Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021, 605-615
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.15096.1792

Perturbed first-order state dependent Moreau's sweeping process

Doria Affane[a], Mustapha Fateh Yarou[b,*]

[a] *LMPA Laboratory, Department of Mathematics, Jijel University*
[b] *LMPA Laboratory, Department of Mathematics, Jijel University*

(Communicated by Hamid Khodaei)

Abstract

In this paper, we deal with the state-dependent nonconvex sweeping process motivated through quasi-variational inequalities arising in the evolution of sandpiles, quasistatic evolution problems with friction, micromechanical damage models for iron materials. We prove the existence of absolutely continuous solution for the problem in presence of a perturbation, that is an external force applied on the system. The perturbation considered here is general and take the form of a sum of a single-valued Carathéodory mapping and a set-valued unbounded mapping.

Keywords: State-dependent sweeping process, prox regular sets, proximal subdifferential, perturbation, ball-compactness.
2010 MSC: Primary 34A60; Secondary 49J52

1. Introduction

The perturbed state-dependent sweeping process is an evolution differential inclusion governed by the normal cone to a mobile set depending on both time and state variables, of the following form:

$$\begin{cases} -\dot{x}(t) \in N_{C(t,x(t))}(x(t)) + F(t, x(t)), & a.e \;\; t \in [T_0, T]; \\ x(t) \in C(t, x(t)), \; \forall t \in [T_0, T], & x(T_0) = a, \end{cases} \tag{1.1}$$

where $N_{C(t,x(t))}(x(t))$ is the normal cone to $C(t, x(t))$ at $x(t)$ and F is a single or set-valued mapping playing the role of a perturbation to the problem, that is an external force applied on the system. This kind of problems was initiated by J. J. Moreau (see [23]) for time-dependent sets $C(t)$ and

*Corresponding author
Email addresses: affanedoria@yahoo.fr (Doria Affane), mfyarou@yahoo.com (Mustapha Fateh Yarou)

Received: June 2018 *Accepted:* August 2021

$F \equiv \{0\}$ to deal with problems arising in elastoplasticity, quasistatics, electrical circuits, hysteresis and dynamics. Since then, various generalizations have been obtained, see for instance [1, 2, 3, 4, 6, 7, 8, 9, 10, 17, 18, 19, 20] and the references therein.

When the moving sets C depends also on the state, one obtain a generalization of the classical sweeping process known as the state-dependent sweeping process. Such problems are motivated by parabolic quasi-variational inequalities arising e.g. in the evolution of sandpiles, and occur also in the treatment of 2-D or 3-D quasistatic evolution problems with friction, as well as in micro-mechanical damage models for iron materials with memory to describe the evolution of the plastic strain in presence of small damages. We refer to [22] for more details. This problem have been studied for the first time for convex sets $C(t,u)$ by Chraibi [15] in $\mathbb{R}^3$, then by Kunze and Monteiro Marques [21] in Hilbert spaces under some compactness condition. After, Chemetov and Monteiro Marques [14] established the existence for prox regular sets $C(t,u)$ with a Carathéodory perturbation $F(t,u(t))$ by applying the Shauder fixed point theorem. By means of a generalized version of the Shauder theorem, Castaing, Ibrahim and Yarou [11] provided an other approach to prove the existence when $F \equiv \{0\}$ and $C(t,u(t))$ is prox regular and ball-compact, and for the perturbed problem (even in presence of a delay). The approach is based on the Moreau catching-up algorithm. For recent results in the study of state-dependent sweeping process, we refer to [4, 24]. Vilches [28] has studied the first order state-dependent sweeping process with single-valued perturbation using the approach of Yoshida regularization. It consists in approaching the problem by a penalized (regularized) one depending on a positif parameter converging to zero.

Usually in mechanical systems and also in planning procedures in mathematical economics, external forces are applied, which leads to consider the sweeping process with set-valued perturbations. Several results have been obtained when the perturbation takes bounded values or satisfies a linear growth condition. Recently, the case of unbounded perturbations has been considered (see for instance [4, 2, 24]). The idea is to take only the element of minimal norm bounded, that is: there exists some real $\alpha > 0$,

$$d(0, F(t,u)) \leq \alpha \quad \text{for all} \quad t \in [T_0, T]\ u \in H \quad \text{with} \quad u \in C(t,u).$$

Our aim in this paper is twofold: taking a perturbation as a sum of two mappings with single and set-values respectively, we generalize all the results obtained in the two cases. Using a different approach, we weaken the hypotheses on the perturbation by taking a Carathéodory single-valued mapping satisfying only a linear growth condition and an unbounded set-valued perturbation; furthermore we extend the approach given in [18] in the case of time-dependent prox regular sets to the time and state-dependent case. The paper is organized as follows. In Section 2, we introduce notation and preliminaries needed throughout the paper. Section 3 is devoted to the study of the existence of solutions for the considered problem.

