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Characteristics of penta- open sets in penta topological spaces

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Abstract

The aim of the presented study is to introduce and verify two new spaces called $\mathcal{P}\_$ compactness and $\mathcal{P}_$connectedness using $\mathcal{P}_$open sets and some of their properties. Moreover, we study the relationship between these spaces. Another purpose of this study is to examine a new form of separation axioms, by using $\mathcal{P}_$open set namely $T_{\mathcal{P}i}$-spaces where $(i = 0, 1, 2)$. The pertinence between them has been discussed and several features of these spaces are demonstrated as well.

Keywords: Penta Topological Space, $\mathcal{P}_$irresolute function, $\mathcal{P}_$compactness, locally $\mathcal{P}_$compact, $\mathcal{P}_$Connectedness, $\mathcal{P}_$separated, strongly $\mathcal{P}_$connected, $T_{\mathcal{P}i}$-space.

1. Introduction

The aims of semi-open sets and their properties were initiated by Levine [7] in 1963.The idea of dealing with single topological space was developed to bi-topological space, tri-topological, quad topological by researchers Kelly [4], Kovar [6] and Mukundan [9], lastly the notion Penta topological space $(\mathbb{X}, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ was introduced by Muhammad and Khan [5] in 2018, where $\mathbb{X}$ is nonempty set together with five topologies $\tau_1, \tau_2, \tau_3, \tau_4 \& \tau_5$. Many researchers verified the basic properties of connectedness and compactness powerful tools in topology. The idea of Hausdorff spaces is almost an integral part of compactness. Many authors in such as Srivastava and Bhatia [13] introduced some kinds of compact spaces in topological space according to the sets. Topological space is said to be compact or have the compact property, if every open cover of $\mathbb{X}$ has a finite sub-cover [2]. Last years

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the generalization of compact spaces and locally generalized to bi-topological and tri-topological setting as in [1, 12, 13]. A topological space $\mathbb{X}$ is said to be disconnected space if $\mathbb{X}$ can be expressed as the union of two disjoint nonempty open subsets of $\mathbb{X}$, otherwise $\mathbb{X}$ is said to be connected space [2]. In 1965, Levine [8] introduced strongly connected in topology. In 1967 Pervin [10] studied connectedness in bi-topological spaces and in 2016 Tapi and others [16, 17] studied Tri-connectedness in Tri-topological spaces and he also introduced connectedness in Quad Topological Spaces [14]. Many papers discussed separation axioms, essentially by replacing open sets, many definitions of separation axioms according to open sets have been introduced by many researchers as El-Tantawy, Hameed, Tapi, and others [3, 15]. In this work we developed compactness and connectedness and separation axioms in Penta topological spaces. we introduced different types of Penta compact and Penta connected and Penta separation axioms in Penta topological spaces, additionally some properties of these spaces are investigated. Throughout this paper. A Penta topological space is denoted by $(\mathbb{X}, \Im_{\mathcal{P}})$ and $(\mathbb{Y}, \Im_{\mathcal{P}})$ or simply by $\mathbb{X}$ and $\mathbb{Y}$. The concept Penta topological space $(\mathbb{X}, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ where $\mathbb{X}$ is non empty set together with five topologies $\tau_1, \tau_2, \tau_3, \tau_4 \& \tau_5$, was introduced by Khan and Khan [5] in 2018. we write $\Im_{\mathcal{P}}$ for Penta Topology ($\mathcal{P}$*_topology*) and $(\mathbb{X}, \Im_{\mathcal{P}})$ for Penta Topological Space where $\Im_{\mathcal{P}} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$.

In the present work , we introduce $\Im_{\mathcal{P}}$ on five different topologies on $\mathbb{X}$, therefore $(\mathbb{X}, \Im_{\mathcal{P}})$ is called Penta Topological Space. The topologies $(\mathbb{X}, \Im_1)$, $(\mathbb{X}, \Im_2)$, $(\mathbb{X}, \Im_3)$, $(\mathbb{X}, \Im_4)$, $(\mathbb{X}, \Im_5)$ are independently satisfying the axioms of topology. The elements of $\Im_1$ are called $\Im_1$- open set and the complement of $\Im_1$-open set is called $\Im_1$-closed set. And the same with the elements of $\Im_2, \Im_3, \Im_4, \Im_5$.

Definition 1.1. *[5]. Let $(\mathbb{X}, \Im_{\mathcal{P}})$ be a Penta Topological Space. Elements of τ_i; $i = \{1, 2, 3, 4, 5\}$ are called τ_i- open sets and their relative complements are called τ_i- closed sets. Also a subset $\mathbb{A}$ of $\mathbb{X}$ is called penta-open ($\mathcal{P}-open$) if $\mathbb{A} \in \cup\tau_i$; $i \in \{1, 2, 3, 4, 5\}$ and its complement is said to be penta-closed ($\mathcal{P}-closed$). $\mathcal{P}-open$ sets satisfies all the axioms of topology. The set of all $\mathcal{P}-open$ sets contains $\tau_1 \cup \tau_2 \cup \tau_3 \cup \tau_4 \cup \tau_5$ So; the family of all $\mathcal{P}-open$ ($\mathcal{P}-closed$) sub sets of $(\mathbb{X}, \Im_{\mathcal{P}})$ will be denoted by $(\mathcal{PO}(\mathbb{X})), (\mathcal{PC}(\mathbb{X}))$.*

Definition 1.2. *[5]. Let $\mathcal{H}$ be a subset of a Penta topological Space $(\mathbb{X}, \Im_{\mathcal{P}})$, then:*

1. *The ($\mathcal{P}$_interior) of $\mathcal{H}$ is the union of all ($\mathcal{P}-open$) subset contained in $\mathcal{H}$ and is denoted by $int_{\mathcal{P}}(\mathcal{H})$. Thus $int_{\mathcal{P}}(\mathcal{H})$ is the largest ($\mathcal{P}-open$) subset of $\mathcal{H}$.*
2. *The ($\mathcal{P}$_closure) of $\mathcal{H}$ is the intersection of all ($\mathcal{P}-closed$) sets containing $\mathcal{H}$ and is denoted by $cl_{\mathcal{P}}(\mathcal{H})$. that is $cl_{\mathcal{P}}(\mathcal{H})$ is the smallest ($\mathcal{P}-closed$) set containing $\mathcal{H}$. Some properties for each $\mathcal{H} \subseteq \mathbb{X}$ [14]*

   i. $(int_{\mathcal{P}}(\mathcal{H}))^c = cl_{\mathcal{P}}(\mathcal{H}^c)$

   ii. *$\mathcal{H}$ is $\mathcal{P}-open$ iff $int_{\mathcal{P}}(\mathcal{H}) = \mathcal{H}$*

   iii. *$\mathcal{H}$ is $\mathcal{P}-closed$ iff $cl_{\mathcal{P}}(\mathcal{H}) = \mathcal{H}$*
3. *The $\mathcal{P}$_neighborhood (in short $G_{\mathcal{P}N}$) $\mathcal{H}$ of a point $x \in \mathbb{X}$ if and only if there exist a $\mathcal{P}-open$ set $\mathcal{G}$ such that $x \in \mathcal{G} \subseteq \mathcal{H}$.*

Definition 1.3. *[5]. A function $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\tilde{\mathcal{P}}})$ is called*

1. *$\mathcal{P}-continuous$ if $\mathfrak{f}^{-1}(\mathcal{H}) \in \mathcal{PO}(\mathbb{X})$, for each $\mathcal{H} \in \mathcal{PO}(\mathbb{Y})$.*
2. *$\mathcal{P}-open$ if, $\mathfrak{f}(\mathcal{H}) \in \mathcal{PO}(\mathbb{Y})$, for each $\mathcal{H} \in \Im_{\mathcal{P}}$.*
3. *$\mathcal{P}-closed$ if any $\Im_{\mathcal{P}}$-closed set $\mathbb{D}$ then $\mathfrak{f}(\mathbb{D}) \in \mathcal{PC}(\mathbb{Y})$.*

4. $\mathcal{P}-$ *homeomorphism if* $\mathfrak{f}$ *is bijective,* $\mathcal{P}-$ *continuous and* $\mathcal{P}-$ *open.*

Proposition 1.4. *A function* $\mathfrak{f}:(\mathbb{X},\Im_{\mathcal{P}})\to(\mathbb{Y},\Im_{\check{\mathcal{P}}})$ *is* $\mathcal{P}-$ *continuous if and only if the inverse image of every* $\mathcal{P}-$ *closed in* $\mathbb{Y}$ *is an* $\mathcal{P}-$ *closed set in* $\mathbb{X}$.

