $\mathcal{FSH}$ *- space* $\widetilde{\mathcal{H}}$, *then*

$$\begin{aligned}\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)}=&\frac{\tilde{1}}{4}[\tilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)-\tilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\\&+\tilde{\imath}\tilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)-\tilde{\imath}\tilde{y}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)]\end{aligned}$$

Verification: *By using linearity of the* $\mathcal{FS}-$ *bilinear form* $\tilde{a}$, *we have*

$$\begin{aligned}&\tilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}+\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)=\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}+\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}+\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\\&=\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)+\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)+\tilde{a}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)+\tilde{a}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\end{aligned}$$

And

$$\begin{aligned}&\tilde{\mathfrak{F}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}-\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)=\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}-\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}-\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\\&=\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)-\tilde{a}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)-\tilde{a}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)+\tilde{a}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)},\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\end{aligned}$$

By subtracting the second from the above equation from the first, and we get

$$\begin{aligned}&\tilde{\mathfrak{F}}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big) - \tilde{\mathfrak{F}}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)} - \widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\\ &= \tilde{2}\tilde{a}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}\big) + \tilde{2}\tilde{a}\big(\tilde{y}_{\mu_2\mathcal{G}(e_2)} - \tilde{x}_{\mu_1\mathcal{G}(e_1)}\big)\end{aligned} \tag{3.1}$$

Replacing $\tilde{y}_{\mu_{2\mathcal{G}(e_2)}}$ by $\tilde{\imath}\tilde{y}_{\mu_{2\mathcal{G}(e_2)}}$ in Eq. (3.1), we obtain

$$\begin{aligned}&\tilde{\mathfrak{F}}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \widetilde{\tilde{\imath}\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big) - \tilde{\mathfrak{F}}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)} - \widetilde{\tilde{\imath}\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\\ &= \tilde{2}\tilde{a}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\imath}\tilde{y}_{\mu_2\mathcal{G}(e_2)}\big) + \tilde{2}\tilde{a}\big(\tilde{\imath}\tilde{y}_{\mu_2\mathcal{G}(e_2)} - \tilde{x}_{\mu_1\mathcal{G}(e_1)}\big)\end{aligned}$$

Or

$$\begin{aligned}&\tilde{\mathfrak{F}}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \widetilde{\tilde{\imath}\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big) - \tilde{\mathfrak{F}}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)} - \widetilde{\tilde{\imath}\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\\ &= \tilde{\imath}\tilde{2}\tilde{a}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}\big) + \tilde{\imath}\tilde{2}\tilde{a}\big(\tilde{y}_{\mu_2\mathcal{G}(e_2)} - \tilde{x}_{\mu_1\mathcal{G}(e_1)}\big)\end{aligned} \tag{3.2}$$

Multiply Eq. (3.1) by $\tilde{\imath}$ and adding it to Eq. (3.2), We obtain a result.

Lemma 3.5. *A $\mathcal{FS}-$ bilinear form $\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)$ is $\mathcal{FS}-$ symmetric if and only if associated $\mathcal{FS}$ - quadratic functional $\widetilde{\tilde{\mathfrak{F}}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)$ is $\mathcal{FS}-$ real.*

Lemma 3.6. *the $\mathcal{FS}-$ bilinear form $\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)$ is $\mathcal{FS}$ - bounded if and only if the associated $\mathcal{FS}$- sesquilinear form $\widetilde{\tilde{\mathfrak{F}}}$ is $\mathcal{FS}-$ bounded. If $\widetilde{\tilde{a}(.,.)}$ is $\mathcal{FS}-$ bounded , then $\widetilde{\|\widetilde{\tilde{\mathfrak{F}}}\|} \geq \widetilde{\|\tilde{a}\|} \geq \tilde{2}\widetilde{\|\widetilde{\tilde{\mathfrak{F}}}\|}$.*

Theorem 3.7. *If $\mathcal{FS}-$ bilinear form $\widetilde{\tilde{a}(.,.)}$ is $\mathcal{FS}-$ bounded and $\mathcal{FS}-$ symmetric, then $\widetilde{\|\tilde{a}\|} = \widetilde{\|\widetilde{\tilde{\mathfrak{F}}}\|}$,such that $\widetilde{\tilde{\mathfrak{F}}}$ is the associated $\mathcal{FS}$ - quadratic functional.*