2. Preliminaries and background

Throughout this paper, let $T > T_0 \geq 0$, $I = [T_0, T]$ be an interval of $\mathbb{R}$ and H be a separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. The closed unit ball of H with center 0 will be denoted by $\mathbb{B}$, and $\overline{B}_H(a,\eta)$ will be the closed ball of center $a \in H$ and radius $\eta > 0$. If A is a subset of H, $\overline{co}\, A$ stands for the closed convex hull of A and $\delta^*(\cdot, A)$ the support function of A that is, for all $\xi \in H$,

$$\delta^*(\xi, A) = \sup_{x \in A} \langle \xi, x \rangle.$$

$L^1_H([T_0,T],dt)$ (shortly $\mathrm{L}^1_H(T_0,T)$) is the Banach space of Lebesgue-Bochner integrable functions $f:[T_0,T]\to H$ and $\mathcal{C}_H(I)$ is the space of continuous mappings $u:I\to H$ equipped with the norm of uniform convergence. A mapping $u:[T_0,T]\to H$ is *absolutely continuous* if there is a function $\dot u\in \mathrm{L}^1_H(T_0,T)$ such that $u(t)=u(T_0)+\int_{T_0}^t \dot u(s)\,ds$, $\forall t\in[T_0,T]$. A set-valued mapping $G:[T_0,T]\times H\to H$ is said to be upper semicontinuous if, for any open subset $\mathcal{V}\subset H$, the set $\{x\in H:\ G(x)\subset\mathcal{V}\}$ is open in H. G is said to be scalarly upper semicontinuous or upper hemicontinuous if, for any $y\in H$, the real-valued function $x\mapsto \delta^*(y,G(x))$ is upper semicontinuous. We refer to [13] for measurable set-valued mappings and convex analysis.
For a given $r\in]0,+\infty]$, a nonempty subset S of a Hilbert space H is r-prox regular or equivalently r-proximally smooth [16, 27] if and only if every nonzero proximal normal to S can be realized by a r-ball. This is equivalent to say that for every $\bar x\in S$, and for every $v\neq 0,\ v\in N^p_S(\bar x)$,

$$\langle \frac{v}{||v||},x'-\bar x\rangle\le \frac{1}{2r}||x'-\bar x||^2$$

for all $x'\in S$, where $N^p_S(\bar x)$ is the proximal normal cone of S at the point $\bar x\in S$ defined by

$$N^p_S(\bar x)=\{\xi\in H:\exists\alpha>0,\bar x\in\ Proj_S(\bar x+\alpha\xi)\},$$

$Proj_S(\cdot)$ stands for the projection on the set S defined by

$$Proj_S(x):=\{y\in S: d_S(x)=\|x-y\|\}$$

and $d_S(\cdot)$ is the distance function to S. We make the convention $\frac{1}{r}=0$ for $r=+\infty$ and recall that for $r=+\infty$, the r-proximal regularity of S is equivalent to the convexity of S. Let $f:H\to\mathbb{R}\cup\{+\infty\}$ be a proper function and $\bar x\in dom f:=\{x\in H:\ f(x)<+\infty\}$, the *proximal subdifferential* of f at $\bar x$ is the set $\partial^p f(\bar x)$ of all elements $v\in H$ for which there exists $\delta>0$ and $\beta>0$ such that

$$f(y)\ge f(\bar x)+\langle v,y-\bar x\rangle-\beta||y-\bar x||^2 \text{ for all } y\in \overline{B}_H(\bar x,\delta).$$

Given a nonempty closed set S and given a point $\bar x\in S$, the *Clarke normal cone* $N_S(\bar x)$ to S at $\bar x$ defined by

$$N_S(\bar x)=\mathrm{cl}_\omega(\mathbb{R}_+\partial\, d_S(\bar x)),$$

where cl_ω denotes the closure with respect to the weak topology of H. With the definition of Clarke normal cones to nonempty closed sets in hand, the *Clarke subdifferential* $\partial f(\bar x)$ of f at a point $\bar x$ (where f is finite) can be defined in terms of Clarke normal cones to the epigraph of the function by

$$\partial f(\bar x)=\{v\in H:(v,-1)\in N_{\mathrm{epi}\,f}((\bar x,f(\bar x)))\},$$

where $\mathrm{epi}\,f$ denotes the epigraph of f, that is, $\mathrm{epi}\,f=\{(\bar x,\lambda)\in H\times\mathbb{R}: f(\bar x)\le\lambda\}$. Further

$$\partial d_S(\bar x)\subset N_S(\bar x)\cap\mathbb{B},\quad \text{for all } \bar x\in S.$$

Let C,C' be two subsets of H, we denote by

$$e(C,C')=\sup\{d_{C'}(a),\ a\in C\}$$

the excess of C over C' and if C and C' are closed,

$$\mathcal{H}(C,C')=\max\{e(C,C'),e(C',C)\}$$

the Hausdorff distance between C and C'. Let us denote, for $r > 0$, by $U_r(C)$ (respectively, by $E_r(C)$) the open tube around the set C (respectively, the open enlargement of C), that is,

$$U_r(C) := \{v \in H : 0 < d_C(v) < r\},$$

respectively,

$$E_r(C) := \{v \in H : d_C(v) < r\}.$$

The following proposition provides some properties of the proximal and Clarke subdifferentials of the function distance $d_C(\cdot)$ when the set C is r-prox regular. It also summarizes some important consequences of the prox regularity property which will be needed in the sequel of the paper. For the proof of these results, we refer to [7, 25].