Proof . Suppose that $\mathbb{D}$ is $\mathcal{P}-closed$ set in $\mathbb{Y}$ then $\mathbb{D}^C$ $\mathcal{P}-open$ set in $\mathbb{Y}$, since $\mathfrak{f}$ is $\mathcal{P}-continuous$, we get $\mathfrak{f}^{-1}(\mathbb{D}^C)\in\mathcal{P}O(\mathbb{X})$, therefore $\mathfrak{f}^{-1}(\mathbb{D}^C)=\mathfrak{f}^{-1}(\mathbb{Y}-\mathbb{D})=\mathfrak{f}^{-1}(\mathbb{Y})-\mathfrak{f}^{-1}(\mathbb{D})=\mathbb{X}-\mathfrak{f}^{-1}(\mathbb{D})=(\mathfrak{f}^{-1}(\mathbb{D}))^C$. Then $(\mathfrak{f}^{-1}(\mathbb{D}))^C$ $\mathcal{P}-open$ in $\mathbb{X}$ therefore $\mathfrak{f}^{-1}(\mathbb{D})$ $\mathcal{P}-closed$ set in $\mathbb{X}$. □

Example 1.5. *A Penta topology* $\Im_{\mathcal{P}}=\{\mathbb{X},\emptyset,\{a\},\{b\},\{b,c\},\{a,d\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$ *on* $\mathbb{X}=\{a,b,c,d\}$*, when* $\Im_1=\{\mathbb{X},\emptyset,\{a\},\{b\},\{a,b\}\},\Im_2=\{\mathbb{X},\emptyset,\{b,c\},\{d\}\}$,
$\Im_3=\{\mathbb{X},\emptyset,\{a\},\{a,b,d\}\},\Im_4=\{\mathbb{X},\emptyset,\{a,d\},\{a,b,d\}\}$ *and* $\Im_5=\{\mathbb{X},\emptyset,\{a,b,c\}\}$.
Let $\Im_{\check{\mathcal{P}}}=\left\{\mathbb{Y},\emptyset,\{\dot a\},\{\dot b\},\{\dot b,\dot a\},\{\dot b,\dot c\},\{\dot a,\dot d\},\{\dot a,\dot b,\dot c\},\{\dot a,\dot b,\dot d\}\right\}$ *on* $\mathbb{Y}=\{\dot a,\dot b,\dot c,\dot d\}$*, when*
$\Im_1=\left\{\mathbb{X},\emptyset,\{\dot a,\dot d\},\{\dot a,\dot b,\dot d\}\right\}$, $\Im_2=\left\{\mathbb{Y},\emptyset,\{\dot b\},\{\dot b,\dot c\}\right\}$, $\Im_3=\left\{\mathbb{Y},\emptyset,\{\dot a\},\{\dot a,\dot b,\dot d\}\right\}$,
$\Im_4=\left\{\mathbb{Y},\emptyset,\{\dot a\},\{\dot b\},\{\dot a,\dot b\}\right\}$, $\Im_5=\left\{\mathbb{Y},\emptyset,\{\dot a,\dot b,\dot c\}\right\}$.
We get
$\mathcal{P}O(\mathbb{X})=\{\mathbb{X},\emptyset,\{a\},\{b\},\{b,c\},\{a,d\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$
$\mathcal{P}C(\mathbb{X})=\{\mathbb{X},\emptyset,\{b,c,d\},\{a,d\},\{c,d\},\{b,c\},\{a,c,d\},\{d\},\{c\}\}$
$\mathcal{P}O(\mathbb{Y})=\left\{\mathbb{Y},\emptyset,\{\dot a\},\{\dot b\},\{\dot b,\dot a\},\{\dot b,\dot c\},\{\dot a,\dot d\},\{\dot a,\dot b,\dot c\},\{\dot a,\dot b,\dot d\}\right\}$
$\mathcal{P}C(\mathbb{Y})=\left\{\mathbb{Y},\emptyset,\{\dot b,\dot c,\dot d\},\{\dot a,\dot c,\dot d\},\{\dot c,\dot d\},\{\dot d,\dot a\},\{\dot b,\dot c\},\{\dot d\},\{\dot c\}\right\}$.
Let $\mathfrak{f}:(\mathbb{X},\Im_{\mathcal{P}})\to(\mathbb{Y},\Im_{\check{\mathcal{P}}})$ *and* $\mathfrak{f}(b)=\dot b$, $\mathfrak{f}(c)=\dot c$, $\mathfrak{f}(d)=\dot d$, $\mathfrak{f}(a)=\dot a$. *Then* $\mathfrak{f}$ *is clearly a* $\mathcal{P}-$ *homeomorphism.*

Definition 1.6. *Let* E *be a subset of* $(\mathbb{X},\Im_{\mathcal{P}})$*, then classes* $\Im_{\mathcal{P}E}$ *of all intersections of* E *with* $\mathcal{P}-$ *open subsets of* $\mathbb{X}$ *belong to* $\Im_{\mathcal{P}}$ *is a topology on* E*,it is called penta- subspace (relative Penta –topological space for* E *with respect to* $\mathcal{P}-$ *open sets). The relative Penta –topological space for* E *is denoted by* $(E,\Im_{\mathcal{P}E})$*, such that* $\Im_{\mathcal{P}E}=\{G\cap E:G\in\Im_{\mathcal{P}}\}$, $\mathcal{P}=\{1,2,3,4,5\}$. *From example 1.5, we get* $\Im_{\mathcal{P}E}=\{E,\emptyset,\{a\},\{b\},\{b,c\},\{a,b\},\{a,b,c\}\}$ *on* $E=\{a,b,c\}$ *then* $(E,\Im_{\mathcal{P}E})$ *is relative* $\mathcal{P}-$ *topology.*

Definition 1.7. *[11]. The subset* $\mathcal{H}\subseteq(\mathbb{X},\Im_{\mathcal{P}})$ *is said to be semi penta open* $(semi_{\mathcal{P}}O)$ *set if* $\mathcal{H}\subseteq cl_{\mathcal{P}}(int_{\mathcal{P}}(\mathcal{H}))$ *and its complement is said to be semi penta-closed* $(semi_{\mathcal{P}}C)$ *set. Therefore; the family of all* $semi_{\mathcal{P}}O.(semi_{\mathcal{P}}C)$ *sub sets of* $(\mathbb{X},\Im_{\mathcal{P}})$ *will be denoted by* $(\mathcal{P}OS(\mathbb{X})),(\mathcal{P}CS(\mathbb{X}))$.
Note: Every $\mathcal{P}-$ *open set is* $(semi_{\mathcal{P}}O)$.

Definition 1.8. *A function* $\mathfrak{f}:(\mathbb{X},\Im_{\mathcal{P}})\to(\mathbb{Y},\Im_{\check{\mathcal{P}}})$ *is called* $\mathcal{P}-$ *irresolute function if* $\mathfrak{f}^{-1}(\mathcal{H})\in\mathcal{P}O(\mathbb{X})$*, for every* $\mathcal{H}\in\mathcal{P}OS(\mathbb{Y})$. *From example 1.5 we get:*

- $\mathcal{P}OS(\mathbb{X})=\{\mathbb{X},\emptyset,\{a\},\{b\},\{b,c\},\{a,d\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$
- $\mathcal{P}CS(\mathbb{X})=\{\mathbb{X},\emptyset,\{c\},\{d\},\{b,c\},\{a,d\},\{c,d\},\{b,c,d\},\{a,c,d\}\}$
- $\mathcal{P}OS(\mathbb{Y})=\left\{\mathbb{Y},\emptyset,\{\dot a\},\{\dot b\},\{\dot b,\dot c\},\{\dot a,\dot b\},\{\dot a,\dot d\},\{\dot a,\dot b,\dot c\},\{\dot a,\dot b,\dot d\right\}$
- $\mathcal{P}CS(\mathbb{Y})=\left\{\mathbb{Y},\emptyset,\{\dot b,\dot c,\dot d\},\{\dot a,\dot c,\dot d\},\{\dot c,\dot d\},\{\dot d,\dot a\},\{\dot b,\dot c\},\{\dot d\},\{\dot c\}\right\}$

Therefore $\mathcal{P}-$ *irresolute.*

2. Penta Connectedness in Penta Topologi cal Space

In this section, we discus $\mathcal{P}-connectedness$ and $\mathcal{P}-disconnectedness$ by means of $\mathcal{P}-separation$.

Definition 2.1. *A Penta Topological space $(\mathbb{X}, \Im_{\mathcal{P}})$, is called Penta _ separated space ($\mathcal{P}-separated$) if and only if there exist $\mathcal{P}-open$ subsets $\mathcal{H}$ and $\mathcal{K}$ of $\mathbb{X}$, such that $\mathcal{H} \cap cl_{\mathcal{P}}(\mathcal{K}) = \emptyset$ and $\mathcal{K} \cap cl_{\mathcal{P}}(\mathcal{H}) = \emptyset$. These two conditions are equivalent to $(\mathcal{H} \cap cl_{\mathcal{P}}(\mathcal{K})) \cup (\mathcal{K} \cap cl_{\mathcal{P}}(\mathcal{H})) = \emptyset$.From example 1.5 we note that $\mathbb{X}$ is $\mathcal{P}-separated$ space because*
$\mathcal{P}O(\mathbb{X}) = \{\mathbb{X}, \emptyset, \{a\}, \{b\}, \{b,c\}, \{a,d\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$. If we take $\{b,c\}$ and $\{a,d\}$ then $(\{b,c\} \cap cl_{\mathcal{P}}(\{a,d\})) \cup (\{a,d\} \cap cl_{\mathcal{P}}(\{b,c\})) = \emptyset$. therefore $\{b,c\}$ and $\{a,d\}$ are $\mathcal{P}-separation$ sets. Also $\{b\} \subseteq \{b,c\}$ and $\{a\} \subseteq \{a,d\}$, then $\{b\}$ and $\{a\}$ are $\mathcal{P}-separation$.

