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Generalized G-Wolfe type fractional symmetric duality theorems over arbitrary cones under (G, ρ, θ)-invexity assumptions

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Abstract

In this paper, we introduce the concept of (G, ρ, θ)-invexity/pseudoinvexity. We formulate duality outcomes for G-Wolfe-type fractional symmetric dual programs over arbitrary cones. In the final section, we discuss the duality theorems under (G, ρ, θ)-invexity/ (G, ρ, θ)-psedoinvexity assumptions.

Keywords: Fractional programming problem, symmetric duality, (G, ρ, θ)-invexity,(G, ρ, θ)-pseudoinvexity, G-Wolfe model
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1. Introduction

Duality in numerical programming has praiseworthy utilize in numerous hypothetical and computational improvements as well as in financial aspects, control hypothesis, business problems and other differing fields. Various authors have considered fractional programming problems containing square root of positive semidefinite quadratic forms like Mond [1] and Zhang and Mond [2]. The ubiquity of this sort of problem lies in the way that in spite of the fact that the target and limitation capacities

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are nondifferentiable, a straightforward definition of the dual might be given.
Kim et al. [3] worked a pair of multiobjective symmetric dual problem under cone constraints with psuedo-invex functions. Devi [4] constructed a pair of second-order symmetric dual programs and derived duality relations with bonvex functions. Chen [5] formulated a pair of multiobjective higher-order dual nonlinear programming and discussed duality theorems under higher-order (F, α, ρ, d)-convexity assumptions. Also, Chen [5] discussed ratio property and established the optimality conditions under higher-order (F, α, ρ, d)-convexity assumptions. Wolfe type second-order symmetric duality has been discussed by Yang et al. [6] for multiobjective programming problems.
Convexity is one of the most much of the time utilized speculation in streamlining hypothesis essentially as a result of a few worldwide properties that it has. Convexity presumptions are frequently not fulfilled in true problem so there was a need to debilitate them. One of the ways was the introduction of generalization of convexity namely quasi/pseudo-convexity. Recently Gutierrez et al. [7] constructed various notions of (K_1, K_2)-pseudoinvexity-I and II with $K_1, K_2 \in \{C_0^c, (\text{int}C)^c\}$ for a locally Lipschitz function by means of the extended Jacobian where $C \subseteq R^n$ is a closed convex pointed cone with non empty interior and $C_0 = C \setminus \{0\}$. They utilized them to consider productivity through variational-like disparities with Lipschitz functions. For more data on fractional programming, readers are advised to see [8, 9, 10, 11, 12, 13, 14, 15].

In this article, we generalized (G, ρ, θ)-invexity/ (G, ρ, θ)-pseudoinvexity assumptions. Generalized G- Wolfe type fractional symmetric dual is proposed over arbitrary cones and duality results are proved by using the above mentioned functions.

2. Preliminaries and Definitions

Let $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open sets and $f(x, y)$ be real valued differentiable function defined on $S_1 \times S_2$. Let $G : R \to R$ be strictly increasing function in their range $G : I_f(S_1 \times S_2) \to R$, where $I_f(S_1 \times S_2)$ is the range of f, η_1, $\eta_2 : S_1 \times S_2 \to R^n$, $\rho \in R$ and $\theta : S_1 \times S_2 \to R$.
Definition 2.1. The function $f(x, y)$ is (G, ρ, θ) -pseudoinvex in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to η_1, if there exist ρ and θ, such that for $x \in S_1$, we have

$$\eta_1^T(x, u)\Big[G'(f(u, v))\nabla_x f(u, v)\Big] + \rho||\theta(x, u)||^2 \geq 0 \Rightarrow [G(f(x, v)) - G(f(u, v))] \geq 0.$$

Remark 2.1. If the above inequality sign changes $\leq$, then the function $f(x, y)$ is (G, ρ, θ) -pseudoincave in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to η_1.
Definition 2.2. The function $f(x, y)$ is (G, ρ, θ)-pseudoinvex in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to η_2, if there exist ρ and θ, such that for $y \in S_2$, we have

$$\eta_2^T(y, v)\Big[G'(f(u, v))\nabla_y f(u, v)\Big] + \rho||\theta(y, v)||^2 \geq 0 \Rightarrow [G(f(u, y)) - G(f(u, v))] \geq 0.$$

