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Existence result of global solutions for a class of generic reaction diffusion systems

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Abstract

In this paper we prove the existence of weak global solutions for a class of generic reaction diffusion systems for which two main properties hold: the quasi-positivity and a triangular structure condition on the nonlinearities. The main result is a generalization of the work already done on models of a single reaction-diffusion equation. The model studied is applied in image recovery and contrast enhancement. It can also be applied to many models in biology and radiology.

Keywords: reaction diffusion system, global existence, Schauder fixed point theorem.
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1. Introduction

Reaction diffusion systems are widely used in biology, ecology, physics and chemistry. What we observe in modern scientific studies is the great interest of scientists in studying this type of system, which confirms once again its importance in developing sciences in all fields. Many models and real examples in various scientific fields, as well as course notes can be found in Kant and Kumar [14], Murray [19], [20], Pierre [22], Quittner and Souplet [23] and the references therein.

This paper reviews one of the major applications of reaction diffusion systems, namely the smoothing and restoration of images. The purpose of image restoration is to estimate the original image from the degraded data. Applications range from medical imaging, astronomical imaging, to forensic science, etc. In recent years, this field has attracted the attention of many researchers in computer vision. This is mainly due to the mathematical formulation framing any PDEs-based approach that

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can give a good justification and explanation of the results obtained through these traditional and heuristic methods in image processing.

There are many important studies and models that have been studied in recent decades on image processing and its applications. The reader can see some of them and some similar models of the problem that we will study in this paper in Alaa et *al.* [1]-[4] and [17], Alvarez et *al.* [5] and [6], Catté et *al.* [9], Weickert et *al.* [11], [26]-[28], Hashemi et *al.* [12], [13], Morfu [18] and the references therein. He will also find some of the methods and techniques used to study these questions. Concerning the fixed point theorems frequently used in the study of this type of problems, we recommend to the reader Du and Rassias [10], Pata [21] and Xu et *al.* [29].

In this paper, we propose a new model of nonlinear generic reaction diffusion system applied to edge detection and image restoration. We tackle the problem of the global existence of solutions for the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div }\left(A\left(|\nabla u_\sigma|\right)\nabla u\right) = f\left(t,x,u,v,w\right) & \text{, in } Q_T \\ \dfrac{\partial v}{\partial t} - \text{div }\left(B\left(|\nabla v_\sigma|\right)\nabla v\right) = g\left(t,x,u,v,w\right) & \text{, in } Q_T \\ \dfrac{\partial w}{\partial t} - d_w\Delta w = h\left(t,x,u,v,w\right) & \text{, in } Q_T \end{cases} \tag{1.1}$$

with the boundary conditions

$$\frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = \frac{\partial w}{\partial \upsilon} = 0 \quad , \quad \text{on } \Sigma_T \tag{1.2}$$

and the initial conditions

$$u\left(0,x\right) = u_0\left(x\right) \quad , \quad v\left(0,x\right) = v_0\left(x\right) \quad , \quad w\left(0,x\right) = w_0\left(x\right) \quad \text{in } \Omega \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $T \in (0,\infty[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω. υ is the outward normal to the domain and $\frac{\partial}{\partial \upsilon}$ is the normal derivative.

Let $\sigma > 0$, ∇u_σ, ∇v_σ are the regularizations by convolution of ∇u and ∇v respectively. We define

$$\nabla u_\sigma = \nabla(G_\sigma * u) \quad \text{and} \quad \nabla v_\sigma = \nabla(G_\sigma * v)$$

where G_σ is the Gaussian function. The anisotropic diffusivities A and B are smooth nonincreasing functions, such that

$$A(0) = B\left(0\right) = 1 \quad \text{and} \quad \lim_{s\to\infty} A(s) = \lim_{s\to\infty} B(s) = 0$$

Note that the function $s \mapsto \dfrac{1}{1+s^2}$ satisfies these conditions.

We have found a good idea to present our work as follows : In the next section, we present our main result. In the third section, we provide some preliminary results on our problem in the case where the nonlinearities are bound which we need later. In the last section, we truncate the problem and show that the approximate problem admits weak solutions using the Schauder fixed point Theorem. Afterward, we will provide some essential compactness and equi-integrability results in order to pass to the limit and rigorously demonstrate the existence of global weak solution to the considered model.