Proposition 2.1. *Let S be a nonempty closed subset in the Hilbert space H and let $r > 0$. If S is r-prox regular, then the following hold:*

a) For any $x \in U_r(S)$, $Proj_S(x)$ exists and is unique, the mapping $Proj_S(\cdot) : U_r(S) \to S$ is locally Lipschitz on $U_r(S)$;

b) For any $v \in U_r(S)$ and $y = Proj_S(v)$ one has $y \in Proj_S\left(y + r\frac{v-y}{\|v-y\|}\right)$;

c) The Clarke and proximal subdifferentials of $d_S(\cdot)$ coincide at all points $v \in E_r(S)$;

d) The Clarke and proximal normal cone to S coincide at all points $u \in S$ and $\alpha\partial^p d_S(u) = N_S^p(u) \bigcap \alpha\mathbb{B}$;

e) Let $C : [T_0, T] \times H \rightharpoonup H$ be r-prox regular and satisfies

$$|d_{C(t,x)}(u) - d_{C(s,y)}(v)| \leq ||u - v|| + \chi(t) - \chi(s) + L||x - y||$$

for all u, x, v, y in H and for all $s \leq t$ in $[T_0, T]$, where $\chi : [T_0, T] \to \mathbb{R}_+$ is a nondecreasing absolutely continuous function and L is a positive constant. Then the convex weakly compact valued mapping $(t, x, y) \to \partial^p d_{C(t,x)}(y)$ satisfies the upper semicontinuity property: Let (t_n, x_n) be a sequence in $[T_0, T] \times H$ converging to some $(t, x) \in [T_0, T] \times H$, and (y_n) be a sequence in H with $y_n \in C(t_n, x_n)$ for all n, converging to $y \in C(t, x)$, then, for any $z \in H$,

$$\limsup_{n\to\infty} \delta^*(z, \partial^p d_{C(t_n,x_n)}(y_n)) \leq \delta^*(z, \partial^p d_{C(t,x)}(y)).$$

3. The main result

Let assume the following assumptions:

($\mathcal{A}_1$) There is some constant $r > 0$ such that, for each $t \in [T_0, T]$ and each $u \in H$, the sets $C(t, u)$ are r-prox regulars.

($\mathcal{A}_2$) For each bounded subset $A \subset H$, the set $C(t, x)$ is relatively ball-compact for all $(t, x) \in [T_0, T] \times A$.

($\mathcal{A}_3$) There is a constant $L \in]0, 1[$ and an absolutely continuous nondecreasing function $\chi : [T_0, T] \to \mathbb{R}_+$ such that, for all $s, t \in [T_0, T], s \leq t$ and any $x, y, u, v \in H$ one has

$$\left|d_{C(t,u)}(x) - d_{C(s,v)}(y)\right| \leq \|x - y\| + \chi(t) - \chi(s) + L\|u - v\|.$$

Let recall the following existence result for the sweeping process without perturbation and with time and state dependent nonconvex moving sets in Hilbert space proved in ([11], Theorem 3.4):

Theorem 3.1. *Assume that $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ hold. Then, for any $a \in C(T_0, a)$, there exists an absolutely continuous solution $u : [T_0, T] \to H$ of the problem*

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) \quad a.e. \quad t \in [T_0, T], \\ u(T_0) = a, \quad u(t) \in C(t, u(t)), \quad \forall t \in [T_0, T], \end{cases}$$

which satisfies

$$\|\dot{u}(t)\| \leq \frac{1}{1-L}\dot{\chi}(t).$$

We start with the case when the perturbation is a single valued integrable function.

Proposition 3.2. *Assume that $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ hold, then, for any mapping $h \in \mathrm{L}^1_H(T_0, T)$ and any $a \in C(T_0, a)$, there exists an absolutely continuous solution $u : [T_0, T] \to H$ of*

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + h(t) \quad a.e. \ in \ [T_0, T] \\ u(T_0) = a, \quad u(t) \in C(t, u(t)), \quad \forall t \in [T_0, T]. \end{cases}$$

Moreover, we have

$$\|\dot{u}(t) + h(t)\| \leq \frac{1}{1-L}(\dot{\chi}(t) + \|h(t)\|) \quad a.e. \ in \ [T_0, T]. \tag{3.1}$$

Proof . Set $\psi(t) = \int_{T_0}^t h(s)ds$, for all $t \in [T_0, T]$, and consider the set-valued mapping $D : [T_0, T] \times H \rightharpoonup H$ defined by

$$D(t, z) = C(t, z - \psi(t)) + \psi(t)$$

for all $(t, z) \in [T_0, T] \times H$. Obviously, D satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, let verify $(\mathcal{A}_3)$. For any $y_1,\ y_2,\ z_1,\ z_2$ in H and all t, s in $[T_0, T]$, we have

$$\begin{aligned} |d_{D(t,z_1)}(y_1) - d_{D(s,z_2)}(y_2)| &\leq |d_{C(t,z_1-\psi(t))}(y_1 - \psi(t)) - d_{C(s,z_2-\psi(s))}(y_2 - \psi(s))| \\ &\leq \|\psi(t) - \psi(s)\| + |d_{C(t,z_1-\psi(t))}(y_1) - d_{C(s,z_2-\psi(s))}(y_2)| \\ &\leq \|\psi(t) - \psi(s)\| + \|y_1 - y_2\| + \chi(t) - \chi(s) + L\|z_1 - \psi(t) - z_2 + \psi(s)\|, \\ &\leq \|y_1 - y_2\| + \chi_1(t) - \chi_1(s) + L\|z_1 - z_2\| \end{aligned}$$