Proof . By Lemma 3.5, $\widetilde{\tilde{\mathfrak{F}}}$ is $\mathcal{FS}-$ real . In view of Lemma 3.6,
To show $\widetilde{\|\tilde{a}\|} \underset{\smile}{\leq} \widetilde{\|\widetilde{\tilde{\mathfrak{F}}}\|}$
Let $\tilde{a}\big(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}\big) = \tilde{\gamma}e^{\tilde{\imath}\tilde{\beta}}$, where $\tilde{\gamma}, \tilde{\beta} \in \mathcal{R}(A)$
So by using the Remark 3.4(4) and bearing in mind that purely imaginary terms are $\tilde{0}$. we have
$\left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\right| = \tilde{\gamma} = \tilde{a}\left(\tilde{\gamma}e^{\tilde{\imath}\tilde{\beta}}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)$
$= \frac{\tilde{1}}{4}\left[\widetilde{\tilde{\mathfrak{F}}}\left(\dot{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}} + \tilde{y}_{\mu_{2\mathcal{G}(e_2)}}\right) - \widetilde{\tilde{\mathfrak{F}}}\left(\dot{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}} - \tilde{y}_{\mu_{2\mathcal{G}(e_2)}}\right)\right]$ where $\dot{\tilde{x}}_{\mu_1\mathcal{G}(e_1)} = \tilde{\gamma}e^{\tilde{\imath}\tilde{\beta}}$. This implies that
$\left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\right| = \frac{\tilde{1}}{4}\left\|\widetilde{\tilde{\mathfrak{F}}}\right\|\left(\left\|\dot{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}} + \tilde{y}_{\mu_{2\mathcal{G}(e_2)}}\right\|^2 + \left\|\dot{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}} + \tilde{y}_{\mu_{2\mathcal{G}(e_2)}}\right\|^2\right)$
By using $\mathcal{FS}-$ parallelogram law, we have
$\left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\right| = \frac{\tilde{1}}{2}\left\|\widetilde{\tilde{\mathfrak{F}}}\right\|\left(\left\|\dot{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\|^2 + \left\|\tilde{y}_{\mu_{2\mathcal{G}(e_2)}}\right\|^2\right)$

$$\sup_{\left\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right\|=\tilde{1}, \left\|\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right\|=\tilde{1}}\left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big)\right| \leq \|\widetilde{\tilde{\mathfrak{F}}}\|$$

Then $\|\widetilde{a}\| \leq \|\widetilde{\tilde{\mathfrak{F}}}\|$ $\square$

Theorem 3.8. *Let $\tilde{\mathcal{T}} \in \tilde{B}(\widetilde{\mathcal{H}})$. Then, the $\widetilde{\tilde{a}(.,)} : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \to \mathcal{C}(A)$ defined by*

$$\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big) = \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle \quad \forall \left(\tilde{x}_{\mu_1\bar{g}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}$$

is $\mathcal{FS}-$ bounded bilinear form on $\widetilde{\mathcal{H}}$, and $\|\widetilde{\tilde{a}}\| = \|\widetilde{\tilde{\mathcal{T}}}\|$. Conversely, let $\widetilde{\tilde{a}(.,.)}$ be $\mathcal{FS}-$ bounded bilinear form on $\widetilde{\mathcal{H}}$. Then, there exists a unique $\mathcal{FS}-$ bounded linear operator $\tilde{\mathcal{T}}$ on $\widetilde{\mathcal{H}}$ such that
$\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\big) = \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle \quad \forall \left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}$

Proof .

1. Let $\tilde{\mathcal{T}} \in \tilde{B}(\widetilde{\mathcal{H}})$. Then, $\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big) = \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle$ satisfies the following condition:
(a)

$$\begin{aligned}
\left(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \tilde{y}_{\mu_2\mathcal{G}(e_2)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right) &= \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \tilde{y}_{\mu_2\mathcal{G}(e_2)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle \\
&= \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle + \left\langle \widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle \\
&= \tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big) + \tilde{a}\big(\widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)
\end{aligned}$$

(b)

$$\begin{aligned}
\tilde{a}\Big(\widetilde{\tilde{\beta}\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\Big) = \left\langle \widetilde{\tilde{\beta}\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle &= \tilde{\beta}\left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle \\
&= \tilde{\beta}\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)
\end{aligned}$$

(c)

$$\left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)\right| = \left|\left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} \right\rangle\right| \leq \left\|\widetilde{\tilde{\mathcal{T}}}\right\| \|\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}}\| \|\widetilde{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\|$$

This implies that

$$\sup_{\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\tilde{z}_{\mu_3 G(e_3)}\| = \tilde{1}} \left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)\right| \leq \|\widetilde{\tilde{\mathcal{T}}}\|$$

Then $\widetilde{\|\tilde{a}\|} \leq \|\widetilde{\tilde{\mathcal{T}}}\|$. In fact $\widetilde{\|\tilde{a}\|} = \|\widetilde{\tilde{\mathcal{T}}}\|$