2. Statement of the main result

2.1. Assumptions

Throughout this note we will assume that : The nonlinear functions $f,\ g,\ h : Q_T \times \mathbb{R}^2 \to \mathbb{R}$ are measurable and $f(t,x,.)\,,\ g(t,x,.)\,,\ h(t,x,.) : \mathbb{R}^3 \to \mathbb{R}$ are continuous. In addition the nonlinearities satisfy the quasi-positivity property

$$f(t,x,0,s,q) \geq 0 \quad , \quad g(t,x,r,0,q) \geq 0 \quad , \quad h(t,x,r,s,0) \geq 0 \quad , \quad \forall r,s,q \geq 0 \tag{2.1}$$

and the triangular structure condition

$$\begin{cases} (f+g+h)(t,x,r,s,q) \leq L_1(1+r+s+q) \\ (g+h)(t,x,r,s,q) \leq L_2(1+r+s+q) \\ h(t,x,r,s,q) \leq L_3(1+r+s+q) \end{cases} \tag{2.2}$$

where L_1, L_2 and L_3 are positive constants. Furthermore,

$$\sup_{|r|+|s|+|q|\leq R} \left(|f(t,x,r,s,q)| + |g(t,x,r,s,q)| + |h(t,x,r,s,q)|\right) \in L^1(Q_T) \tag{2.3}$$

for $R > 0$. The initial conditions u_0, v_0, w_0 are only assumed to be square integrable.

In Pierre [22], we find some examples of reaction diffusion systems as models for very different applications and for which the two properties (2.1) and (2.2) hold. We also refer to Murray's books [19] and [20], in which we find many important models in multiple fields. An interesting example where the result of this paper can be applied is the Modified Fitz-Hugh-Nagumo Model for image restoration. To learn more about this model, we refer to Alaa and Zirhem [2].

We introduce the notion of solution to the problem $(1.1) - (1.3)$, $(2.1) - (2.3)$ used here :

Definition 2.1. *We say that* (u,v,w) *is a weak solution of the system* $(1.1) - (1.3)$ *under the assumptions* $(2.1) - (2.3)$*, if*

(i) $u,\ v,\ w \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$,

(ii) $\forall \phi,\ \psi,\ \eta \in C^1(Q_T)$*, such that* $\phi(\cdot,T) = 0$*,* $\psi(\cdot,T) = 0$ *and* $\eta(\cdot,T) = 0$*, we have*

$$\begin{aligned} \int_{Q_T} -u\frac{\partial \phi}{\partial t} + A(|\nabla u_\sigma|)\nabla u \nabla \phi &= \int_{Q_T} f(t,x,u,v,w)\,\phi + \int_\Omega u_0\phi(0,\cdot) \\ \int_{Q_T} -v\frac{\partial \psi}{\partial t} + B(|\nabla v_\sigma|)\nabla v \nabla \psi &= \int_{Q_T} g(t,x,u,v,w)\,\psi + \int_\Omega v_0\psi(0,\cdot) \\ \int_{Q_T} -w\frac{\partial \eta}{\partial t} + d_w \nabla w \nabla \eta &= \int_{Q_T} h(t,x,u,v,w)\,\eta + \int_\Omega w_0\eta(0,\cdot) \end{aligned}$$

where $f,\ g,\ h \in L^1(Q_T)$.

2.2. The main result

Now, we can state the main result of this work :

Theorem 2.2. *Under the assumptions* $(2.1) - (2.3)$ *and for a continuous functions* f*,* g *and* h *as described above. The reaction diffusion system* $(1.1) - (1.3)$ *admits a global weak solution* (u,v,w) *in the sense defined in Definition 2.1 for all* $u_0, v_0, w_0 \in L^2(\Omega)$ *such that* u_0, v_0, w_0 *are positive.*

3. Preliminary results for bounded nonlinearities

Before treating the nonlinear case, we will prove an existence result for bounded nonlinearities. In what follows, we denote $\mathcal{V} = H^1(\Omega)$ and $\mathcal{H} = L^2(\Omega)$.

Theorem 3.1. *Under the above assumptions on the nonlinearities, if there exist $M_1,\ M_2,\ M_3 \geq 0$, such that for almost every $(t,x) \in Q_T$,*

$$|f(t,x,r,s,q)| \leq M_1 \quad , \quad |g(t,x,r,s,q)| \leq M_2 \quad , \quad |h(t,x,r,s,q)| \leq M_3 \quad , \quad \forall r,s,q \in \mathbb{R}$$

then for every $u_0,\ v_0,\ w_0 \in L^2(\Omega)$, there exists a weak solution (u,v,w) to the considered system $(1.1)-(1.3)$. Moreover there exists a constant C depends on M_1, M_2, M_3, σ, T, $\|u_0\|_{L^2(\Omega)}$, $\|v_0\|_{L^2(\Omega)}$ and $\|w_0\|_{L^2(\Omega)}$, such that

$$\|(u,v,w)\|_{L^\infty(0,T;\mathcal{H})^2} + \|(u,v,w)\|_{L^2(0,T;\mathcal{V})^2} \leq C$$

Furthermore, if $u_0,\ v_0,\ w_0$ are positive and $f,\ g,\ h$ are quasi-positive then $u(t,x) \geq 0$, $v(t,x) \geq 0$ and $w(t,x) \geq 0$ for a.e. $(t,x) \in Q_T$.