Remark 2.2.
1. *Every two disjoint $\mathcal{P}-open$ ($\mathcal{P}-closed$) subsets of any space are ($\mathcal{P}-separated$).*
2. *let $\mathcal{H}$ and $\mathcal{K}$ be ($\mathcal{P}-separated$) subsets of $\mathbb{X}$,then if $\mathcal{D} \subseteq \mathcal{H}$ and $\mathcal{S} \subseteq \mathcal{K}$, then $\mathcal{D}$ and $\mathcal{S}$ are also ($\mathcal{P}-separated$).*

Theorem 2.3. *Every two $\mathcal{P}-open$ ($\mathcal{P}-closed$) subsets of $\mathbb{X}$ are ($\mathcal{P}-separated$),if and only if they are disjoint.*

Proof . Any two ($\mathcal{P}-separated$) $\mathcal{H}, \mathcal{K}$ sets are disjoint, then two sets $\mathcal{P}-open$ ($\mathcal{P}-closed$) are ($\mathcal{P}-separated$) (because if $\mathcal{H}, \mathcal{K}$ are both disjoint ($\mathcal{P}-closed$), then $\mathcal{H} \cap \mathcal{K} = U$, $cl_{\mathcal{P}}(\mathcal{H}) = \mathcal{H}$, $cl_{\mathcal{P}}(\mathcal{K}) = \mathcal{K}$ and so that $\mathcal{H} \cap cl_{\mathcal{P}}(\mathcal{K}) = \emptyset, \mathcal{K} \cap cl_{\mathcal{P}}(\mathcal{H}) = \emptyset$).
Conversely, if $\mathcal{H}, \mathcal{K}$ are both disjoint $\mathcal{P}-open$, then $\mathcal{H}^c$ and $\mathcal{K}^c$ are both $\mathcal{P}-closed$, so that; $\mathcal{H} \cap \mathcal{K} = \emptyset \rightarrow \mathcal{H} \subseteq \mathcal{K}^c$ and $\mathcal{K} \subseteq \mathcal{H}^c$, also $cl_{\mathcal{P}}(\mathcal{K}^c) = \mathcal{K}^c$ and $cl_{\mathcal{P}}(\mathcal{H}^c) = \mathcal{H}^c$. We get $cl_{\mathcal{P}}(\mathcal{K}) \subseteq cl_{\mathcal{P}}(\mathcal{H}^c) = \mathcal{H}^c$ and $cl_{\mathcal{P}}(\mathcal{H}) \subseteq cl_{\mathcal{P}}(\mathcal{K}^c) = \mathcal{K}^c$. Hence $\mathcal{H}$ and $\mathcal{K}$ are ($\mathcal{P}-separated$) so that $\mathcal{H} \cap cl_{\mathcal{P}}(\mathcal{K}) = \emptyset$, $\mathcal{K} \cap cl_{\mathcal{P}}(\mathcal{H}) = \emptyset$.
□

Definition 2.4. *A Penta Topological Space $(\mathbb{X}, \Im_{\mathcal{P}})$ is said to be Penta_ connected ($\mathcal{P}-connected$) if there exist two $\mathcal{P}-open$ subsets $\mathcal{H}$ and $\mathcal{K}$ of $\mathbb{X}$, provided that $\mathcal{H} \cap \mathcal{K} = \emptyset$ and $\mathcal{K} \cup \mathcal{H} \neq \mathbb{X}$. A Penta Topological Space is Penta_ disconnected ($\mathcal{P}-disconn.$) if it does not ($\mathcal{P}-connected$).*

Theorem 2.5. *Let $(\mathbb{X}, \Im_{\mathcal{P}})$ be ($\mathcal{P}-connected$) space, then at least one of the five topologies is connected.*

Proof . suppose that $\Im_i$ is disconnected, where $\Im_i = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, then $\mathbb{X} = \mathcal{H} \cup \mathcal{K}$ where $\mathcal{H}$ and $\mathcal{K}$ are disjoint non-empty open sets. Since every open set is a $\mathcal{P}-open$ set, then $(\mathbb{X}, \Im_{\mathcal{P}})$ is not ($\mathcal{P}-connected$) space (contradiction by hypothesis). Then at least one of the five topologies is connected. □

Remark 2.6. *The converse of theorem 2.5 is not necessary true.*

Example 2.7. *From example 1.5 we note that $(\mathbb{X}, \Im_{\mathcal{P}})$ is not ($\mathcal{P}-connected$) space in spite of there exist at least one of five topologies is connected.*

Theorem 2.8. *Let E be a subset of $(\mathbb{X}, \Im_{\mathcal{P}})$. Then E is $\mathcal{P}-disconnected$ set iff it can be expressed as the union of two non-empty ($\mathcal{P}-separated$) sub sets of $\mathbb{X}$.*

Proof . Let E be $\mathcal{P}-disconnected$ set, then $E = \mathcal{H}\cup\mathcal{K}$ where $\mathcal{H}$ and $\mathcal{K}$ are disjoint non-empty subsets of $\Im_{\mathcal{P}E}-\mathcal{P}-closed$ sets. $\mathcal{H},\mathcal{K}$ are ($\mathcal{P}-separated$) subsets of $\mathbb{X}$, thus $\mathcal{H}\cap cl_{\mathcal{P}}(\mathcal{K}) = (\mathcal{H}\cap E)\cap cl_{\mathcal{P}}(\mathcal{K}) = \mathcal{H}\cap cl_{\mathcal{P}\Im_{\mathcal{P}E}}(\mathcal{K}) = \mathcal{H}\cap\mathcal{K} = \emptyset$, similarly $\mathcal{K}\cap cl_{\mathcal{P}\Im_{\mathcal{P}E}}(\mathcal{H}) = \emptyset$. Then $(\mathcal{H}\cap cl_{\mathcal{P}}(\mathcal{K}))\cup(\mathcal{K}\cap cl_{\mathcal{P}}(\mathcal{H})) = \emptyset$. Conversely, suppose that $E = \mathcal{H}\cup\mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are non- empty $\mathcal{P}-separated$ subsets of $\mathbb{X}$. We have $\mathcal{H}\cap cl_{\mathcal{P}}(\mathcal{K}) = (\mathcal{H}\cap E)\cap cl_{\mathcal{P}}(\mathcal{K}) = \emptyset$ and similarly $\mathcal{K}\cap cl_{\mathcal{P}\Im_{\mathcal{P}E}}(\mathcal{H}) = \emptyset$, so E is a union of non-empty ($\mathcal{P}-separated$) subsets of E. Thus E is $\mathcal{P}-disconnected.$ □

The following example constructs a $\mathcal{P}-connected$ space.

Example 2.9. *From example 1.5 we note that $(\mathbb{X},\Im_{\check{\mathcal{P}}})$ is ($\mathcal{P}-connected$), because $\mathcal{P}-open$ sets of $\mathbb{Y} = \left\{\mathbb{Y},\emptyset,\{\dot{a}\},\{\dot{b}\},\{\dot{b},\dot{a}\},\{\dot{b},\dot{c}\},\{\dot{a},\dot{d}\},\{\dot{a},\dot{b},\dot{c}\},\{\dot{a},\dot{b},\dot{d}\}\right\}$ on $\mathbb{Y} = \{\dot{a},\dot{b},\dot{c},\dot{d}\}$, if we take $\mathcal{H} = \{\dot{a}\}$ and $\mathcal{K} = \{\dot{b},\dot{c}\}$. That is ascertain $\mathcal{P}-connection$ condition.*

Remark 2.10.
1. *Let $\mathbb{X}$ be $\mathcal{P}-connected$ space, then any of the five topologies is not necessary to connected space. As in the example 1.5.*
2. *$\mathbb{X}$ is $\mathcal{P}-connected$ set, iff it is not the union of two non-empty ($\mathcal{P}-separated$) sets.*
3. *If $\mathbb{X}$ is the union of two disjoint non- empty $\mathcal{P}-open$ sub sets then $\mathbb{X}$ is $\mathcal{P}-disconnected.$*
4. *If E is $\mathcal{P}-connected$ set of $\mathbb{X}$ and $\mathcal{H},\mathcal{K}$ are ($\mathcal{P}-separated$) sets of $\mathbb{X}$ with $E\subseteq\mathcal{H}\cup\mathcal{K}$, then either $E\subseteq\mathcal{H}$ or $E\subseteq\mathcal{K}$.*
5. *If E subset of $\mathbb{X}$ is a $\mathcal{P}-connected$, then $cl_{\mathcal{P}}(E)$ is $\mathcal{P}-connected.$*

We know that if $\mathcal{H}$ and $\mathcal{K}$, $\mathcal{P}-connected$ sets then, $\mathcal{H}\cup\mathcal{K}$ is $\mathcal{P}-disconnected$ set but by adding some condition we can prove that $\mathcal{P}-connected$ sets by the following theorem.