Remark 2.2. If the above inequality sign changes $\leq$, then the function $f(x, y)$ is (G, ρ, θ) -pseudoincave in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to η_2.
Definition 2.3. The function $f(x, y)$ is (G, ρ, θ) -invex in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to η_1 if there exist ρ and θ, such that for $x \in S_1$, we have

$$[G(f(x, v)) - G(f(u, v))] \geq \eta^T(x, u)\Big[G'(f(u, v))\nabla_x f(u, v)\Big] + \rho||\theta(x, u)||^2.$$

Remark 2.3. If the above inequality sign changes $\leq$, then the function $f(x, y)$ is (G, ρ, θ) -incave in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to η_1.

Definition 2.4. The function $f(x,y)$ is (G,ρ,θ)-invex in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to η_2, if there exist ρ and θ, such that for $y \in S_2$, we have

$$[G(f(u,y)) - G(f(u,v))] \geq \eta_2^T(y,v)\Big[G'(f(u,v))\nabla_y f(u,v)\Big] + \rho||\theta(y,v)||^2.$$

Remark 2.4. If the above inequality sign changes $\leq$, then the function $f(x,y)$ is (G,ρ,θ) -incave in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to η_2.
Definition 2.5. The positive polar cone $S^*$of a cone $S \subseteq R^s$ is defined by

$$S^* = \{y \in R^s : x^T y \geq 0, \text{ for } x \in S\}.$$

3. *G*-Wolfe Type Fractional Symmetric Pair of Primal-Dual Model

The application of non-linear programming methods for the optimum design of statically indeterminate structures is discussed, with special emphasis on the design of elastic grillages loaded laterally and in plane. In the following section, we formulate the following pair of G-Wolfe type fractional symmetric dual programming problem over arbitrary cones:

Primal Problem (FWP):

$$\text{Min } \frac{G(f(x,y)) - y^T G'(f(x,y))\nabla_y f(x,y)}{G(g(x,y)) - y^T G'(g(x,y))\nabla_y g(x.y)}$$

Subject to

$$-[(G(f(x,y)) - y^T G'(f(x,y))\nabla_y f(x,y))G'(g(x,y))\nabla_y g(x,y) - (G(g(x,y))$$
$$-y^T G'(g(x,y))\nabla_y g(x,y))G'(f(x,y))\nabla_y f(x,y)] \in C_2^*,$$
$$x \in C_1.$$

Dual Problem (FWD):

$$\text{Max } \frac{G(f(u,v)) - u^T[G'(f(u,v))\nabla_x f(u,v)]}{G(g(u,v)) - u^T[G'(g(u,v))\nabla_x g(u,v)]}$$

Subject to

$$[(G(f(u,v)) - u^T G'(f(u,v))\nabla_x f(u,v))G'(g(u,v))\nabla_x g(u,v) - (G(g(u,v))$$
$$-u^T G'(g(u,v))\nabla_x g(u,v))G'(f(u,v))\nabla_x f(u,v)] \in C_1^*,$$
$$v \in C_2,$$

where $f : S_1 \times S_2 \to R$ and $g : S_1 \times S_2 \to R_+ \setminus \{0\}$ are differentiable functions. The above primal-dual programs can be re-written as:

(EFWP) Min w

Subject to

$$(G(f(x,y)) - y^T G'(f(x,y))\nabla_y f(x,y)) - w(G(g(x,y)) - y^T G'(g(x,y))\nabla_y g(x,y)) = 0, \tag{3.1}$$

$$-\left[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)\right] \in C_2^*, \tag{3.2}$$

$$x \in C_1. \tag{3.3}$$

(EFWD) Min t

Subject to

$$(G(f(u,v)) - u^T G'(f(u,v))\nabla_x f(u,v)) - t(G(g(u,v)) - u^T G'(g(u,v))\nabla_x g(u,v)) = 0, \tag{3.4}$$

$$[(G'(f(u,v))\nabla_x f(u,v)) - t(G'(g(u,v))\nabla_x g(u,v))] \in C_1^*, \tag{3.5}$$

$$v \in C_2. \tag{3.6}$$

Let P^0 and Q^0 be the sets of feasible solution of (EFWP) and (EFWD), respectively.