Proof . We introduce the space

$$\mathcal{W}(0,T) = \left\{ u,v,w \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) \mid \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \in L^2(0,T;\mathcal{V}') \right\}$$

Let $z = (z_1, z_2, z_3) \in \mathcal{W}(0,T)$ and let (u,v,w) be the solution of a linearization of problem $(1.1)-(1.3)$, $(2.1)-(2.3)$ given by

$$\begin{cases} (u,v,w) \in L^2(0,T;\mathcal{V}) \cap C(0,T;\mathcal{H}) \\ \forall \phi,\psi,\eta \in C^1(Q_T) \text{ such that } \phi(\cdot,T)=0, \psi(\cdot,T)=0, \eta(\cdot,T)=0 \\ \int_{Q_T} -u\dfrac{\partial \phi}{\partial t} + A\left(|\nabla(z_1)_\sigma|\right)\nabla u \nabla \phi = \int_{Q_T} f(t,x,z_1,z_2,z_3)\,\phi + \int_\Omega u_0 \phi(0,\cdot) \\ \int_{Q_T} -v\dfrac{\partial \psi}{\partial t} + B\left(|\nabla(z_2)_\sigma|\right)\nabla v \nabla \psi = \int_{Q_T} g(t,x,z_1,z_2,z_3)\,\psi + \int_\Omega v_0 \psi(0,\cdot) \\ \int_{Q_T} -w\dfrac{\partial \eta}{\partial t} + d_w \nabla w \nabla \eta = \int_{Q_T} h(t,x,z_1,z_2,z_3)\,\eta + \int_\Omega w_0 \eta(0,\cdot) \end{cases} \tag{3.1}$$

The application $z \in \mathcal{W}(0,T) \mapsto (u,v,w) \in \mathcal{W}(0,T)$ is clearly well defined. In fact, z_1, z_2, z_3 are in $L^\infty(0,T;\mathcal{H})$, G_σ is $C^\infty(Q_T)$ therefore $A\left(|\nabla(z_1)_\sigma|\right)$ and $B\left(|\nabla(z_2)_\sigma|\right)$ are in $C^\infty(Q_T)$ and since A and B are nonincreasing, it results

$$a \leq A\left(|\nabla z_\sigma|\right) \leq b \quad \text{and} \quad c \leq B\left(|\nabla z_\sigma|\right) \leq d \tag{3.2}$$

where $b, d > 0$ and $a,\ c$ are a positive constants which only depend on A and B respectively. The property (3.2) with the fact that nonlinearities are bounded implies that the differential operators in (3.1) are continuous and coercive thus by application of the standard theory of Partial Differential Equations, we obtain (u,v,w) the solution of the linearized problem (3.1). To learn more about this existence, we refer to Amann [7], Benilan and Brezis [8] and Lions [16].

Now, we establish some important estimates to reformulate the problem in the form of a fixed point problem. The following result holds for $t \in [0,T]$.

$$\begin{cases} \frac{1}{2}\int_\Omega u^2(t) + \int_{Q_T} A\left(|\nabla (z_1)_\sigma|\right)|\nabla u|^2 = \frac{1}{2}\int_\Omega u_0^2 + \int_{Q_T} u \ f\left(t,x,z_1,z_2,z_3\right) \\ \frac{1}{2}\int_\Omega v^2(t) + \int_{Q_T} B\left(|\nabla (z_2)_\sigma|\right)|\nabla v|^2 = \frac{1}{2}\int_\Omega v_0^2 + \int_{Q_T} v \ g\left(t,x,z_1,z_2,z_3\right) \\ \frac{1}{2}\int_\Omega w^2(t) + d_w\int_{Q_T} |\nabla w|^2 = \frac{1}{2}\int_\Omega w_0^2 + \int_{Q_T} w \ h\left(t,x,z_1,z_2,z_3\right) \end{cases} \tag{3.3}$$

Consequently,

$$\begin{cases} \int_\Omega u^2(t) \le M_1 + \int_{Q_T} u^2 + \int_\Omega u_0^2 \\ \int_\Omega v^2(t) \le M_2 + \int_{Q_T} v^2 + \int_\Omega v_0^2 \\ \int_\Omega w^2(t) \le M_3 + \int_{Q_T} w^2 + \int_\Omega w_0^2 \end{cases} \tag{3.4}$$