Theorem 2.11. *Let $\mathcal{H},\mathcal{K}$ be $\mathcal{P}-connected$ sets and $\mathcal{H}\cap\mathcal{K}\neq\emptyset$, then $\mathcal{H}\cup\mathcal{K}$ is $\mathcal{P}-connected$ set.*

Proof . Assume that $\mathcal{H},\mathcal{K}\subseteq\mathbb{X}$, $\mathcal{H},\mathcal{K}$ are $\mathcal{P}-connected$ and $\mathcal{H}\cap\mathcal{K}\neq\emptyset$.
Suppose that $\mathcal{H}\cup\mathcal{K}$ is $\mathcal{P}-disconnected$,
if $\mathbb{X},\mathbb{Y}$ are two disjoint non empty $\mathcal{P}-open$ sets, $\mathbb{X},\mathbb{Y}\in\Im_{\mathcal{P}(\mathcal{H}\cup\mathcal{K})}$ then $\mathcal{H}\cup\mathcal{K} = \mathbb{X}\cup\mathbb{Y}$; so, $\mathcal{H}\subseteq\mathcal{H}\cup\mathcal{K}\to\mathcal{H}\subseteq\mathbb{X}\cap\mathbb{Y}\to\mathcal{H}\subseteq\mathbb{X}$ or $\mathcal{H}\subseteq\mathbb{Y}$ (because $\mathcal{H}$ is $\mathcal{P}-connected$).
Also $\mathcal{K}\subseteq\mathcal{H}\cup\mathcal{K}\to\mathcal{K}\subseteq\mathbb{X}\cap\mathbb{Y}\to\mathcal{K}\subseteq\mathbb{X}$ or $\mathcal{K}\subseteq\mathbb{Y}$ (because $\mathcal{K}$ is $\mathcal{P}-connected$).
Now, either $\mathcal{H}\subseteq\mathbb{X}\wedge\mathcal{K}\subseteq\mathbb{X}\to\mathcal{H}\cup\mathcal{K}\subseteq\mathbb{X}\to\mathbb{Y}=\emptyset$ contradiction.
Or $\mathcal{H}\subseteq\mathbb{Y}\wedge\mathcal{K}\subseteq\mathbb{Y}\to\mathcal{H}\cup\mathcal{K}\subseteq\mathbb{Y}\to\mathbb{X}=\emptyset$ contradiction.
Or $\mathcal{H}\subseteq\mathbb{Y}\wedge\mathcal{K}\subseteq\mathbb{X}\to\mathcal{H}\cap\mathcal{K}\subseteq\mathbb{X}\cap\mathbb{Y}=\emptyset\to\mathbb{X}\cap\mathbb{Y}=\emptyset$ contradiction.
Or $\mathcal{H}\subseteq\mathbb{X}\wedge\mathcal{K}\subseteq\mathbb{Y}\to\mathcal{H}\cap\mathcal{K}\subseteq\mathbb{X}\cap\mathbb{Y}=\emptyset\to\mathbb{X}\cap\mathbb{Y}=\emptyset$ contradiction. Hence $\mathcal{H}\cup\mathcal{K}$ is $\mathcal{P}-connected.$ □

And by generalizing the above theorem to any family of $\mathcal{P}-connected$ sets we obtain the following theorem.

Theorem 2.12. *The union of any family of $\mathcal{P}-connected$ sets have non-empty intersection $\mathcal{P}-connected$ sets.*

Proof . Let $\mathcal{M}_i:\ i\in\mathbb{N}$ is non-empty of $\mathcal{P}-connected$ subset of $\mathbb{X}$ and suppose that $\bigcup_{i\in\wedge}\mathcal{M}_i$ is $\mathcal{P}-disconnected$, then $\bigcup_{i\in\wedge}\mathcal{M} = \mathcal{H}\cup\mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are $\mathcal{P}-separated$ sets in $\mathbb{X}$. Since $\bigcap_{i\in\wedge}\mathcal{M}_i\neq\emptyset$, we get $x\in\bigcap_{i\in\wedge}\mathcal{M}_i$. Since $x\in\bigcup_{i\in\wedge}\mathcal{M}_i$ either $x\in\mathcal{H}$ or $x\in\mathcal{K}$ if $x\in\mathcal{H}\wedge x\in\mathcal{M}_i,\forall i\in\mathbb{N}$. By (Remark 2.10[9]) $\mathcal{M}_i\subseteq\mathcal{H}$ or $\mathcal{M}_i\subseteq\mathcal{K}$ and since $\mathcal{H}\cap\mathcal{K}\neq\emptyset$. Therefore $\bigcup_{i\in\wedge}\mathcal{M}_i\subseteq\mathcal{H}$ (because $\mathcal{M}_i\subseteq\mathcal{H}$ for all $i\in\mathbb{Z}$) that leads to $\mathcal{K}$ is empty. this is a contradiction.
By similar discussion $\mathcal{H}$ is also empty and this is a contradiction.
Then $\bigcup_{i\in\wedge}\mathcal{M}_i$ is $\mathcal{P}-connected$ sets. □

Theorem 2.13. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be a surjective $\mathcal{P}-$ continuous function and $\mathcal{H}, \mathcal{K}$ are $\mathcal{P}-$ separated sets in $\mathbb{Y}$, then $\mathfrak{f}^{-1}(\mathcal{H}), \mathfrak{f}^{-1}(\mathcal{K})$ are $\mathcal{P}-$ separated in $\mathbb{X}$.*

Proof . Since $\mathcal{H}, \mathcal{K}$ are $\mathcal{P}-$ *separated* sets in $\mathbb{Y}$, if $\mathfrak{f}$ is surjective, then $\mathfrak{f}^{-1}(\mathcal{H}), \mathfrak{f}^{-1}(\mathcal{K})$ are non-empty in $\mathbb{X}$. Suppose that $\mathfrak{f}^{-1}(\mathcal{H}), \mathfrak{f}^{-1}(\mathcal{K})$ are not $\mathcal{P}-$ *separated* in $\mathbb{X}$, then $\mathfrak{f}^{-1}(\mathcal{H}) \cap cl_{\mathcal{P}}(\mathfrak{f}^{-1}(\mathcal{K})) \neq \emptyset$, then we obtain that $\mathfrak{f}^{-1}(\mathcal{H}) \cap \mathfrak{f}^{-1}(cl_{\mathcal{P}}(\mathcal{K})) \neq \emptyset$, thus $\mathcal{H} \cap cl_{\mathcal{P}}(\mathcal{K}) \neq \emptyset$. Similarly, $\mathcal{K} \cap cl_{\mathcal{P}}(\mathcal{H}) \neq \emptyset$. So we get $\mathcal{H}, \mathcal{K}$ are not $\mathcal{P}-$ *separated* sets in $\mathbb{Y}$, which contradict the hypothesis. Hence $\mathfrak{f}^{-1}(\mathcal{H}), \mathfrak{f}^{-1}(\mathcal{K})$ are $\mathcal{P}-$ *separated* in $\mathbb{X}$. $\square$

Theorem 2.14. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be a $\mathcal{P}-$ continuous function and $\mathbb{X}$ is $\mathcal{P}-$ connected, then $\mathbb{Y}$ is connected.*

Proof . Suppose that $\mathbb{Y}$ is not connected, let $\mathbb{Y} = \mathcal{H} \cup \mathcal{K}$ where $\mathcal{H}$ and $\mathcal{K}$ are disjoint non-empty open in $\mathbb{Y}$. Since $\mathfrak{f}$ is $\mathcal{P}-$ *continuous*, $\mathbb{X} = \mathfrak{f}^{-1}(\mathcal{H}) \cup \mathfrak{f}^{-1}(\mathcal{K})$ where $\mathfrak{f}^{-1}(\mathcal{H})$ and $\mathfrak{f}^{-1}(\mathcal{K})$ are disjoint non-empty $\mathcal{P}-$ *open* sets in $\mathbb{X}$. Hence $\mathbb{Y}$ is connected. $\square$

Proposition 2.15. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be a bijective $\mathcal{P}-$ continuous function and E is $\mathcal{P}-$ connected in $\mathbb{X}$, then $\mathfrak{f}(E)$ is $\mathcal{P}-$ connected in $\mathbb{Y}$.*

Proof . Suppose that $\mathfrak{f}(E)$ is $\mathcal{P}-$ *disconnected* in $\mathbb{Y}$. Then $\mathfrak{f}(E) = \mathcal{H} \cup \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ two non-empty $\mathcal{P}-$ *separated* in $\mathbb{Y}$. By theorem 2.14, we have $\mathfrak{f}^{-1}(\mathcal{H}), \mathfrak{f}^{-1}(\mathcal{K})$ are $\mathcal{P}-$ *separated* in $\mathbb{X}$. Since $\mathfrak{f}$ is bijective, then $E = \mathfrak{f}^{-1}(\mathfrak{f}(E)) = \mathfrak{f}^{-1}(\mathcal{H}) \cup \mathfrak{f}^{-1}(\mathcal{K})$. Hence E is not $\mathcal{P}-$ *connected* in $\mathbb{X}$, which contradict the hypothesis. Thus $\mathfrak{f}(E)$ is $\mathcal{P}-$ *connected* in $\mathbb{Y}$. $\square$