where $\chi_1(t) = \int_{T_0}^t (\dot{\chi}(s) + (1 + L)\|h(s)\|)ds$ is nondecreasing absolutely continuous. Hence, D satisfies $(\mathcal{A}_3)$, as $a \in D(T_0, a) = C(T_0, a)$, from Theorem 3.1 there exists an absolutely continuous solution y to the state-dependent sweeping process

$$\begin{cases} -\dot{y}(t) \in N_{D(t,y(t))}(y(t)) \quad \text{a.e. in} \ [T_0, T] \\ y(T_0) = a, \quad y(t) \in D(t, y(t)), \end{cases}$$

which verify

$$\|\dot{y}(t)\| \leq \frac{1}{1-L}\dot{\chi}_1(t).$$

Furthermore, the mapping $u(t) = y(t) - \psi(t)$ is solution of

$$-\dot{u}(t) - h(t) = -\dot{y}(t) \in N_{C(t,y(t)-\psi(t))}(y(t) - \psi(t)) := N_{C(t,u(t))}u(t)$$

and satisfies $\|\dot{u}(t) + h(t)\| \leq \frac{1}{1-L}(\dot{\chi}(t) + \|h(t)\|)$ a.e. $t \in [T_0, T]$. □

Now, we are able to give our main result in the paper, which is an existence result for a class of first-order differential inclusions. The perturbation is supposed to be the sum of a Carathéodory mapping and a scalarly upper semicontinuous set-valued mapping.

Theorem 3.3. *Assume that* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *hold. Let* $g : [T_0, T] \times H \to H$ *be a map such that*

i) g *is a Carathéodory mapping on* $[T_0, T] \times H$;

ii) there exists a non-negative function $\alpha(\cdot) \in \mathrm{L}^1_{\mathbb{R}_+}(T_0, T)$ *such that, for all* $t \in [T_0, T]$ *and for all* $x \in H$*, we have*
$$\|g(t, x)\| \leq \alpha(t)(1 + \|x\|).$$

Let $G : [T_0, T] \times H \rightharpoonup H$ *be a set-valued mapping with nonempty closed convex values such that*

iii) G *is* $\mathcal{L}([T_0, T] \otimes \mathcal{B}(H))$ *measurable and scalarly upper semicontinuous on* H;

iv) there exists a real $\gamma > 0$*, such that, for all* $(t, x) \in [T_0, T] \times H$,
$$d(0, G(t, u)) \leq \gamma(1 + \|x\|).$$

Then, for any $a \in C(T_0, a)$*, there exists an absolutely continuous mapping* $u : [T_0, T] \to H$ *solution of*
$$(\mathcal{SP}) \begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + G(t, u(t)) + g(t, u(t)) \;\; a.e. \;\, in \;\; [T_0, T]; \\ u(T_0) = a, \;\; u(t) \in C(t, u(t)), \;\; \forall t \in [T_0, T]. \end{cases}$$

Proof . For each $(t, x) \in [T_0, T] \times H$, denote by $P(t, x)$ the element of minimal norm of the closed convex set $G(t, x)$ of H, we put $f(t, x) = g(t, x) + P(t, x)$ and $\beta(t) = \alpha(t) + \gamma$, by *ii)* and *iv)*, we get for all $(t, x) \in [T_0, T] \times H$,
$$\|f(t, x)\| \leq \beta(t)(1 + \|x\|). \tag{3.2}$$
We suppose that
$$\int_{T_0}^{T} \beta(s) ds \leq \frac{1 - L}{4}, \tag{3.3}$$
and consider, for every $n \in \mathbb{N}$, a partition of $[T_0, T]$ defined by
$$t_i^n = i\frac{T - T_0}{n} \quad (0 \leq i \leq n).$$

We are going to construct a sequence of maps $(u_n(\cdot))$ in $\mathcal{C}_H(T_0, T)$ via Proposition 3.2 by considering a perturbation f with fixed second variable u in each subinterval. So, for $a \in C(T_0, a)$, let us consider the following problem on the interval $[T_0, t_1^n]$:
$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + f(t, a) \qquad \text{a.e.} \;\; t \in [T_0, t_1^n]; \\ u(T_0) = a \in C(T_0, a), \end{cases}$$

where $f(\cdot, a)$ is a mapping depending only on t and is $L^1_H(T_0, t_1^n)$. By Proposition 3.2, it has an absolutely continuous solution that we denote by $u_0^n(\cdot) : [T_0, t_1^n] \to H$. According to (3.1) this solution satisfies
$$\|\dot{u}_0^n(t) + f(t, a)\| \leq \frac{1}{1 - L}(\dot{\chi}(t) + \|f(t, a)\|) \quad \text{a.e.} \;\; t \in [T_0, t_1^n]. \tag{3.4}$$
Now, since $u_0^n(t_1^n) \in C(t_1^n, u_0^n(t_1^n))$ is well defined, let consider in the interval $[t_1^n, t_2^n]$ the problem
$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + f(t, u_0^n(t_1^n)) \qquad \text{a.e.} \;\; t \in [t_1^n, t_2^n]; \\ u(t_1^n) = u_0^n(t_1^n) \in C(t_1^n, u_0^n(t_1^n)) \end{cases}$$