Using Gronwall's inequality, we get

$$\begin{cases} \int_{Q_T} u^2 \le \left(e^T - 1\right)\left(M_1 + \int_\Omega u_0^2\right) \\ \int_{Q_T} v^2 \le \left(e^T - 1\right)\left(M_2 + \int_\Omega v_0^2\right) \\ \int_{Q_T} w^2 \le \left(e^T - 1\right)\left(M_3 + \int_\Omega w_0^2\right) \end{cases}$$

By substituting the above expression in (3.3), we obtain

$$\begin{cases} \sup\limits_{0\le t\le T}\int_\Omega u^2(t) \le M_1 + \left(e^T - 1\right)\left(M_1 + \int_\Omega u_0^2\right) + \int_\Omega u_0^2 := C_u \\ \sup\limits_{0\le t\le T}\int_\Omega v^2(t) \le M_2 + \left(e^T - 1\right)\left(M_2 + \int_\Omega v_0^2\right) + \int_\Omega v_0^2 := C_v \\ \sup\limits_{0\le t\le T}\int_\Omega w^2(t) \le M_3 + \left(e^T - 1\right)\left(M_3 + \int_\Omega w_0^2\right) + \int_\Omega w_0^2 := C_w \end{cases}$$

Therefore by setting $C_1 = \max\{C_u, C_v, C_w\}$, we obtain

$$\|(u,v,w)\|_{L^\infty(0,T;\mathcal{H})^3} \le C_1$$

Using (3.3) and (3.4) , we deduce

$$\begin{cases} \int_{Q_T} u^2 + |\nabla u|^2 \le \dfrac{M_1 + \int_{Q_T} u^2 + \int_\Omega u_0^2}{\min\left\{\frac{1}{2}, a\right\}} \le C'_u \\ \int_{Q_T} v^2 + |\nabla v|^2 \le \dfrac{M_2 + \int_{Q_T} v^2 + \int_\Omega v_0^2}{\min\left\{\frac{1}{2}, b\right\}} \le C'_v \\ \int_{Q_T} w^2 + |\nabla w|^2 \le \dfrac{M_3 + \int_{Q_T} w^2 + \int_\Omega w_0^2}{\min\left\{\frac{1}{2}, d_w\right\}} \le C'_w \end{cases}$$

Setting $C_2 = \max\{C'_u, C'_v, C'_w\}$, we conclude that

$$\|(u,v,w)\|_{L^2(0,T;\mathcal{V})^3} \le C_2$$

Next we estimate $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ in $L^2(0,T;\mathcal{V}')$. We know that

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\ \left(A\left(|\nabla u_\sigma|\right)\nabla u\right) + f\left(t,x,u,v,w\right) \\ \dfrac{\partial v}{\partial t} = \operatorname{div}\ \left(B\left(|\nabla v_\sigma|\right)\nabla v\right) + g\left(t,x,u,v,w\right) \\ \dfrac{\partial w}{\partial t} = d_w \Delta w + h\left(t,x,u,v,w\right) \end{cases}$$

It follows that

$$\begin{cases} \left\|\dfrac{\partial u}{\partial t}\right\|_{L^2(0,T;\mathcal{V}')} \leq C \left\|\nabla u\right\|_{L^2(Q_T)} + M_1 T \\ \left\|\dfrac{\partial v}{\partial t}\right\|_{L^2(0,T;\mathcal{V}')} \leq C' \left\|\nabla v\right\|_{L^2(Q_T)} + M_2 T \\ \left\|\dfrac{\partial w}{\partial t}\right\|_{L^2(0,T;\mathcal{V}')} \leq d_w \left\|\nabla w\right\|_{L^2(Q_T)} + M_3 T \end{cases}$$

and

$$\begin{cases} \left\|\dfrac{\partial u}{\partial t}\right\|_{L^2(0,T;\mathcal{V}')} \leq CC_1 + M_1 T \\ \left\|\dfrac{\partial v}{\partial t}\right\|_{L^2(0,T;\mathcal{V}')} \leq C'C_1 + M_2 T \\ \left\|\dfrac{\partial w}{\partial t}\right\|_{L^2(0,T;\mathcal{V}')} \leq d_w C_1 + M_3 T \end{cases}$$

Eventually,

$$\left\|\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right)\right\|_{L^2(0,T;\mathcal{V}')} \leq \max\left\{CC_1 + M_1 T, C'C_1 + M_2 T, d_w C_1 + M_3 T\right\} := C_3$$