2. For the converse, let $\widetilde{\tilde{a}(.,)}$ be $\mathcal{FS}-$ bounded bilinear form on $\widetilde{\mathcal{H}}$. For any $\tilde{y}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{H}}$, we define $\tilde{f}_{\tilde{y}_{\mu_2\mathcal{G}(e_2)}}$ on $\widetilde{\mathcal{H}}$ as follows:

$$\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) = \tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big) \tag{3.3}$$

We have

$$\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1) + \tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) = \tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1} + \tilde{y}_{\mu_2\mathcal{G}(e_2)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)$$

$$\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{\beta}\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) = \tilde{a}\Big(\widetilde{\tilde{\beta}\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\Big) = \tilde{\beta}\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)$$

$$\left|\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right)\right| = \left|\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big)\right| \leq \widetilde{\|\tilde{a}\|}\,\widetilde{\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\|}\,\widetilde{\|\tilde{z}_{\mu_3\mathcal{G}(e_3)}\|}$$

$$\Longrightarrow \left\|\widetilde{\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}}\right\| \leq \widetilde{\|\tilde{a}\|}\,\widetilde{\|\tilde{z}_{\mu_3\mathcal{G}(e_3)}\|}$$

Thus, $\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}$ is $\mathcal{FS}-$ bounded bilinear form on $\widetilde{\mathcal{H}}$. By $\mathcal{FS}-$ Riesz representation theorem, there exists a unique $\tilde{\mathcal{T}}\tilde{\tilde{Z}}_{\mu_3\mathcal{G}(e_3)} \in \widetilde{\mathcal{H}}$ such that

$$\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) = \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle \tag{3.4}$$

$$\text{And } \left\|\widetilde{\tilde{\mathcal{T}}\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right\| = \left\|\widetilde{\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}}\right\| \leq \widetilde{\|\tilde{a}\|}\,\widetilde{\|\tilde{z}_{\mu_3\mathcal{G}(e_3)}\|} \tag{3.5}$$

The $\tilde{\mathcal{T}} : \tilde{z}_{\mu_3\mathcal{G}(e_3)} \longrightarrow \tilde{\mathcal{T}}\tilde{z}_{\mu_3\mathcal{G}(e_3)}$ defined by $\tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle$ is $\mathcal{FS}-$ linear in view of the following relations:

$$\begin{aligned}
\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{\beta}\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle &= \tilde{f}_{\tilde{\beta}\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) = \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\beta}\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right\rangle \\
&= \tilde{\beta}\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right\rangle = \tilde{\beta}\tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) \\
&= \tilde{\beta}\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle = \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\beta}\tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle \\
&= \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{\beta}\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle = \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\beta}\tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{\beta}\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) - \tilde{\beta}\tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) = \right\rangle \tilde{0} \quad \forall \tilde{z}_{\mu_3\mathcal{G}(e_3)} \in \widetilde{\mathcal{H}} \\
&\Longrightarrow \tilde{\mathcal{T}}\left(\tilde{\beta}\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) = \tilde{\beta}\tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)
\end{aligned}$$

Now

$$\begin{aligned}
&\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)} + \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right)\right\rangle = \tilde{f}_{\tilde{y}_{\mu_2\mathcal{G}(e_2)}+\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) = \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)} + \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right\rangle \\
&= \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)} + \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right\rangle + \left\langle \widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)} + \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right\rangle = \tilde{f}_{\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) + \tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) \\
&= \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right)\right\rangle + \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle = \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\widetilde{\tilde{y}_{\mu_2\mathcal{G}(e_2)}}\right) + \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle \\
&\Longrightarrow \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)+\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right) - \left(\tilde{\mathcal{T}}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right) + \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right)\right\rangle = \tilde{0} \quad \forall \tilde{z}_{\mu_3\mathcal{G}(e_3)} \in \widetilde{\mathcal{H}}
\end{aligned}$$

Which gives $\tilde{\mathcal{T}}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)+\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\right) = \tilde{\mathcal{T}}\left(\tilde{y}_{\mu_2\mathcal{G}(e_2)}\right) + \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)$

Equation (3.5) implies that $\widetilde{\|\tilde{\mathcal{T}}\|} \leq \widetilde{\|\tilde{a}\|}$. By Eqs. (3.3) and (3.4), we have

$$\tilde{a}\big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big) = \tilde{f}_{\tilde{z}_{\mu_3\mathcal{G}(e_3)}}\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}\right) = \left\langle \widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle \quad \forall \big(\widetilde{\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}}\big) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}$$