which admits a solution denoted by $u_1^n(\cdot) : [t_1^n, t_2^n] \to H$ with $u_1^n(t_1^n) = u_0^n(t_1^n)$ and satisfying

$$\|\dot{u}_1^n(t) + f(t, u_0^n(t_1^n))\| \leq \frac{1}{1-L}(\dot{\chi}(t) + \|f(t, u_0^n(t_1^n))\|) \quad \text{a.e. in } [t_1^n, t_2^n]. \tag{3.5}$$

And so on, for each n, there exists a finite sequence of absolutely continuous mappings $u_i^n(\cdot) : [t_i^n, t_{i+1}^n] \to H$, $(0 \leq i \leq n-1)$ such that, for each $i \in \{0, ..., n-1\}$,

$$\begin{cases} -\dot{u}_i^n(t) \in N_{C(t,u_i^n(t))}(u_i^n(t)) + f(t, u_{i-1}^n(t_i^n)) & \text{a.e. } t \in [t_i^n, t_{i+1}^n], \\ u_i^n(t_i^n) = u_{i-1}^n(t_i^n) \in C(t_i^n, u_{i-1}^n(t_i^n)), \end{cases}$$

where $u_{-1}^n(0) = a$ and

$$\|\dot{u}_i^n(t)\| \leq \frac{1}{1-L}(\dot{\chi}(t) + 2\|f(t, u_{i-1}^n(t_i^n))\|) \quad \text{a.e. in } [t_i^n, t_{i+1}^n]. \tag{3.6}$$

Define the functions $u_n : [T_0, T] \to H$ and $\theta_n : [T_0, T] \to [T_0, T]$ by

$$u_n(t) = u_i^n(t); \ \forall t \in [t_i^n, t_{i+1}^n], \ i = 0, ..., n-1$$

$$\begin{cases} \theta_n(t) = t_i^n, \ \forall t \in]t_i^n, t_{i+1}^n], \ i = 0, ..., n-1 \\ \theta_n(T_0) = T_0, \end{cases} \tag{3.7}$$

so, $u_n(\cdot)$ is absolutely continuous on $[T_0, T]$, and one has

$$\begin{cases} -\dot{u}_n(t) \in N_{C(t,u_n(t))}(u_n(t)) + f(t, u_n(\theta_n(t))) & \text{a.e. } t \in [T_0, T] \\ u_n(T_0) = a \end{cases} \tag{3.8}$$

with

$$\|\dot{u}_n(t)\| \leq \frac{1}{1-L}(\dot{\chi}(t) + 2\|f(t, u_n(\theta_n(t)))\|) \quad \text{a.e. } t \in [T_0, T]. \tag{3.9}$$

It follows that

$$\|u_n(t_{i+1}^n)\| \leq \|u_n(t_i^n)\| + \frac{1}{1-L}\int_{t_i^n}^{t_{i+1}^n} (\dot{\chi}(s) + 2\|f(s, u_n(t_i^n))\|)ds. \tag{3.10}$$

By iteration, we obtain

$$\begin{aligned} \|u_n(t_{i+1}^n)\| &\leq \|a\| + \frac{1}{1-L}\sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n} (\dot{\chi}(s) + 2\|f(s, u_n(t_k^n))\|)ds \\ &\leq \|a\| + \frac{1}{1-L}\int_{T_0}^{t_{i+1}^n} \dot{\chi}(s)ds + \frac{2}{1-L}\sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n} \|f(s, u_n(t_k^n))\|ds \\ &\leq \|a\| + \frac{1}{1-L}\int_{T_0}^{t_{i+1}^n} \dot{\chi}(s)ds + \frac{2}{1-L}\sum_{k=0}^{i}(1 + \|u_n(t_k^n)\|)\int_{t_k^n}^{t_{k+1}^n} \beta(s)ds. \end{aligned}$$

Then

$$\|u_n(t_{i+1}^n)\| \leq \|a\| + \frac{1}{1-L}\int_{T_0}^{t_{i+1}^n} \dot{\chi}(s)ds + \frac{2}{1-L}(1 + \max_{0\leq k\leq i}\|u_n(t_k^n)\|)\int_{T_0}^{t_{i+1}^n} \beta(s)ds$$

for each $i = 0, ..., n-1$ and thus

$$\max_{0\leq k\leq n}\|u_n(t_k^n)\| \leq \|a\| + \frac{1}{1-L}\int_{T_0}^{T} \dot{\chi}(s)ds + \frac{2}{1-L}(1 + \max_{0\leq k\leq n}\|u_n(t_k^n)\|)\int_{T_0}^{T} \beta(s)ds.$$