Now, we can apply the Schauder fixed point Theorem in the functional space

$$\begin{aligned} \mathcal{W}_0(0,T) = \quad & \{u,v,w \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) : \|(u,v,w)\|_{L^\infty(0,T;\mathcal{H})^3} \leq C_1; \\ & \|(u,v,w)\|_{L^2(0,T;\mathcal{V})^3} \leq C_2; \quad \left\|\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right)\right\|_{L^2(0,T;\mathcal{V}')} \leq C_3, \\ & u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0\} \end{aligned}$$

We can easily verify that $\mathcal{W}_0(0,T)$ is a nonempty closed convex in $\mathcal{W}(0,T)$. To use Schauder's Theorem, we will show that the application

$$F : z \in \mathcal{W}_0(0,T) \longrightarrow F(z) = (u,v,w) \in \mathcal{W}_0(0,T)$$

is weakly continuous.

Let us consider a sequence $z_n \in \mathcal{W}_0(0,T)$ such that z_n converges weakly in $W_0(0,T)$ toward z, and let $F(z_n) = (u_n, v_n, w_n)$. Thus,

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \text{div } \left(A\left(|\nabla z_{1n_\sigma}|\right)\nabla u_n\right) + f\left(t,x,u_n,v_n,w_n\right) \\ \dfrac{\partial v_n}{\partial t} = \text{div } \left(B\left(|\nabla z_{2n_\sigma}|\right)\nabla v_n\right) + g\left(t,x,u_n,v_n,w_n\right) \\ \dfrac{\partial w_n}{\partial t} = d_w \Delta w_n + h\left(t,x,u_n,v_n,w_n\right) \end{cases} \tag{3.5}$$

Based on the previous estimation, (u_n, v_n, w_n) is bounded in $L^2(0,T;\mathcal{V})^3$ and $\left(\dfrac{\partial u_n}{\partial t}, \dfrac{\partial v_n}{\partial t}, \dfrac{\partial w_n}{\partial t}\right)$ is bounded in $L^2(0,T;\mathcal{V}')^3$. Then, by using Aubin-Simon compactness in Simon [25], we have that (u_n, v_n, w_n) is relatively compact on $\left(L^2(Q_T)\right)^3$; which allows us to extract a subsequence denoted $z_n = (u_n, v_n, w_n)$, such that:

- $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ in $L^2(0,T;\mathcal{V})$,
- $f\left(t,x,z_n\right) \to f\left(t,x,z\right)$, $g\left(t,x,z_n\right) \to g\left(t,x,z\right)$ and $h\left(t,x,z_n\right) \to h\left(t,x,z\right)$ in $\left(L^2(Q_T)\right)$,
- $u_n \to u$, $v_n \to v$ and $w_n \to w$ in $L^2(0,T;\mathcal{H})$ and a.e in Q_T,
- $\nabla u_n \rightharpoonup \nabla u$, $\nabla v_n \rightharpoonup \nabla v$ and $\nabla w_n \rightharpoonup \nabla w$ in $L^2(0,T;\mathcal{H})$,
- $z_n \longrightarrow z$ in $L^2(0,T;\mathcal{H})$ and a.e in Q_T,
- $A\left(|\nabla z_{1n\sigma}|\right) \to A\left(|\nabla z_{1\sigma}|\right)$ and $B\left(|\nabla z_{2n\sigma}|\right) \to B\left(|\nabla z_{2\sigma}|\right)$ in $L^2(0,T;\mathcal{V})$,
- $\dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t}$, $\dfrac{\partial v_n}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t}$ and $\dfrac{\partial w_n}{\partial t} \rightharpoonup \dfrac{\partial w}{\partial t}$ in $L^2(0,T;\mathcal{V}')$.

Using this convergences, we can pass to the limit in (3.5) and show that the limit u, v and w are solutions of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \text{div } \left(A\left(|\nabla z_{1\sigma}|\right)\nabla u\right) + f\left(t,x,z_n\right) \\ \dfrac{\partial v}{\partial t} = \text{div } \left(B\left(|\nabla z_{2\sigma}|\right)\nabla v\right) + g\left(t,x,z_n\right) \\ \dfrac{\partial w}{\partial t} = d_w \Delta w + h\left(t,x,z_n\right) \end{cases}$$

Thus $F(z) = (u, v, w)$, then F is weakly continuous which proves the desired results.