Theorem 2.16. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be surjective $\mathcal{P}-$ irresolute and $\mathbb{X}$ is $\mathcal{P}-$ connected, then $\mathbb{Y}$ is $\mathcal{P}-$ connected.*

Proof . Suppose that $\mathbb{Y}$ is $\mathcal{P}-$ *disconnected*, let $\mathbb{Y} = \mathcal{H} \cup \mathcal{K}$ where $\mathcal{H}$ and $\mathcal{K}$ are disjoint non-empty $\mathcal{P}-$ *open* in $\mathbb{Y}$. Since $\mathfrak{f}$ is $\mathcal{P}-$ *irresolute* and onto, then $\mathbb{X} = \mathfrak{f}^{-1}(\mathcal{H}) \cup \mathfrak{f}^{-1}(\mathcal{K})$ where $\mathfrak{f}^{-1}(\mathcal{H})$ and $\mathfrak{f}^{-1}(\mathcal{K})$ are disjoint non-empty $\mathcal{P}-$ *open* sets in $\mathcal{K}$ (every $\mathcal{P}-$ *open* set is $semi_{\mathcal{P}}O$.).
Since $\mathbb{X}$ is $\mathcal{P}-$ *connected* then we get that $\mathbb{Y}$ is $\mathcal{P}-$ *connected.* $\square$

Definition 2.17. *A space $(\mathbb{X}, \Im_{\mathcal{P}})$ is called strongly $\mathcal{P}-$ connected briefly ($\mathcal{P}\_\mathcal{SC}$.) iff it is not a disjoint union of countably many but more one $\mathcal{P}-$ closed set.*
By another words, if $\mathbb{X} \neq \cup F_i$, where F_i are disjoint non empty $\mathcal{P}-$ closed sets of $\mathbb{X}$.

Remark 2.18. *A subset E of a $(\mathbb{X}, \Im_{\mathcal{P}})$ is said to be strongly $\mathcal{P}-$ connected iff $E \subseteq \mathcal{H}$ or $E \subseteq \mathcal{K}$ whenever $E \subseteq \mathcal{H} \cup \mathcal{K}$, $\mathcal{H}$ and $\mathcal{K}$ are $\mathcal{P}-$ open sets in $\mathbb{X}$.*

Proposition 2.19. *Let E be $\mathcal{P}\_\mathcal{SC}$., then E is $\mathcal{P}-$ connected.*

Proof . Suppose that E is $\mathcal{P}-$ *disconnected*, then $\exists$ $\mathcal{P}-$ *open* sets $\mathcal{H}$ and $\mathcal{K}$ for which $E = (E \cap \mathcal{H}) \cup (E \cap \mathcal{K})$, such that $E \cap \mathcal{H} \neq \emptyset$, $E \cap \mathcal{K} \neq \emptyset$, $E \cap (\mathcal{H} \cap \mathcal{K}) = \emptyset$ then $E \subseteq (\mathcal{H} \cup \mathcal{K})$, but by definition 2.17, $E \nsubseteq \mathcal{H}$ and $E \nsubseteq \mathcal{K}$ contradicts E being $\mathcal{P}-$ *connected.* $\square$

Remark 2.20. *A set E will be weakly $\mathcal{P}-$ disconnected iff it is not $\mathcal{P}\_\mathcal{SC}$.*

By adding some condition to the $\mathcal{P}-$ *irresolute* function we obtain the following Theorem.

Theorem 2.21. *Any surjective $\mathcal{P}-$ irresolute image of a strongly $\mathcal{P}-$ connected space is strongly $\mathcal{P}-$ connected.*

Proof . Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be surjective $\mathcal{P}-$ *irresolute* function and assume that $\mathfrak{f}(\mathbb{X})$ is weakly $\mathcal{P}-$ *disconnected*, since $\mathfrak{f}$ is $\mathcal{P}-$ *irresolute* using definition 2.17, then the inverse image of $\mathcal{P}-$ *semi open* sets is $\mathcal{P}-$ *open* sets, so $\mathbb{X}$ is a disjoint union of $\mathcal{P}-$ *open* sets. Hence $\mathfrak{f}(\mathbb{X})$ is $\mathcal{P}\_\mathcal{SC}$.
$\square$

3. Penta Compactness in Penta Topological Space

Definition 3.1. *A collection* $\{G_i : i \in \wedge\}$ *of* $\mathcal{P}-open$ *sets in* $(\mathbb{X}, \Im_{\mathcal{P}})$ *is called a* $\mathcal{P}-open$ *cover of a subset* E *of* $\mathbb{X}$ *if* $E \subseteq \{G_i : i \in \wedge\}$.

Definition 3.2. *A* $(\mathbb{X}, \Im_{\mathcal{P}})$ *is said to be* $\mathcal{P}-compact$ *if every* $\mathcal{P}-open$ *cover of* $\mathbb{X}$ *has a finite sub cover.*

Note: For each $\{G_i : i \in \wedge\}$ of $\mathcal{P}-open$ sets for which $E \subseteq \cup\{G_i : i \in \wedge\}, \exists G_{i1}, G_{i2}, ..., G_{in}$ among the G_i's, such that $E \subseteq G_{i1} \cup G_{i2} \cup ... \cup G_{in}$. Then a space $(\mathbb{X}, \Im_{\mathcal{P}})$ is $\mathcal{P}-compact$ iff for each $\{G_i : i \in \wedge\}$ of $\mathcal{P}-open$ sets for which $\mathbb{X} = \cup\{G_i : i \in \wedge\}$, there exist finitely many sets $G_{i1}, G_{i2}, ..., G_{in}$ among the G_i's such that $\mathbb{X} = G_{i1} \cup G_{i2} \cup ... \cup G_{in}$.

Theorem 3.3. *Every* $\mathcal{P}-compact$ *space is compact space.*

Proof . Let $(\mathbb{X}, \Im_{\mathcal{P}})$ be a $\mathcal{P}-compact$ space.
Assume that $(\mathbb{X}, \Im_i)$ is not compact, where $\Im_i = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$. Then every open cover of $\mathbb{X}$ has not finite sub cover. Since every open set is a $\mathcal{P}-open$ set, so $(\mathbb{X}, \Im_{\mathcal{P}})$ is not $\mathcal{P}-compact$ space (contradiction by hypothesis). Hence $(\mathbb{X}, \Im_i)$ is compact. □

Remark 3.4. *The converse is not true.*

Example 3.5. *Let* $\Im_{\mathcal{P}} = \{\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{I}_{rr}, \mathbb{Z}, \mathbb{N}, (-n, n)\}$ *be a Penta topology on* $\mathbb{R}$, *when co-finite topolog y* $= \{G \subseteq \mathbb{R}; G^c \text{ } is \text{ } finite\} \cup \{\emptyset\}$ *such that* $\Im_1 = \{\mathbb{R}, \emptyset, \mathbb{Q}\}$, $\Im_1 = \{\mathbb{R}, \emptyset, \mathbb{I}_{rr}\}$, $\Im_3 = \{\mathbb{R}, \emptyset, \mathbb{Z}\}$,$\Im_4 = \{\mathbb{R}, \emptyset, \mathbb{N}\}$ *and the usual topology* $\Im_5 = \{\mathbb{R}, \emptyset, u \subseteq \mathbb{R}; u = (-n, n) isanopeninteraval\}$. *We can show that* $(\mathbb{R}, \Im_{\mathcal{P}})$ *is* $\mathcal{P}-compact$ *space but* $\Im_5$ *is not compact, because* $C = \{(-n, n) : n \in \mathbb{N}\}$ *is open cover of* $\mathbb{R}$ *but it hasn't sub cover.*

Proposition 3.6. *Every* $\mathcal{P}-closed$ *subset of a* $\mathcal{P}-compact$ *space is* $\mathcal{P}-compact$ *space.*

Proof . Let $(\mathbb{X}, \Im_{\mathcal{P}})$ be a $\mathcal{P}-compact$ space and $A \subseteq \mathbb{X}$ be $\mathcal{P}-closed$, suppose that $\{G_i : i \in \wedge\}$ is $\mathcal{P}-open$ set of A, then $A \subseteq \cup\{G_i : i \in \wedge\}$, it is clearly that $\mathbb{X} - A$ is $\mathcal{P}-open$ set. This shows that the family consisting of the sets $\mathbb{X} - A$ and G_i's is an $\mathcal{P}-open$ covers of $\mathbb{X}$, which are known to be $\mathcal{P}-compact$ Hence these covers has finite subcovers, say $\mathbb{X} - A, G_{i1}, G_{i2}, ..., G_{in}$, we get $\mathbb{X} = (\mathbb{X} - A) \cup (\bigcup_{r=1}^{n} G_{ir})$. We claim that $A \subseteq \bigcup_{r=1}^{n} G_{ir}$ and assume that there exist $a \in A$, such that $a \notin \bigcup_{r=1}^{n} G_{ir}$ and $a \notin \mathbb{X} - A$. Now the family consisting of $\mathbb{X} - A, G_{i1}, G_{i2}, ..., G_{in}$ is not an $\mathcal{P}-open$ covers of $\mathbb{X}$. That is a contradiction. Hence $A \subseteq \bigcup_{r=1}^{n} G_{ir}$ hold, such that $\{G_{ir} : r = 1, 2, ..., n\}$ are a $\mathcal{P}-open$ covers of A. So $\{G_i\}$ of A, has finite subcover. Then A is a $\mathcal{P}-compact$. □