Taking in account (3.3), we obtain

$$\max_{0\le k\le n}\|u_n(t_k^n)\|\le\|a\|+\frac{1}{2}+\frac{1}{1-L}\int_{T_0}^{T}\dot{\chi}(s)ds+\frac{1}{2}\max_{0\le k\le n}\|u_n(t_k^n)\|$$

and hence

$$\|u_n(\theta_n(t))\|\le 2(\|a\|+\frac{1}{2}+\frac{1}{1-L}\int_{T_0}^{T}\dot{\chi}(s)ds):=m. \tag{3.11}$$

By (3.9), (3.11) and (3.2), one has for any n and almost all t

$$\|f(t,u_n(\theta_n(t)))\|\le(1+m)\beta(t) \tag{3.12}$$

$$\|\dot{u}_n(t)\|\le\frac{1}{1-L}(\dot{\chi}(t)+2(1+m)\beta(t)):=m_1(t). \tag{3.13}$$

Thus by (3.12) and (3.13) we may suppose that $(\dot{u}_n)$ $\sigma(L^1_H(T_0,T),L^\infty_H(T_0,T))$-converges in $\mathrm{L}^1_H(T_0,T)$ to a function z with $||z(t)||\le m_1(t)$ for a.e. $t\in[T_0,T]$ (see e.g. [12], Proposition 6.2.3), and (u_n) converges pointwisely on $[T_0,T]$ with respect to the weak topology to an absolutely continuous function u

$$u(t)=a+\int_{T_0}^{t}\dot{u}(s)\,ds,\ \forall t\in[T_0,T]$$

for all $t\in[T_0,T]$ with $\dot{u}=z$. Further, according to (3.11), we have by construction

$$u_n(\theta_n(t))\in C(\theta_n(t),u_n(\theta_n(t)))\bigcap\overline{\mathbb{B}}_H(0,m). \tag{3.14}$$

Then, $(u_n(\theta_n(t))$ is relatively compact for every $t\in[T_0,T]$ in H since

$$D(t):=\cup_n C(\theta_n(t),u_n(\theta_n(t)))$$

is ball-compact thanks to $(\mathcal{A}_2)$ and (3.11). As $\theta_n(t)\to t$, it follows that

$$\lim_{n\to\infty}||u_n(\theta_n(t))-u_n(t)||=0 \tag{3.15}$$

and $(u_n(t))$ is relatively compact. By (3.13), $(u_n(\cdot))$ is equicontinuous, thus relatively compact in $\mathcal{C}_H(T_0,T)$, consequently, (u_n) converges in $\mathcal{C}_H(T_0,T)$ to the absolutely continuous function u. Furthermore, for all $t\in[T_0,T]$

$$\|g(t,u_n(\theta_n(t)))\|\le(1+m)\alpha(t),$$

by the continuity of the mapping $g(t,\cdot)$ we get

$$g(t,u_n(\cdot))\to g(t,u(\cdot)),$$

and

$$\|g(t,u(t))\|\le(1+m)\alpha(t).$$

In the other hand we have for all $t\in[T_0,T]$

$$||P(t,u_n(\theta_n(t)))\|\le(1+m)\gamma$$

for all $n \geq n_0$ and for all $t \in [T_0, T]$, we put $(P(\cdot, u_n(\theta_n(\cdot)))) = (\rho_n(\cdot))$ so $(\rho_n(\cdot))$ is bounded, taking a subsequence if necessary, we may conclude that $(\rho_n(\cdot))$ converges $\sigma(L^1_H(T_0,T), L^\infty_H(T_0,T))$ to some mapping $\rho \in L^1_H(T_0,T)$ with

$$\|\rho(t)\| \leq \gamma(1+m).$$

Now, we proceed to prove that $\dot{u}(t) \in -N_{C(t,u(t))}(u(t)) + G(t,u(t)) + g(t,u(t))$ a.e. $t \in [T_0,T]$. First, we check that $u(t) \in C(t,u(t))$, for all $t \in [T_0,T]$. Indeed, for every $t \in [T_0,T]$ and for every n, we have

$$\begin{aligned} d_{C(t,u(t))}(u_n(t)) &\leq ||u_n(t) - u_n(\theta_n(t))|| + d_{C(t,u(t))}(u_n(\theta_n(t))) \\ &\leq ||u_n(t) - u_n(\theta_n(t))|| + \mathcal{H}(C(\theta_n(t), u_n(\theta_n(t))), C(t,u(t))) \\ &\leq ||u_n(t) - u_n(\theta_n(t))|| + \chi(\theta_n(t)) - \chi(t) + L||u_n(\theta_n(t)) - u(t)||. \end{aligned}$$