Note that the condition of quasi-positivity (2.1) leads to the positivity of u, v and w. For more details, we refer to Alaa et *al.* [4] and [17]. □

4. Existence result for unbounded nonlinearities

4.1. Approximating scheme

First, we truncate f, g and h using truncation function $\Psi_n \in C_c^\infty(\mathbb{R})$, such that $0 \le \Psi_n \le 1$ and

$$\Psi_n\left(r\right) = \begin{cases} 0 & \text{if} \quad |r| \le n \\ 1 & \text{if} \quad |r| \ge n+1 \end{cases}$$

We can say that the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \text{div } \left(A\left(|\nabla u_{n\sigma}|\right)\nabla u_n\right) + f_n\left(t,x,u_n,v_n,w_n\right) \\ \dfrac{\partial v_n}{\partial t} = \text{div } \left(B\left(|\nabla v_{n\sigma}|\right)\nabla v_n\right) + g_n\left(t,x,u_n,v_n,w_n\right) \\ \dfrac{\partial w_n}{\partial t} = d_w \Delta w_n + h_n\left(t,x,u_n,v_n,w_n\right) \end{cases} \tag{4.1}$$

where

$$\begin{aligned} f_n(t,x,u_n,v_n,w_n) &= \Psi_n(|u_n|+|v_n|+|w_n|)\cdot f(t,x,u_n,v_n,w_n)\\ g_n(t,x,u_n,v_n,w_n) &= \Psi_n(|u_n|+|v_n|+|w_n|)\cdot g(t,x,u_n,v_n,w_n)\\ h_n(t,x,u_n,v_n,w_n) &= \Psi_n(|u_n|+|v_n|+|w_n|)\cdot h(t,x,u_n,v_n,w_n)\end{aligned}$$

admits a weak solution by means of Theorem 3.1.

4.2. A priori estimates

In what follows, In what follows, C denotes a constant independent of n. Now we show that up to a subsequences (u_n, v_n, w_n) converges to the weak solution (u, v, w) of problem $(1.1)-(1.3)$, $(2.1)-(2.3)$. For this, we will prove some important results.

Lemma 4.1. *Under the assumptions of the main result and for (u_n, v_n, w_n) a weak solution of the truncated problem, there exists $C > 0$, such that*

$$\|u_n+v_n+w_n\|_{L^2(Q_T)} \le C\left[1+\|v_n\|_{L^2(Q_T)}+\|w_n\|_{L^2(Q_T)}\right]$$

Proof . This estimate is based on the duality method, for more details, see Pierre [22]. Let $\theta \in \mathcal{C}_c^\infty(Q_T)$ be such that $\theta \ge 0$ and let ϕ be a solution of

$$\begin{cases} \dfrac{-\partial\phi}{\partial t} - \operatorname{div}\ (A(|\nabla u_{n\sigma}|)\,u_n\nabla\phi) = \theta \\ \dfrac{\partial\phi}{\partial n} = 0 \\ \phi(T,0) \end{cases} \tag{4.2}$$

We know that there exists $C > 0$ such that $\|\phi\|_{H^2(Q_T)} \le C\|\theta\|_{L^2(Q_T)}$. Further details can be found in Ladyzhenskaya et *al.* [15] and Schmitt [24]. We set $W = \exp(-L_1t)(u_n+v_n+w_n)$, by the mass control the following inequality holds,

$$\begin{aligned} &\int_{Q_T}\partial_t W\phi + \int_{Q_T}\exp(-L_1t)\left[\operatorname{div}\ (A(|\nabla u_{n\sigma}|)u_n) + \operatorname{div}\ (B(|\nabla v_{n\sigma}|)v_n) + d_w\Delta w_n\right]\phi \\ \le\ &\int_{Q_T} L_1\exp(-L_1t)\,\phi \end{aligned}$$

Integrating by parts and using (4.2) , we get

$$\begin{aligned} \int_{Q_T} W\theta \le \int_{Q_T}&\exp(-L_1t)[d_w\Delta\phi - A(|\nabla u_{n\sigma}|)\Delta\phi - \nabla A(|\nabla u_{n\sigma}|)\nabla\phi - \\ &B(|\nabla v_{n\sigma}|)\Delta\phi - \nabla B(|\nabla v_{n\sigma}|)\nabla\phi]w_n \\ &+\int_{Q_T} L_1\exp(-L_1t)\phi + \int_\Omega (u_0+v_0+w_0)\,\phi(0,\cdot) \end{aligned}$$

where $A(|\nabla u_{n\sigma}|)$, $B(|\nabla v_{n\sigma}|)$, $\nabla A(|\nabla u_{n\sigma}|)$ and $\nabla B(|\nabla v_{n\sigma}|)$ are bounded independently of n in $L^\infty(Q_T)$; hence