Then, for every fixed $\tilde{z}_{\mu_3\mathcal{G}(e_3)} \in \widetilde{\mathcal{H}}$, we get $\widetilde{\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle} = \widetilde{\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{S}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle}$
$\Longrightarrow \left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{(\tilde{\mathcal{T}} - \tilde{\mathcal{S}})}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle = \tilde{0}$
This implies that $(\tilde{\mathcal{T}} - \tilde{\mathcal{S}})\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right) = \tilde{0} \quad \forall \tilde{z}_{\mu_3\mathcal{G}(e_3)} \in \widetilde{\mathcal{H}}$, i.e., $\tilde{\mathcal{T}} = \tilde{\mathcal{S}}$.
This proves that there exists a unique $\tilde{\mathcal{T}} \in \tilde{B}(\widetilde{\mathcal{H}})$ such that

$$\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} = \widetilde{\left\langle \tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{\mathcal{T}}\left(\tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)\right\rangle}$$

□

Corollary 3.9. *Let $\tilde{\mathcal{T}} \in \tilde{B}(\widetilde{\mathcal{H}})$. Then, the , the $\tilde{b}\widetilde{(.,.)} : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \to \mathcal{C}(A)$ defined by $\tilde{b}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} = \left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{z}}_{\mu_{3\mathcal{G}(e_3)}}\right\rangle$ is $\mathcal{FS}-$ bounded bilinear form on $\widetilde{\mathcal{H}}$ and $\widetilde{\|\tilde{b}\|} = \widetilde{\|\tilde{\mathcal{T}}\|}$. Conversely, let $\tilde{b}\widetilde{(.,.)}$ be $\mathcal{FS}-$ bounded bilinear form on $\widetilde{\mathcal{H}}$. Then, there exists a unique $\mathcal{FS}$- bounded linear operator $\tilde{\mathcal{T}}$ on $\widetilde{\mathcal{H}}$ such that*

$$\tilde{b}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{z}_{\mu_3\mathcal{G}(e_3)}\right)} = \widetilde{\left\langle \tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\mathcal{T}}\tilde{z}_{\mu_{3\mathcal{G}(e_3)}}\right\rangle}, \quad \forall \left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{y}_{\mu_2\mathcal{G}(e_2)}\right) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}}$$

Corollary 3.10. *If $\tilde{\mathcal{T}} \in \tilde{B}\widetilde{(\mathcal{H})}$, then*

$$\begin{aligned}\widetilde{\|\tilde{\mathcal{T}}\|} &= \sup_{\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\tilde{y}_{\mu_2\mathcal{G}(e_2)}\| = \tilde{1}} \left|\widetilde{\left\langle \tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\mathcal{T}}\tilde{z}_{\mu_{3\mathcal{G}(e_3)}}\right\rangle}\right| \\ &= \sup_{\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\tilde{y}_{\mu_2\mathcal{G}(e_2)}\| = \tilde{1}} \left|\widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{z}_{\mu_{3\mathcal{G}(e_3)}}\right\rangle}\right|\end{aligned}$$

Theorem 3.11. *Let $\tilde{\mathcal{T}} \in \tilde{B}(\widetilde{\mathcal{H}})$. Then, the following statements are equivalent:*

1. *$\tilde{\mathcal{T}}$ is $\mathcal{FS}-$ self $-$ adjoint.*
2. *The $\mathcal{FS}$-bilinear from $\tilde{a}\widetilde{(.,.)}$ on $\widetilde{\mathcal{H}}$ defined by $\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_2\mathcal{G}(e_2)}\right)} = \left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{y}}_{\mu_{2\mathcal{G}(e_2)}}\right\rangle$ is $\mathcal{FS}-$ symmetric.*
3. *The $\mathcal{FS}$-bilinear from $\tilde{\tilde{\mathfrak{F}}}\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}\right)$ on $\widetilde{\mathcal{H}}$ defined by $\tilde{\tilde{\mathfrak{F}}}\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}\right) = \widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle}$ is $\mathcal{FS}-$ real.*