Theorem 3.1 (Weak duality theorem). Let $(x,y,w) \in P^0$ and $(u,v,t) \in Q^0$. Let

- *(i)* $f(.,v)$ be (G,ρ_1,θ_1)- invex and $g(.,v)$ be (G,ρ_2,θ_2)- incave at u for fixed v with respect to η_1,
- *(ii)* $f(x,.)$ be $(G,\rho_3,\theta_3)-$incave and $g(x,.)$ be $(G,\rho_4,\theta_4)-$ invex at y for fixed x with respect to η_2,
- *(iii)* $\eta_1(x,u) + u \in C_1$ and $\eta_2(v,y) + y \in C_2$,
- *(iv)* $G(g(x,v)) > 0$,
- *(v)* $(\rho_1||\theta_1(x,u)||^2 - t\rho_2||\theta_2(x,u)||^2) \geq 0$,
- *(vi)* $(\rho_3||\theta_3(x,u)||^2 - w\rho_4||\theta_4(x,u)||^2) \leq 0$,

then, $w \geq t$.

Proof. From hypothesis (i), we have

$$G(f(x,v)) - G(f(u,v)) \geq \eta_1^T(x,u)G'(f(u,v))\nabla_x f(u,v) + \rho_1||\theta_1(x,u)||^2 \tag{3.7}$$

and

$$-G(g(x,v)) + G(g(u,v)) \geq -\eta_1^T(x,u)G'(g(u,v))\nabla_x g(u,v) - \rho_2||\theta_2(x,u)||^2. \tag{3.8}$$

Multiplying by t in inequality (3.8) and combining with (3.7), we have

$$\begin{aligned} &G(f(x,v)) - tG(g(x,v)) - G(f(u,v)) + tG(g(u,v)) \\ &\quad \geq \eta_1^T(x,u)[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)] + \rho_1||\theta_1(x,u)||^2 - t\rho_2||\theta_2(x,u)||^2. \end{aligned}$$

Using hypothesis (v), the above inequality follows that

$$\begin{aligned} &G(f(x,v)) - tG(g(x,v)) - G(f(u,v)) + tG(g(u,v)) \\ &\qquad \geq \eta_1^T(x,u)[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)]. \end{aligned} \tag{3.9}$$

Next, by hypothesis (ii), gives

$$-G(f(x,v))+G(f(x,y)) \geq -\eta_2^T(v,y)G'(f(x,y))\nabla_y f(x,y) - \rho_3||\theta_3(v,y)||^2 \tag{3.10}$$

and

$$G(g(x,v))-G(g(x,y)) \geq \eta_2^T(v,y)[G'(g(x,y))\nabla_y g(x,y)] + \rho_4||\theta_4(v,y)||^2. \tag{3.11}$$

Multiplying by w in inequality (3.11) and combining with (3.10), we obtain

$$\begin{aligned}&-G(f(x,v))+wG(g(x,v))-G(f(x,y))+wG(g(x,y))\\ &\quad\geq -\eta_2^T(v,y)[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)] - \rho_3||\theta_3(v,y)||^2 + w\rho_4||\theta_4(v,y)||^2.\end{aligned}$$

Using hypothesis (vi), the above inequality follows that

$$\begin{aligned}&-G(f(x,v))+wG(g(x,v))-G(f(x,y))+wG(g(x,y))\\ &\quad\geq -\eta_2^T(v,y)[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)].\end{aligned} \tag{3.12}$$

On adding inequalities (3.9) and (3.12), we have

$$\begin{aligned}&G(f(x,v))-tG(g(x,v))-G(f(u,v))+tG(g(u,v))-G(f(x,v))\\ &+wG(g(x,v))-G(f(x,y))-wG(g(x,y))\\ &\qquad\geq \eta_1^T(x,u)[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)]\\ &\qquad - \eta_2^T(v,y)[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)].\end{aligned} \tag{3.13}$$

From dual constraint (3.5) and hypothesis (iii), we get

$$(\eta_1(x,u)+u)^T[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)] \geq 0,$$

or

$$\begin{aligned}&\eta_1^T(x,u)[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)]\\ &\qquad\geq -u^T[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)].\end{aligned} \tag{3.14}$$