$$\begin{aligned} \int_{Q_T} W\theta &\le C\left[1+\|u_0+v_0+w_0\|_{L^2(\Omega)}+\|v_n\|_{L^2(Q_T)}+\|w_n\|_{L^2(Q_T)}\right]\|\phi\|_{H^2(Q_T)} \\ &\le C\left[1+\|v_n\|_{L^2(Q_T)}+\|w_n\|_{L^2(Q_T)}\right]\|\theta\|_{L^2(Q_T)} \end{aligned}$$

which by duality completes the proof. □

Lemma 4.2. *Let (u_n, v_n, w_n) be the solution of the approximate problem* (4.1). *Then*
(i) There exists a constant M depending only on $\int_\Omega u_0$, $\int_\Omega v_0$, $\int_\Omega w_0$, L_1, T and $|\Omega|$, such that

$$\int_{Q_T} (u_n + v_n + w_n) \leq M \quad , \quad \forall t \in [0, T]$$

(ii) There exists $C_1 > 0$, such that

$$\int_{Q_T} \left(|\nabla u_n|^2 + |\nabla v_n|^2 + |\nabla w_n|^2\right) \leq C_1$$

(iii) There exists $C_2 > 0$, such that

$$\int_{Q_T} (|f_n| + |g_n| + |h_n|) \leq C_2$$

Proof . (i) The triangular structure of problem (1.1) − (1.3), (2.1) − (2.3) implies that

$$\frac{\partial u_n}{\partial t} + \frac{\partial v_n}{\partial t} + \frac{\partial w_n}{\partial t} - \operatorname{div}\ (A(|\nabla u_{n\sigma}|)\nabla u_n)$$
$$-\operatorname{div}\ (B(|\nabla v_{n\sigma}|)\nabla v_n) - d_w \Delta w_n \leq L_1 (1 + u_n + v_n + w_n)$$

The integration over Q_T leads to

$$\int_\Omega (u_n + v_n + w_n)(t) \leq \int_\Omega (u_0 + v_0 + w_0) + L_1 \int_{Q_T} (1 + u_n + v_n + w_n)$$

According to Gronwall's Lemma, we get

$$\int_\Omega (u_n + v_n + w_n)(t) \leq \left[\int_\Omega (u_0 + v_0 + w_0) + L_1 |Q_T|\right] \exp(L_1 T)$$

It is what we want to prove.
(ii) We have

$$\frac{\partial w_n}{\partial t} - d_w \Delta w_n = h_n \leq L_3 (1 + u_n + v_n + w_n)$$

The integration over Q_T leads to

$$\frac{1}{2}\int_{Q_T} \left(w_n^2\right)_t + d_w \int_{Q_T} |\nabla w_n|^2 \leq L_3 \int_{Q_T} (1 + u_n + v_n + w_n) w_n$$

According to Young's inequality and Lemma 4.1, we get

$$\begin{aligned} \frac{1}{2}\int_\Omega w_n^2 + d_w \int_{Q_T} |\nabla w_n|^2 &\leq \frac{1}{2}\int_\Omega w_0^2 + L_3 \left[\int_{Q_T} (1 + u_n + v_n + w_n)^2 + \int_{Q_T} w_n^2\right] \\ &\leq \frac{1}{2}\int_\Omega w_0^2 + C \int_{Q_T} w_n^2 \end{aligned}$$

and by Gronwall's Lemma, we deduce that

$$\int_{Q_T} w_n^2 \leq C$$

which ensures that $\int_{Q_T} |\nabla w_n|^2$ and $\int_{Q_T} w_n^2$ are bounded. Now let us show that $\int_{Q_T} |\nabla v_n|^2$ are bounded. We have $v_n + w_n$ satisfies

$$\partial_t(v_n + w_n) - \text{div } \left(B\left(|\nabla v_{n_\sigma}|\right)\nabla v_n\right) - d_w \Delta w_n = g_n + h_n \leq L_2\left(1 + u_n + v_n + w_n\right)$$

Letting $\gamma = \exp\left(-L_2 t\right)\left(v_n + w_n\right)$, he comes

$$\int_{Q_T} \frac{\partial \gamma}{\partial t}\, \gamma + I + \int_{Q_T} \exp\left(-L_2 t\right) d_w \nabla w_n \nabla(v_n + w_n) \leq \int_{Q_T} \exp\left(-L_2 t\right) L_2 \gamma \tag{4.3}$$

where

$$\begin{aligned} I &= \int_{Q_T} \exp\left(-L_2 t\right) B\left(|\nabla v_{n_\sigma}|\right) \nabla v_n \nabla(v_n + w_n) \\ &= \int_{Q_T} \exp\left(-L_2 t\right) B\left(|\nabla v_{n_\sigma}|\right) |\nabla(v_n + w_n)|^2 \\ &\quad - \int_{Q_T} \exp\left(-L_2 t\right) B\left(|\nabla v_{n_\sigma}|\right) \nabla w_n \nabla(v_n + w_n) \end{aligned}$$