Proposition 3.7. *Every* $\mathcal{P}-closed$ *subset of a* $\mathcal{P}-compact$ *space is* $\mathcal{P}-compact$ *relative* $\mathcal{P}-$ *topology.*

Proof . Let A be a $\mathcal{P}-closed$ subset of a $\mathcal{P}-compact$ $(\mathbb{X}, \Im_{\mathcal{P}})$, then $\mathbb{X}-A$ is $\mathcal{P}-open$ set in $(\mathbb{X}, \Im_{\mathcal{P}})$, so $\{G_i : i \in \wedge\}$ be $\mathcal{P}-open$ cover of A such that $A \subseteq \cup\{G_i : i \in \wedge\}$ and $(\mathbb{X}-A) \cup \{G_i : i \in \wedge\} = \mathbb{X}$. Hence $A \subseteq \cup\{G_i : i \in \wedge\}$ (because $\mathbb{X}$ is $\mathcal{P}-compact$, $\exists A \subseteq (\mathbb{X} - A) \cup \{G_i : i \in \wedge\} = \mathbb{X}$). Then A is $\mathcal{P}-compact$ relative $\mathbb{X}$. □

Theorem 3.8. $\mathcal{P}-continuous$ *image of a* $\mathcal{P}-compact$ *space is* $\mathcal{P}-compact$.

$$\begin{tikzcd} \mathcal{P}\_\text{compact space} \arrow[r] \arrow[d] & \text{compact space} \arrow[d] \\ \text{locally } \mathcal{P}\_\text{compact space} & \text{locally compact space} \arrow[l] \end{tikzcd}$$

Figure 1: Relationship between the compactness space

Proof . Let $\mathfrak{f} : \mathbb{X} \to \mathbb{Y}$ be a $\mathcal{P} - continuous$ function and let $\{G_i : \ i \in \wedge\}$ be a $\mathcal{P} - cover$ of E. then $\{\mathfrak{f}^{-1}(G_i) : i \in \wedge\}$ is a $\mathcal{P} - open$ cover of $\mathbb{X}$, $(\mathfrak{f}^{-1}(G))$ is a $\mathcal{P} - open$ set in $\mathbb{X}$, since $\mathbb{X}$ is $\mathcal{P} - compact$ it has a finite sub_cover say $\mathbb{X} = \mathfrak{f}^{-1}(G_{i1}) \cup ... \cup \mathfrak{f}^{-1}(G_{in}) = \mathfrak{f}^{-1}(G_{i1} \cup ... \cup G_{in})$. So that $\mathbb{X} = G_{i1} \cup G_{i2} \cup ... \cup G_{in}$. Hence $\mathbb{Y}$ is $\mathcal{P} - compact$. □

Proposition 3.9. *Any surjective $\mathcal{P} - irresolute$ image function of a $\mathcal{P} - compact$ space is $\mathcal{P} - compact$.*

Proof . Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be surjective $\mathcal{P} - irresolute$ function and $\mathbb{X}$ is $\mathcal{P} - compact$ space then to prove $\mathbb{Y}$
$\mathcal{P} - compact$ space, suppose that $\{G_i : \ i \in \wedge\}$ is $\mathcal{P} - open$ sets of $\mathbb{Y}$, then $\{\mathfrak{f}^{-1}(G_i) : i \in \wedge\}$ is $\mathcal{P} - open$ sets of $\mathbb{X}$ (because $\mathfrak{f}$ is $\mathcal{P} - irresolute$). Since $\mathbb{X}$ is $\mathcal{P} - compact$ has a finite subcover, we get $\{\mathfrak{f}^{-1}(G_1), \mathfrak{f}^{-1}(G_2), ..., \mathfrak{f}^{-1}(G_n)\}$ then it leads to $\{G_1, ..., G_n\}$ is a finite subcover of $\mathbb{Y}$, we get $\mathbb{Y}$ is $\mathcal{P} - compact$. □

Definition 3.10. *A Penta Topological Space $(\mathbb{X}, \Im_{\mathcal{P}})$ is called locally $\mathcal{P} - compact$ if every point in $\mathbb{X}$ has at least one G_{PN} has at least one $\mathcal{P} - compact$ sub spaces.*

Theorem 3.11. *Every $\mathcal{P} - compact$ space is locally $\mathcal{P} - compact$.*

Proof . Let $(\mathbb{X}, \Im_{\mathcal{P}})$ be $\mathcal{P} - compact$ space, then $\mathbb{X}$ is both $\mathcal{P} - closed$ and $\mathcal{P} - open$ set and has at last one G_{PN} whose $\mathcal{P} - closure$ is $\mathcal{P} - compact$. Hence $(\mathbb{X}, \Im_{\mathcal{P}})$ is locally $\mathcal{P} - compact$. □

Remark 3.12.

1. *The converse of theorem 3.11 is not true, this can be seen through the following example.*
   Example.
   Let $\mathbb{X}$ be infinite set with discrete penta topological space, then $\mathbb{X}$ is not $\mathcal{P} - compact$ but the collection of all singleton sets is an infinite $\mathcal{P} - open$ cover of $\mathbb{X}$ which cannot has a finite subcover, by hypotheses $\mathbb{X}$ is locally $\mathcal{P} - compact$, hence for each point of $x \in \{x\}$ has a G_{PN} whose $\mathcal{P} - closure$ is $\mathcal{P} - compact$.
2. *Every locally compact space is locally $\mathcal{P} - compact$ space, from definition locally compact space and every open set is $\mathcal{P} - open$ set.*

By theorems 3.3,3.11 and remarks 3.12, present the following diagram that illustrates the relationship between the compactness space. We get Figure 1

4. Penta Separation Axioms in Penta Topological Spaces

Definition 4.1. *A space* $(\mathbb{X}, \Im_{\mathcal{P}})$ *is said to be*

i. $T_{\mathcal{P}0}$*-space if for every pair of distinct points* v, u *of* $\mathbb{X}$*, there exists a* $\mathcal{P}-$ *open set containing one of them but not the other.*

ii. $T_{\mathcal{P}1}$*-space if for every pair of distinct points* v, u *of* $\mathbb{X}$*, there exists two* $\mathcal{P}-$ *open sets containing one of the two points but not the other.*

iii. $T_{\mathcal{P}2}$*-space if for every pair of distinct points* v, u *of* $\mathbb{X}$*, there exists two distinct* $\mathcal{P}-$ *open sets* $\mathcal{H}, \mathcal{K}$*, such that* $v \in \mathcal{H}, u \in \mathcal{K}$*.*

Results 4.2. *A Penta Topological Spaces* $(\mathbb{X}, \Im_{\mathcal{P}})$*:*

First case: *Every* T_0*- Topological Space is a* $\Im_{\mathcal{P}0}$*- Topological Space.*

Second case: *Every* T_1*- Topological Space is a* $\Im_{\mathcal{P}1}$*- Topological Space.*

Third case: *If there is no one of the topologies is a* T_0- *space, then the Penta Topological Space is a* $\Im_{\mathcal{P}0}$*- Topological Space.*

Fourth case: *If there is no one of the topologies is a* T_1- *space, then the Penta Topological Space is a* $\Im_{\mathcal{P}1}$*- Topological Space.*

Fifth case: *If at least one of the topologies is a* T_i- *space* $\forall i = 0, 1$*, then the Penta Topological Space is a* $\Im_{\mathcal{P}i}$*- Topological Space.*

Let us discus the following examples for above cases:

Example 4.3. *Let* $\mathbb{X} = \{a, b, c, d\}$*.*

1. *For First and second cases.*
$\Im_1 = \{\mathbb{X}, \emptyset, \{b\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b, d\}, \{b, c, d\}\}$, $\Im_2 = \{\mathbb{X}, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$
$\Im_3 = \{\mathbb{X}, \emptyset, \{a\}, \{b, d\}, \{a, d\}, \{a, b, d\}, \{b, c, d\}\}$, $\Im_4 = \{\mathbb{X}, \emptyset, \{b\}, \{d\}, \{b, d\}, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$
$\Im_5 = \{\mathbb{X}, \emptyset, \{a\}, \{b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b\}\}$*, are five Topological Spaces, then*
$\Im_{\mathcal{P}} = \{\mathbb{X}, \emptyset, \{a\}, \{d\}, \{b\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$
Therefore $(\mathbb{X}, \Im_{\mathcal{P}})$ *is* $T_{\mathcal{P}i}-$ *space with the five topological spaces is* T_i- *spaces.* $\forall i = 0, 1$
2. *For the third and Fourth cases.*
$\Im_1 = \{\mathbb{X}, \emptyset, \{b\}, \{a, b, c\}\}$, $\Im_2 = \{\mathbb{X}, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$
$\Im_3 = \{\mathbb{X}, \emptyset, \{a\}, \{b, c, d\}\}$, $\Im_4 = \{\mathbb{X}, \emptyset, \{d\}, \{a, d\}\}$, $\Im_5 = \{\mathbb{X}, \emptyset, \{b\}, \{b, d\}, \{b, c\}, \{b, c, d\}\}$*, are five Topological Spaces. Then*
$\Im_{\mathcal{P}} = \{\mathbb{X}, \emptyset, \{a\}, \{d\}, \{b\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.
Therefore $(\mathbb{X}, \Im_{\mathcal{P}})$ *is* $T_{\mathcal{P}i}-$ *space, but there exist* $(\mathbb{X}, \Im_2)$ *is* T_i- *spaces.* $\forall i = 0, 1$
3. *For the fifth case.*
$\Im_1 = \{\mathbb{X}, \emptyset, \{b\}, \{a, b\}\}$, $\Im_2 = \{\mathbb{X}, \emptyset, \{a, d\}, \{a, b, d\}\}$, $\Im_3 = \{\mathbb{X}, \emptyset, \{b, d\}, \{b, c, d\}\}$,
$\Im_4 = \{\mathbb{X}, \emptyset, \{a\}, \{d\}, \{a, d\}\}$ *and* $\Im_5 = \{\mathbb{X}, \emptyset, \{b, c\}, \{a, b, c\}\}$*, are five Topological Spaces, then*
$\Im_{\mathcal{P}} = \{\mathbb{X}, \emptyset, \{a\}, \{d\}, \{b\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.
Therefore $(\mathbb{X}, \Im_{\mathcal{P}})$ *is* $T_{\mathcal{P}i}-$ *space but the five topological spaces are not* T_i- *spaces.* $\forall i = 0, 1$

Results 4.4. *A Penta Topological Spaces* $(\mathbb{X}, \Im_{\mathcal{P}})$*:*

$$\begin{array}{ccccc} \exists\ T_2 - \text{space} & \rightarrow & T_1 - \text{space} & \rightarrow & T_0 - \text{space} \\ \downarrow & & \downarrow & & \downarrow \\ T_{\mathcal{P}2}\text{- space} & \rightarrow & T_{\mathcal{P}1}\text{ - space} & \rightarrow & T_{\mathcal{P}0}\text{- space} \end{array}$$

Figure 2:

First case: *If there is no one of the topologies is a $T_2 - space$ then the Penta Topological Space is a $\Im_{\mathcal{P}2}$- Topological Space.*

Second case: *If at least one of the topologies is a $T_2 - space$ then the Penta Topological Space is a $\Im_{\mathcal{P}2}$- Topological Space.*

Example 4.5. *A Penta topology*
$\Im_{\mathcal{P}} = \{\mathbb{X}, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b,d\}, \{b,c\}, \{a,d\}, \{a,b\}, \{a,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}\}$ *on* $\mathbb{X} = \{a,b,c,d\}$, *when*

I. $\Im_1 = \{\mathbb{X}, \emptyset, \{a\}, \{b\}, \{d\}, \{b,d\}, \{b,c\}, \{a,d\}, \{a,b\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}\}$
$\Im_2 = \{\mathbb{X}, \emptyset, \{a,c\}, \{b,d\}\}$, $\Im_3 = \{\mathbb{X}, \emptyset, \{c\}, \{c,d\}, \{a,c,d\}\}$, $\Im_4 = \{\mathbb{X}, \emptyset, \{c\}, \{d\}, \{c,d\}\}$ *and* $\Im_5 = \{\mathbb{X}, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ *are five Topological Spaces, then* $(\mathbb{X}, \Im_{\mathcal{P}})$ *is* $T_{\mathcal{P}2} - space$ *but not necessary that one of the five topologies* $T_2 - space$. *It is clear that* $(\mathbb{X}, \Im_1)$ *is* $T_2 - space$.

II. $\Im_1 = \{\mathbb{X}, \emptyset, \{b\}, \{a,d\}, \{b,c\}, \{a,b,d\}, \{b,c,d\}\}$, $\Im_2 = \{\mathbb{X}, \emptyset, \{a,c\}, \{b,d\}\}$,
$\Im_3 = \{\mathbb{X}, \emptyset, \{c\}, \{c,d\}, \{a,c,d\}\}$,$\Im_4 = \{\mathbb{X}, \emptyset, \{c\}, \{d\}, \{c,d\}\}$ *and*
$\Im_5 = \{\mathbb{X}, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ *are five Topological Spaces.*
Then $(\mathbb{X}, \Im_{\mathcal{P}})$ *is* $T_{\mathcal{P}2} - space$ *but the five topologies are* $T_2 - space$.

By results 4.2,4.4 and definition 4.1, we obtain the following diagram which illustrates the relationship between the types of separation axioms. By using case (1,5) we get Figure 2.

Proposition 4.6. *A space* $(\mathbb{X}, \Im_{\mathcal{P}})$ *is a* $T_{\mathcal{P}0} - space$ *if and only if for every distinct points* v, u *of* $\mathbb{X}$, $cl_{\mathcal{P}}\{v\} \neq cl_{\mathcal{P}}\{u\}$.

Proof . For every v, u of $\mathbb{X}$ and $v \neq u$, whenever $\mathbb{X}$ is a $T_{\mathcal{P}0} - space$ there exist a $\mathcal{P} - open$ set such that $v \in \mathcal{H}, u \in \mathbb{X} \backslash \mathcal{H}$, hence $\{u\} \subseteq \mathbb{X} \backslash \mathcal{H}$ is a $\mathcal{P} - closed$ set, $cl_{\mathcal{P}}\{u\} \subseteq \mathbb{X} \backslash \mathcal{H}$, so $v \notin cl_{\mathcal{P}}\{u\}$. Then $cl_{\mathcal{P}}\{v\} \neq cl_{\mathcal{P}}\{u\}$.
Conversely, assume that $v \neq u$, then $cl_{\mathcal{P}}\{v\}$ and $cl_{\mathcal{P}}\{u\}$, are distinct sets, $\exists \rho \in \mathbb{X}$ belong to one sets $\rho \in cl_{\mathcal{P}}\{v\}$ and $\rho \notin cl_{\mathcal{P}}\{u\}$. Now $v \notin cl_{\mathcal{P}}\{u\}$ (because $v \in cl_{\mathcal{P}}\{u\}$ then $cl_{\mathcal{P}}\{v\} \subseteq cl_{\mathcal{P}}(cl_{\mathcal{P}}\{u\}) = cl_{\mathcal{P}}\{u\}$). Also $\rho \in cl_{\mathcal{P}}\{v\} \subseteq cl_{\mathcal{P}}\{u\}$ which is a contradiction, therefore $v \in (cl_{\mathcal{P}}\{u\})^c$, so $(cl_{\mathcal{P}}\{u\})^c$ is $\mathcal{P} - open$ set contained one but not the other. Then $\mathbb{X}$ is a $T_{\mathcal{P}0} - space$. □

Theorem 4.7. *A Penta topological space* $\mathbb{X}$ *is a* $T_{\mathcal{P}1} - space$, *iff every singleton is* $\mathcal{P} - closed$ *sets.*

Proof . Obvious. □

Theorem 4.8. *A strongly* $\mathcal{P} - connected$ $T_{\mathcal{P}1} - space$ *has at most one point in uncountable space.*

Proof . By theorem 4.7, we have a singleton set in a $T_{\mathcal{P}1} - space$ is $\mathcal{P} - closed$ set. Therefore, we get $T_{\mathcal{P}1} - space$ cannot have countably many but more than one point. □

By adding some conditions to the function, we get the following theorems.

Theorem 4.9. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be a bijective $\mathcal{P} - open$ function and $\mathbb{X}$ is a $T_i - space$ then $\mathbb{Y}$ is $T_{\mathcal{P}i} - space$, where $i = 0, 1, 2$.*

Proof . We prove the case $i = 2$.
Let v_2, u_2 be two points in $\mathbb{Y}$ and $v_2 \neq u_2$, since $\mathfrak{f}$ is bijective, then $\exists v_1, u_1 \in \mathbb{X}$ and $\mathfrak{f}(v_1) = v_2, \mathfrak{f}(u_1) = u_2$. But $\mathbb{X}$ is a T_2, then $\exists$ two disjoint open sets $\mathcal{H}, \mathcal{K} \in \mathbb{X}$, whenever $v_1 \in \mathcal{H}, u_1 \in \mathcal{K}$. Then $\mathfrak{f}(\mathcal{H}), \mathfrak{f}(\mathcal{K})$ are $\mathcal{P} - open$ sets in $\mathbb{Y}$ (because every $\mathcal{P} - open$ is $semi_{\mathcal{P}}0.$, and $\mathfrak{f}$ is $\mathcal{P} - open$) we get $v_2 \in \mathfrak{f}(\mathcal{H}), u_2 \in \mathfrak{f}(\mathcal{K})$ and $\mathfrak{f}(\mathcal{H}) \cap \mathfrak{f}(\mathcal{K}) = \emptyset$. Hence $\mathbb{Y}$ is $T_{\mathcal{P}2} - space$. □