Similarly, from inequality (3.2) and hypothesis (iii), we get

$$-(\eta_2^T(v,y)+y)^T[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)] \geq 0,$$

or

$$\begin{aligned}&-\eta_2^T(v,y)[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)]\\ &\qquad\geq y^T[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)].\end{aligned} \tag{3.15}$$

From inequalities (3.13), (3.14) and (3.15), we get

$$G(f(x,v))-tG(g(x,v))-G(f(u,v))+tG(g(u,v))-G(f(x,v))+wG(g(x,v))$$

$$- G(f(x,y)) + wG(g(x,y)) \geq u^T[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)]$$
$$+ y^T[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)].$$

Using equations (3.1) and (3.4), it follows that

$$(w-t)G(g(x,v)) \geq 0.$$

From hypothesis (iv), above inequality gives

$$w \geq t.$$

Hence, completes the results. □

Remark 3.1 Since every invex function is pseudoinvex. So, we can easily follow that every (G,ρ,θ)-invex function is (G,ρ,θ)-pseudoinvex, therefore above weak duality theorem follows on the same pattern.

Theorem 3.2 (Weak duality): Let $(x,y,w) \in P^0$ and $(u,v,t) \in Q^0$. Let

(i) $f(.,v)$ be (G,ρ_1,θ_1)- pseudoinvex and $g(.,v)$ be (G,ρ_2,θ_2)- pseudoincave at u for fixed v with respect to η_1,

(ii) $f(x,.)$ be $(G,\rho_3,\theta_3)-$ pseudoincave and $g(x,.)$ be $(G,\rho_4,\theta_4)-$ pseudoinvex at y for fixed x with respect to η_2,

(iii) $\eta_1(x,u) + u \in C_1$ and $\eta_2(v,y) + y \in C_2$,

(iv) $G(g(x,v)) > 0$,

(v) $(\rho_1||\theta_1(x,u)||^2 - t\rho_2||\theta_2(x,u)||^2) \geq 0$,

(vi) $(\rho_3||\theta_3(x,u)||^2 - w\rho_4||\theta_4(x,u)||^2) \leq 0$.

Then, $w \geq t$.

Proof: The proof follows on the same pattern of theorem 3.1.

Theorem 3.3 (Strong duality theorem). Let f and g be differentiable functions. Let $(\bar{r},\bar{s},\bar{w})$ be an optimal solution of (EFWP). Suppose that

(i) $[G''(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})(\nabla_s f(\bar{r},\bar{s}))^T + G'(f(\bar{r},\bar{s}))\nabla_{ss} f(\bar{r},\bar{s}) - \bar{w}\{G''(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})(\nabla_s g(\bar{r},\bar{s}))^T$

$+ G'(g(\bar{y},\bar{y}))\nabla_{ss} g(\bar{r},\bar{s})\}]$ is non-singular,

(ii) $\big(\bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s}) - \bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s}))G(f(\bar{r},\bar{s})) + (\bar{s}^T G'(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})$

$- \bar{r}^T G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s}))G(g(\bar{r},\bar{s})\big) = 0.$

Then, $(\bar{r},\bar{s},\bar{w})\in Q^0$ and objective values of (EFWP) and (EFWD) are equal. Moreover, if all the hypotheses of weak duality theorem are satisfied, then $(\bar{r},\bar{s},\bar{w},\bar{q}=0)$ is an optimal solution of (EFWD).

Proof: Since $(\bar{r},\bar{s},\bar{w})$ is an optimal solution of (EFWP), $\alpha\in R,\beta\in R,\gamma\in C_2,\mu\in R$ such that the following Fritz John necessary conditions [16] are satisfied at $(\bar{r},\bar{s},\bar{w})$:

$$[\beta(G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})-\bar{w}G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s}))+(\gamma-\beta\bar{s})^T(G''(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})$$

$$\nabla_r f(\bar{r},\bar{s})+G'(f(\bar{r},\bar{s}))\nabla_{sr}f(\bar{r},\bar{s})-\bar{w}(G''(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})\nabla_r g(\bar{r},\bar{s})$$
$$+G'(g(\bar{r},\bar{s}))\nabla_{sr}g(\bar{r},\bar{s}))-\mu]=0, \tag{3.16}$$