Taking into account (3.15) and passing to the limit when $n \to \infty$, in the preceding inequality, we get $u(t) \in C(t,u(t))$. Recall that (3.8), (3.12), (3.13) and Proposition 2.1 entails for a.e. $t \in [T_0,T]$

$$-\dot{u}_n(t) - f(t, u_n(\theta_n(t))) \in N^p_{C(t,u_n(t))} u_n(t) \cap m_2(t)\mathbb{B} \tag{3.16}$$

$$= m_2(t)\partial^p d_{C(t,u_n(t))}(u_n(t))$$

with

$$m_2(t) = m_1(t) + (1+m)\beta(t). \tag{3.17}$$

Putting $p_n(t) = f(t, u_n(\theta_n(t)))$ and $p(t) = f(t,u(t))$, remark that $(\dot{u}_n + p_n, \rho_n)$ weakly converges in $L^1_{H\times H}(T_0,T)$ to $(\dot{u}+p, \rho)$. An application of the Mazur's theorem to $(\dot{u}_n + p_n, \rho_n)$ provides a sequence (w_n, ζ_n) with

$$w_n \in co\{\dot{u}_l + p_l : \ l \geq n\} \quad \text{and} \quad \zeta_n \in co\{\rho_j : \ j \geq n\}$$

such that (w_n, ζ_n) converges strongly in $L^1_{H\times H}(T_0,T)$ to $(\dot{u}+p, \rho)$. We can extract from (w_n, ζ_n) a subsequence which converges a.e. to $(\dot{u}+p,\rho)$. Then, there is a Lebesgue negligible set $S \subset [T_0,T]$ such that for every $t \in [T_0,T] \setminus S$

$$\dot{u}(t) + p(t) \in \bigcap_{n\geq 0} \overline{\{w_k(t) : \ k \geq n\}} \subset \bigcap_{n\geq 0} \overline{co}\{\dot{u}_k(t) + p_k(t) : \ k \geq n\} \tag{3.18}$$

and

$$\rho(t) \in \bigcap_{n\geq 0} \overline{\{\zeta_k(t) : \ k \geq n\}} \subset \bigcap_{n\geq 0} \overline{co}\{\zeta(t) : \ k \geq n\}. \tag{3.19}$$

Fix any $t \in [T_0,T] \setminus S, n \geq n_0$ and $\mu \in H$, then the relation 3.18 gives

$$\begin{aligned} \langle \mu, -\dot{u}(t) - p(t)\rangle &\leq \limsup_{n\to\infty} \delta^*(\mu, m_2(t)\partial d_{C(t,u_n(t))}(u_n(t)) \\ &\leq \delta^*(\mu, m_2(t)\partial d_{C(t,u(t))}(u(t)), \end{aligned}$$

according to $(\mathcal{A}_3)$ and Proposition 2.1, taking the supremum over $\mu \in H$, we deduce that

$$\delta(-\dot{u}(t) - p(t), m_2(t)\partial d_{C(t,u(t))}(u(t)) = \delta^{**}(-\dot{u}(t) - p(t), m_2(t)\partial d_{C(t,u(t))}(u(t)) \leq 0,$$

which entails

$$\begin{aligned} -\dot{u}(t) &\in m_2(t)\partial d_{C(t,u(t))}(u(t)) + f(t,u(t)) \\ &\in N_{C(t,u(t))}(u(t)) + f(t,u(t)) \\ &\in N_{C(t,u(t))}(u(t)) + \rho(t) + g(t,u(t)). \end{aligned}$$

Further, the relation (3.19) gives

$$\langle \mu, \rho(t) \rangle \leq \limsup_{n\to\infty} \delta^*(\mu, G(t, u_n(\theta_n(t)))),$$

since $\delta^*(\mu, G(t, \cdot))$ is upper semicontinuous on H, then

$$\langle \mu, \rho(t) \rangle \leq \delta^*(\mu, G(t, u(t))),$$

so, we get $d(\rho(t), G(t, u(t))) \leq 0$, consequently

$$\rho(t) \in G(t, u(t)) \ \text{ a.e. } \ t \in [T_0, T].$$

Finally, to turn to the general case when condition (3.3) is not satisfied, we subdivide I into intervals satisfying (3.3) and, thanks to the foregoing, one construct an absolutely continuous solution in each subinterval, then by continuity the problem $(\mathcal{SP})$ admits a solution on $[T_0, T]$. □

Remark 3.4. *We cannot replace the constant γ in the condition (iv) of Theorem 3.3 by an integrable mapping since the distance function (which define the element of minimal norm of G) is not necessary continuous. This is possible in finite dimensional setting, see ([26], Lemma 2.2) or if G has compact values, ([5], Corollary 1.4.17,) we get then the following result*

Corollary 3.5. *Assume that $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ and assumptions on g hold. Let $G : [T_0, T] \times \mathbb{R}^n \rightharpoonup \mathbb{R}^n$ be a set-valued mapping with nonempty closed convex values such that*

- *i) G is Carathéodory on $[T_0, T] \times \mathbb{R}^n$;*
- *ii) there exists a non-negative function $\gamma(\cdot) \in \mathrm{L}^1_{\mathbb{R}_+}(T_0, T)$ such that, for all $t \in [T_0, T]$ and for all $x \in \mathbb{R}^n$, we have*
$$d(0, G(t, u)) \leq \gamma(t)(1 + \|x\|).$$

Then, for any $a \in C(T_0, a)$, there exists an absolutely continuous mapping $u : [T_0, T] \to \mathbb{R}^n$ solution of $(\mathcal{SP})$. Moreover, we have for a.e. $t \in [T_0, T]$

$$\|d(0, G(t, u))\| \leq (1 + m)\gamma(t)$$

and

$$\|\dot{u}(t) + g(t, u(t)) + d(0, G(t, u))\| \leq \frac{1}{1 - L}\Big(\dot{\chi}(t) + (1 + m)(\gamma(t) + \alpha(t))\Big),$$

where

$$m = 2\Big(\|a\| + \frac{1}{2} + \frac{1}{1 - L}\int_{T_0}^{T} \dot{\chi}(s)ds\Big).$$

Acknowledgements: Research supported by the General direction of scientific research and technological development (DGRSDT) under project PRFU No. C00L03UN 180120180001.