Theorem 4.10. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be an injective $\mathcal{P} - continuous$ function and $\mathbb{Y}$ is $T_i - space$, then $\mathbb{X}$, is $T_{\mathcal{P}i} - space$, where $i = 0, 1, 2$.*

Proof . We prove the case $i = 1$
Since $\mathbb{Y}$ is T_1 and let v, u of $\mathbb{X}$ and $v \neq u$, there exist two disjoint $\mathcal{P} - open$ sets $\mathcal{H}, \mathcal{K} \in \mathbb{Y}$ (because every $\mathcal{P} - open$ is $semi_{\mathcal{P}}0.$) such that $\mathfrak{f}(v) \in \mathcal{H}, \mathfrak{f}(u) \in \mathcal{K}, \mathfrak{f}(v) \neq \mathfrak{f}(u)$, since $\mathfrak{f}$ is $\mathcal{P} - continuous$, then $\mathfrak{f}^{-1}(\mathcal{H})$ and $\mathfrak{f}^{-1}(\mathcal{K})$ are $\mathcal{P} - open$ sets of $mathbbX$, we get $v \in \mathfrak{f}^{-1}(\mathcal{H}), u \in \mathfrak{f}^{-1}(\mathcal{K})$. Hence $\mathbb{X}$ is $T_{\mathcal{P}1} - space$. □

Theorem 4.11. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be an injective $\mathcal{P} - continuous$ function and $\mathbb{Y}$ is $T_{\mathcal{P}i} - space$, then $\mathbb{X}$, is $T_{\mathcal{P}i} - space$, where $i = 0, 1, 2$.*

Proof . We prove the case $i = 2$
Let v, u of $\mathbb{X}$ and $v \neq u$, since $\mathfrak{f}$ is one to one, then $\mathfrak{f}(v) \neq \mathfrak{f}(u)$ in $\mathbb{Y}$. But $\mathbb{Y}$, is $T_{\mathcal{P}2} - space$, then there exist two disjoint $\mathcal{P} - open$ sets $\mathcal{H}, \mathcal{K} \in \mathbb{Y}$, whenever $\mathfrak{f}(v) \in \mathcal{H}, \mathfrak{f}(u) \in \mathcal{K}$. Then $\mathfrak{f}^{-1}(\mathcal{H}), \mathfrak{f}^{-1}(\mathcal{K})$ a $\mathcal{P} - open$ (because $\mathfrak{f}$ is $\mathcal{P} - continuous$), we get $v \in \mathfrak{f}^{-1}(\mathcal{H}), u \in \mathfrak{f}^{-1}(\mathcal{K})$ and $\mathfrak{f}^{-1}(\mathcal{H}) \cap \mathfrak{f}^{-1}(\mathcal{K}) = \emptyset$. So $\mathbb{X}$ is $T_{\mathcal{P}2} - space$ □

Theorem 4.12. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be an injective $\mathcal{P} - irresolute$ function and $\mathbb{Y}$, is $T_{\mathcal{P}i} - space$. Then $\mathbb{X}$, is $T_{\mathcal{P}i} - space$, where $i = 0, 1, 2$.*

Proof . We prove the case $i = 0$
Let v, u in $\mathbb{X}$ and $v \neq u$, since $\mathfrak{f}$ is one to one, then $\mathfrak{f}(v) \neq \mathfrak{f}(u)$ in $\mathbb{Y}$, $\mathbb{Y}$ is $T_{\mathcal{P}0} - space$, then $\exists$ a $\mathcal{P} - open$ set $\mathcal{H} \in \mathbb{Y}$, whenever $\mathfrak{f}(v) \in \mathcal{H}, \mathfrak{f}(u) \notin \mathcal{H}$. Then $\mathfrak{f}^{-1}(\mathcal{H})$ is $semi_{\mathcal{P}}0$. Set (because $\mathfrak{f}$ is $\mathcal{P} - irresolute$ and every $\mathcal{P} - open$ is $semi_{\mathcal{P}}0.$ set), we get $v \in \mathfrak{f}^{-1}(\mathcal{H}), u \notin \mathfrak{f}^{-1}(\mathcal{H})$. So $\mathbb{X}$ is $T_{\mathcal{P}0} - space$. □

Proposition 4.13. *Let $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be $\mathcal{P} - homeomorphim$ and $\mathbb{Y}$ is $T_{\mathcal{P}2} - space$, then $\mathbb{X}$ $T_{\mathcal{P}2} - space$.*

Proof . We prove the case $i = 0$
Suppose that $v_1, v_2 \in \mathbb{X}$, with $v_1 \neq v_2$. We get $\mathfrak{f}(v_1) \neq \mathfrak{f}(v_2)$ and $\mathfrak{f}(v_1), \mathfrak{f}(v_2) \in \mathbb{Y}$, since $\mathbb{Y}$ is $T_{\mathcal{P}2} - space$ there exist two $\mathcal{P} - open$ sets $\mathcal{H}, \mathcal{K} \in \mathbb{Y}$ such that $\mathfrak{f}(v_1) \in \mathcal{H}, \mathfrak{f}(v_2) \in \mathcal{K}$ and $\mathcal{H} \cap \mathcal{K} = \emptyset$. Now $v_1 \in \mathfrak{f}^{-1}(\mathcal{H}), v_2 \in \mathfrak{f}^{-1}(\mathcal{K})$ and $\mathfrak{f}^{-1}(\mathcal{H}) \cap \mathfrak{f}^{-1}(\mathcal{K}) = \mathfrak{f}^{-1}(\mathcal{H} \cap \mathcal{K}) = \mathfrak{f}^{-1}(\emptyset) = \emptyset$. Hence $\mathbb{X}$ is $T_{\mathcal{P}2} - space$.
□

Theorem 4.14. *Every $\mathcal{P} - compact$ subset of $T_{\mathcal{P}2} - space$ is $\mathcal{P} - closed$ set.*

Proof . Let E be a $\mathcal{P}-compact$ subset of $T_{\mathcal{P}2}-space$ $(\mathbb{X}, \Im_{\mathcal{P}})$ and suppose that $u \notin E$, then $\exists$ distinct $\mathcal{P}-open$ set contains u *and* $v \in E$. We obtain $\mathcal{H}, \mathcal{K}$ $\mathcal{P}-open$ sets contains u and v respectively. $\exists$ $\{G_v : v \in E\}$ is $\mathcal{P}-cover$ of E by $\mathcal{P}-open$ sets in $\mathbb{X}$, we get $\exists$ finitely many of them, $G_{v1}, ..., G_{vn}$ is $\mathcal{P}-cover$ of E, thus $G = \bigcup_{i=1}^{n} G_{vi}$ contains E and disjoint from $\mathcal{P}-open$ set $C = \bigcap_{i=1}^{n} C_{ui}$. Taking the intersection of $\mathcal{P}-open$ sets contains u, if $w \in G$, then $w \in G_{vi}$, then $w \notin C_{vi}$ and $w \notin C$. Then C is $\mathcal{P}-open$ set and $u \in C$ disjoint from E. □

Example 4.15. *From example 1.5, since* $\mathbb{X}$ *is finite, then* $(\mathbb{X}, \Im_{\mathcal{P}})$ $\mathcal{P}-compact$ *but* $(\mathbb{X}, \Im_{\mathcal{P}})$ *is not* $T_{\mathcal{P}2}-space$.

Results 4.16. *All discrete spaces are locally* $\mathcal{P}-compact$ *and* $T_{\mathcal{P}2}-space$ *these are* $\mathcal{P}-compact$ *if and only if they are finite.*

5. Conclusions

The main results of this paper are stated as below:

1. the concepts of connectedness, compactness and separation axioms on Penta topological space developed with some theorems and the relationship between them.
2. If $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be bijective $\mathcal{P}-open$ function and $\mathbb{X}$ is a $T_i-space$, then $\mathbb{Y}$ is $T_{\mathcal{P}i}-space$, where $i = 0, 1, 2$.
3. If $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be injective $\mathcal{P}-continuous$ function and $\mathbb{Y}$ is a $T_i-space$, then $\mathbb{X}$ is $T_{\mathcal{P}i}-space$, where $i = 0, 1, 2$.
4. If $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be injective $\mathcal{P}-continuous$ function and $\mathbb{Y}$ is a $T_{\mathcal{P}i}-space$, then $\mathbb{X}$ is $T_{\mathcal{P}i}-space$, where $i = 0, 1, 2$.
5. If $\mathfrak{f} : (\mathbb{X}, \Im_{\mathcal{P}}) \to (\mathbb{Y}, \Im_{\check{\mathcal{P}}})$ be injective $\mathcal{P}-irresolute$ function and $\mathbb{Y}$ is a $T_{\mathcal{P}i}-space$, then $\mathbb{X}$ is $T_{\mathcal{P}i}-space$, where $i = 0, 1, 2$.
6. If $\mathcal{P}-connected$ space then there exist at least one of the five topologies is connected.
7. Every $\mathcal{P}-compact$ space is locally $\mathcal{P}-compact$ space.
8. Every locally compact space is locally $\mathcal{P}-compact$ space.

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