$$(\gamma-\beta\bar{s})^T[G''(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})(\nabla_s f(\bar{r},\bar{s}))^T+G'(f(\bar{r},\bar{s}))\nabla_{ss}f(\bar{r},\bar{s})$$
$$-\bar{w}(G''(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})(\nabla_s g(\bar{r},\bar{s}))^T+G'(g(\bar{r},\bar{s}))\nabla_{ss}g(\bar{r},\bar{s}))]=0, \tag{3.17}$$

$$\gamma^T[G'(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})-\bar{w}G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})]=0, \tag{3.18}$$

$$\alpha-\beta[G(g(\bar{r},\bar{s}))-\bar{s}^TG'(g(\bar{r},\bar{s})\nabla_s g(\bar{r},\bar{s}))]-\gamma G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})=0, \tag{3.19}$$

$$\mu^T\bar{r}=0, \tag{3.20}$$

$$(\alpha,\beta,\gamma,\mu)\neq 0,\ (\alpha,\beta,\gamma,\mu)\geq 0. \tag{3.21}$$

Since $\Big(G''(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})(\nabla_s f(\bar{r},\bar{s}))^T+G'(f(\bar{r},\bar{s}))\nabla_{ss}f(\bar{r},\bar{s})-\bar{w}(G''(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})$ $(\nabla_s g(\bar{r},\bar{s}))^T+G'(g(\bar{r},\bar{s}))\nabla_{ss}g(\bar{r},\bar{s}))\Big)$ is non-singular, it follows from (3.17) that

$$\gamma=\beta\bar{s}. \tag{3.22}$$

Next, our aim to show that $\beta\neq 0$. If possible, then suppose that $\beta=0$, then from (3.22), we get $\gamma=0$. From (3.19), we have $\alpha=0$, which contradicts (3.21). This combined with (3.16), we find that $\mu=0$. Hence, $\beta\neq 0 \implies \beta>0$. Now, it gives that (3.17) and (3.22) and in particular, by (3.22), $\beta>0$ and since $\gamma\geq 0$, hence we have $\bar{s}\geq 0$.
From inequality (3.16), we get

$$G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})-\bar{w}G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})=\frac{\mu}{\beta}\geq 0, \tag{3.23}$$

or

$$G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})-\bar{w}G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})\in C_2^*. \tag{3.24}$$

Therefore, $(\bar{r},\bar{s},\bar{w})\in Q^0$.

Next, we have to claim that the objective values of the problem are equal. It is sufficient to show that

$$\frac{G(f(\bar{r},\bar{s}))-\bar{r}^TG'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})}{G(g(\bar{r},\bar{s}))-\bar{r}^TG'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})}=\frac{G(f(\bar{r},\bar{s}))-\bar{s}^TG'(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})}{G(g(\bar{r},\bar{s}))-\bar{s}^TG'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})}$$

Now, multiplying (3.23), by $\bar{r}^T$ and using (3.20), we have

$$\frac{\bar{r}^T G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})}{\bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})} = \bar{w}. \tag{3.25}$$

Further, using (3.22) and (3.18), we get

$$\frac{\bar{s}^T G'(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})}{\bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})} = \bar{w}. \tag{3.26}$$

From (3.25) and (3.26), we have

$$\frac{\bar{r}^T G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})}{\bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})} = \frac{\bar{s}^T G'(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})}{\bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})}$$

i.e.

$$\begin{aligned}&\big(\bar{r}^T G'((\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})\big)\big(\bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})\big)\\&\qquad = \big(\bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})\big)\big(\bar{s}^T G'((\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})\big).\end{aligned} \tag{3.27}$$

By hypothesis (ii), we get

$$\begin{aligned}&\bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})G(f(\bar{r},\bar{s})) + \bar{s}^T G'((\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})G(g(\bar{r},\bar{s}))\\&\qquad = \bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})G(f(\bar{r},\bar{s})) + \bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})G(g(\bar{r},\bar{s})).\end{aligned} \tag{3.28}$$