Proof . (1) $\Rightarrow$ (2) : $\overline{\tilde{\tilde{\mathfrak{F}}}\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}\right)} = \overline{\widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle}} = \widetilde{\left\langle \tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\mathcal{T}}\tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle}$
$= \widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle} = \tilde{\tilde{\mathfrak{F}}}\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}\right)$. In view of Lemma 3.7, we obtain the result
(3) $\Rightarrow$ (2) : By Lemma (3.7) $\tilde{\tilde{\mathfrak{F}}}\left(\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}\right) = \widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle}$ is $\mathcal{FS}$- real if and only if the $\mathcal{FS}$-bilinear from $\tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_2\mathcal{G}(e_2)}\right)} = \left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{y}}_{\mu_{2\mathcal{G}(e_2)}}\right\rangle$ is $\mathcal{FS}$- symmetric
(2) $\Rightarrow$ (1) : $\overline{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{y}}_{\mu_{2\mathcal{G}(e_2)}}\right\rangle \tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_2\mathcal{G}(e_2)}\right)}} = \tilde{a}\widetilde{\left(\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{b}_{\mu_2\mathcal{G}(e_2)}\right)}$
$= \widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle} = \widetilde{\left\langle \tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{\mathcal{T}}\tilde{\tilde{x}}_{\mu_{1\mathcal{G}(e_1)}}\right\rangle}$. This shows that $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^*$ that $\tilde{\mathcal{T}}$ is $\mathcal{FS}$-self-adjoint □

Corollary 3.12. *If $\tilde{\mathcal{T}}$ is $\mathcal{FS}-$ bounded self $-$ adjoint on $\widetilde{\mathcal{H}}$, then*

$$\widetilde{\|\tilde{\mathcal{T}}\|} = \sup_{\|\tilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \tilde{1}} \left| \widetilde{\left\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_1\mathcal{G}(e_1)}, \tilde{x}_{\mu_1\mathcal{G}(e_1)} \right\rangle} \right|$$

4. Conclusions

The necessity of combining the two concepts fuzzy and soft sets was to provide more accurate and general results. The other concepts consequently followed this structure resulting like fuzzy soft normed spaces, fuzzy soft inner product space, fuzzy soft Hilbert space and fuzzy soft bounded linear operators. We studied and discussion new kind of sesquilinear functional which is fuzzy soft sesquilinear functional and given some properties with characterization, also theorems related on fuzzy soft sesquilinear functional have been given. Additionally we presented the relationship between this kind and other kinds. types.

5. Open problem

There are other issues that can be studied using the fuzzy soft notice, including fuzzy soft quasi normal, fuzzy soft N-normal, fuzzy soft b-matric space etc.

References

[1] S. Bayramov and C. Gunduz, *Soft locally compact spaces and soft paracompact spaces,* J. Math. Syst. 3 (2013) 122–130.

[2] T. Beaula and M. M. Priyanga, *A new notion for fuzzy soft normed linear space,* Int. J. Fuzzy Math. Arch. 9(1) (2015) 81–90.

[3] S. Das and S.K. Samanta, *On soft inner product spaces,* Ann. Fuzzy Math. Inf. 6(1) (2013) 151–170.

[4] N. Faried, M.S.S. Ali and H.H. Sakr, *On fuzzy soft Hermition operators*, Sci. J. 9(1) (2020) 73–82.

[5] N. Faried, M. S. S. Ali and H. H. Sakr, *On fuzzy soft linear operators in fuzzy soft Hilbert spaces*, Abst. Appl. Anal. 2020 (2020).

[6] N. Faried, M.S.S. Ali and H.H. Sakr, *Fuzzy soft Hilbert spaces,* Math. Stat. 8(3) (2020).

[7] A.Z. Khameneh, A. Kiliçman and A.R. Salleh, *Parameterized norm and parameterized fixed- point theorem by using fuzzy soft set theory*, arXiv, 15, (2013) 2.9, 2.10, 2.11, 2.12, 2.13, 2.14.

[8] P.K. Maji, R. Biswas and A.R. Roy, *Fuzzy soft set,* J. Fuzzy Math. 9(3) (2001) 677–692.

[9] D. Molodtsov, *Soft set theory-First results*, Comput. Math. Appl. 37 (1999) 19–31.

[10] T. J. Neog, D. K. Sut, and G. C. Hazarika, *Fuzzy soft topological spaces*, Int. J. Latest Trend. Math. 2(1) (2012) 54–67.

[11] S. Das and S.K. Samanta, *Projection operators on soft inner product spaces*, Ann. Fuzzy Math. Inf. 11(5) (2016) 809–827.

[12] M.I. Yazar, C.G. Aras and S. Bayramov, *Results on soft Hilbert spaces*, TWMS J. App. Eng. Math. 9(1) (2019) 159–164.

[13] M. I. Yazar, C. G. Aras, S. Bayramov and C. Gunduz, *A new view on soft normed spaces*, Int. Math. For. 9(24) (2014) 1149–1159.

[14] L.A. Zadeh, *Fuzzy sets*, Inf. Cont. 8(3) (1965) 338–353.