References

[1] S. Adly, F. Nacry and L. Thibault, *Discontinuous sweeping process with prox-regular sets,* ESAIM Control Optim. Calc. Var. 23(4) (2017) 1293–1329.

[2] D. Affane, M. Aissous and M. F. Yarou, *Existence results for sweeping process with almost convex perturbation,* Bull. Math. Soc. Sci. Math. Roumanie, 2 (2018) 119–134.

[3] D. Affane, M. Aissous and M. F. Yarou, *Almost mixed semi-continuous perturbation of Moreau's sweeping process,* Evol. Equ. Control Theory, 1 (2020), 27–38.

[4] D. Affane and M. F. Yarou, *Unbounded perturbation for a class of variational inequalities*, Discuss. Math. Diff. inclu. control optim. 37 (2017) 83–99.

[5] J. P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhauser, Boston Basel Berlin, 1990.

[6] H. Benabdellah, *Existence of solutions to the nonconvex sweeping process*, J. Diff. Equa. 164 (2000) 286–295.

[7] M. Bounkhel and L. Thibault, *Nonconvex sweeping process and prox-regularity in Hilbert space*, J. Nonlin. Convex Anal. 6 (2) (2005) 359–374.

[8] M. Bounkhel and M. F. Yarou, *Existence results for first and second order nonconvex sweeping process with delay*, Portug. Math. 61 (2) (2004) 2007–2030.

[9] B. Brogliato, *The absolute stability problem and the Lagrange-Dirichlet theorem with monotone multivalued mappings*, System Contr. Letters. 51 (5) (2004) 343–353.

[10] C. Castaing, A. G. Ibrahim and M. F. Yarou, *Existence problems in second order evolution inclusions: discretization and variational approach,* Taiwanese J. Math. 12 (6) (2008) 1435-1477.

[11] C. Castaing, A. G. Ibrahim and M. F. Yarou, *Some contributions to nonconvex sweeping process*, J. Nonlin. Convex Anal. 10 (2009) 1–20.

[12] C. Castaing, P. Raynaud de Fitte and M. Valadier, *Young measures on Topological Spaces With Applications in Control Theory and Probability Theory*, Kluwer Academic Publishers, Dordrecht, 2004.

[13] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lect. Notes in Math., Springer Verlag, Berlin 580, 1977.

[14] N. Chemetov and M. D. P. Monteiro Marques, *Non-convex Quasi-variational Differential Inclusions*, Set-Valued Variat. Anal. 5 (3) (2007) 209–221.

[15] K. Chraibi, *Resolution du problème de rafle et application a un problème de frottement*, Topol. Metho. Nonlin. Anal. 18 (2001) 89–102.

[16] F. H. Clarke, R. J. Stern and P. R. Wolenski, *Proximal smoothness and the lower C^2 property*, J. Convex Anal 2 (1995) 117–144.

[17] G. Colombo and V. V. Goncharov, *The sweeping processes without convexity* , Set-Vaued Anal. 7 (4) (1999) 357–374.

[18] J. F. Edmond and L. Thibault, *Relaxation and optimal control problem involving a perturbed sweeping process*, Math. Program. Ser. B 104 (2005) 347–373.

[19] D. Goeleven and B. Brogliato, *Stability and instability matrices for linear evolution variational inequalities*, IEEE Transactions Automatic Control 49 (4) (2004) 521–534.

[20] P. Krejci and V. Recupero, *Comparing BV solutions of rate independent processes*, J. convex Anal. 21 (1) (2014) 121–146.

[21] M. Kunze and M. D. P. Monteiro Marques, *On parabolic quasi-variational inequalities and state-dependent sweeping processes*, Topol. Metho. Nonlin. Anal. 12 (1998) 179–191.

[22] M. Kunze and M. D. P. Monteiro Marques, *An introduction to Moreau's sweeping process*, Lect. Notes in Physics, Springer Verlag, Berlin, **551**, 2000, 1–60.

[23] J. J. Moreau, *Evolution problem asssociated with a moving convex set in a Hilbert space*, J. Diff. Equ. 26 (1977) 347–374.

[24] J. Noel and L. Thibault, *Nonconvex sweeping process with a moving set depending on the state*, Vietnam Journ. Math. 42 (2014) 595–612.

[25] R. A. Poliquin, R. T. Rockafellar and L. Thibault, *Local differentiability of distance functions*, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[26] G. V. Smirnov, *Introduction to the theory of differential inclusions*, Graduate Studies in Mathematics V. 41, American Mathematical Society, 2002.

[27] L. Thibault, *Sweeping process with regular and nonregular sets*, J. Diff. Equa. 193 (2003) 1–26.

[28] E. Vilches, *Inclusions différentielles sur les espaces de Hilbert avec des cones normaux à des ensembles non réguliers*, Thesis, Université de Bourgogne, France, 2017.