On subtracting (3.28) from (3.27) and after this we adding $G(f(\bar{r},\bar{s}))G(g(\bar{r},\bar{s}))$ of both sides, we have

$$\begin{aligned}&G(f(\bar{r},\bar{s}))G(g(\bar{r},\bar{s})) - G(f(\bar{r},\bar{s}))\bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s}) - \bar{s}^T G'(g(\bar{r},\bar{s}))\\&\nabla_s f(\bar{r},\bar{s})G(g(\bar{r},\bar{s})) + \bar{r}^T G'((\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})\bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})\\&\quad = G(f(\bar{r},\bar{s}))G(g(\bar{r},\bar{s})) - \bar{r}^T G'((\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})G(g(\bar{r},\bar{s})) - \bar{s}^T G'(g(\bar{r},\bar{s}))\\&\qquad \nabla_s g(\bar{r},\bar{s})G(f(\bar{r},\bar{s})) + \bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})\bar{s}^T G'((\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s}).\end{aligned}$$

This can be rewritten as:

$$\frac{G(f(\bar{r},\bar{s})) - \bar{r}^T G'(f(\bar{r},\bar{s}))\nabla_r f(\bar{r},\bar{s})}{G(g(\bar{r},\bar{s})) - \bar{r}^T G'(g(\bar{r},\bar{s}))\nabla_r g(\bar{r},\bar{s})} = \frac{G(f(\bar{r},\bar{s})) - \bar{s}^T G'(f(\bar{r},\bar{s}))\nabla_s f(\bar{r},\bar{s})}{G(g(\bar{r},\bar{s})) - \bar{s}^T G'(g(\bar{r},\bar{s}))\nabla_s g(\bar{r},\bar{s})}.$$

Under the weak duality theorem, if $(\bar{r},\bar{s},\bar{w})$ is not an optimal solution of (EFWD), then there are other $(u,v,W) \in Q^0$ such that $\bar{w} \geq W$. Since, $(\bar{r},\bar{s},\bar{w}) \in P^0$. So, we obtain that $\bar{w} \geq W$, which is a contradiction. Thus,$(\bar{r},\bar{s},\bar{w})$ is an optimal solution of (EFWD). Hence, the result.

Theorem 3.4 (Strict converse duality). Let f and g be differentiable functions. Let $(\bar{v},\bar{w},\bar{t})$ be an optimal solution of (EFWD). Suppose that

(i) $\Big(G''(f(\bar{v},\bar{w}))\nabla_x f(\bar{v},\bar{w})(\nabla_x f(\bar{v},\bar{w}))^T + G'(f(\bar{v},\bar{w}))\nabla_{xx} f(\bar{v},\bar{w}) - \bar{t}\big(G''(g(\bar{v},\bar{w}))$
$\nabla_x g(\bar{v},\bar{w})(\nabla_x g(\bar{v},\bar{w}))^T + G'(g(\bar{v},\bar{w}))\nabla_{xx} g(\bar{v},\bar{w})\big)\Big)$ is non-singular,

$$(ii)\ \big(\bar{u}^T G'(g(\bar{v},\bar{w}))\nabla_x g(\bar{v},\bar{w}) - \bar{v}^T G'(g(\bar{v},\bar{w}))\nabla_y g(\bar{v},\bar{w})\big)G(f(\bar{v},\bar{w}))$$

$$+ \big(\bar{v}^T G'(f(\bar{v},\bar{w}))\nabla_y f(\bar{v},\bar{w}) - \bar{u}^T G'(f(\bar{v},\bar{w}))\nabla_x f(\bar{v},\bar{w})\big)G(g(\bar{v},\bar{w})) = 0.$$

Then, there exists $(\bar{v},\bar{w},\bar{t}) \in P^0$ and objective values are equal. Moreover, if all the hypotheses of weak duality theorem are satisfied, then $(\bar{v},\bar{w},\bar{t})$ is an optimal solution of (EFWP).

Proof: Proof follows on the lines of Theorem 3.3, due to symmetric programming problem.

Conclusion

In this article, we considered fractional dual symmetric programming problem and derived duality theorems under (G,ρ,θ)-invexity conditions. The present work can be extended to multiobjective symmetric fractional dual programs. This may be taken as the future task of the researchers.

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