Since $B\left(|\nabla v_{n_\sigma}|\right) \geq c$, we have

$$I \geq c \int_{Q_T} |\nabla(v_n + w_n)|^2 - \int_{Q_T} \exp\left(-L_2 t\right) B\left(|\nabla v_{n_\sigma}|\right) \nabla w_n \nabla(v_n + w_n)$$

Substituting in (4.3), he comes

$$\begin{aligned} & \frac{1}{2}\int_{\Omega} \gamma^2\left(T\right) + c \int_{Q_T} |\nabla(v_n + w_n)|^2 \\ \leq\ & C + \int_{Q_T} \exp\left(-L_2 t\right)\left(d_w - B\left(|\nabla v_{n_\sigma}|\right)\right) \nabla w_n \nabla(v_n + w_n) \end{aligned}$$

According to Young's inequality on $|\nabla v_n \nabla(v_n + w_n)|$, we have

$$\begin{aligned} c\int_{Q_T} |\nabla(v_n + w_n)|^2 &\leq C + \int_{Q_T} \exp\left(-L_2 t\right)\left(d_w - d\right)\left[\frac{|\nabla v_n|^2}{2\varepsilon} + \frac{\varepsilon\, |\nabla(v_n + w_n)|^2}{2}\right] \\ &\leq C\left(1 + \frac{\exp\left(-L_2 t\right)\left(d_w - d\right)}{2\varepsilon C}\left[\int_{Q_T} |\nabla v_n|^2 + \varepsilon^2 \int_{Q_T} |\nabla(v_n + w_n)|^2\right]\right) \\ &\leq C\left(1 + C\left(\varepsilon\right)\left[\int_{Q_T} |\nabla v_n|^2 + \varepsilon^2 \int_{Q_T} |\nabla(v_n + w_n)|^2\right]\right) \end{aligned}$$

Hence by choosing a suitable ε we deduce that $\int_{Q_T} |\nabla(v_n + w_n)|^2$ is bounded and because $\int_{Q_T} |\nabla w_n|^2$ is bounded, $\int_{Q_T} |\nabla v_n|^2$ is bounded as well.

In the same way, taking $u_n + v_n + w_n$, we deduce that $\int_{Q_T} |\nabla(v_n + w_n + w_n)|^2$ is bounded and because $\int_{Q_T} |\nabla w_n|^2$ and $\int_{Q_T} |\nabla v_n|^2$ are bounded, we conclude that $\int_{Q_T} |\nabla u_n|^2$ is bounded as well.

(iii) For w_n solution of

$$\frac{\partial w_n}{\partial t} - d_w \Delta w_n = h_n \leq L_3\left(1 + u_n + v_n + w_n\right)$$

We can write

$$\frac{\partial w_n}{\partial t} - d_w \Delta w_n + L_3\left(1+u_n+v_n+w_n\right) - h_n = L_3\left(1+u_n+v_n+w_n\right)$$

which implies

$$\int_{Q_T} \frac{\partial w_n}{\partial t} + \int_{Q_T} \left[L_3\left(1+u_n+v_n+w_n\right)\right] - h_n = \int_{Q_T} L_3\left(1+u_n+v_n+w_n\right)$$

Then

$$\int_{\Omega} w_n(T) - \int_{\Omega} w_n(0) + \int_{Q_T} \left[L_3\left(1+u_n+v_n+w_n\right) - h_n\right] = \int_{Q_T} L_3\left(1+u_n+v_n+w_n\right)$$

We know that $\int_{Q_T} L_3(1+u_n+v_n+w_n)$ is bounded, which follows that

$$\left\|L_3\left(1+u_n+v_n+w_n\right) - h_n\right\|_{L^1(Q_T)} \leq C$$

Therefore

$$\|h_n\|_{L^1(Q_T)} \leq C_h$$

Since $L_2(1+u_n+v_n+w_n) - g_n - h_n \geq 0$, we obtain the same for $g_n + h_n$, hence

$$\|g_n\|_{L^1(Q_T)} \leq C_g$$

and since $L_1(1+u_n+v_n+w_n) - f_n - g_n - h_n \geq 0$, we obtain the same for $f_n + g_n + h_n$, hence

$$\|f_n\|_{L^1(Q_T)} \leq C_f$$

□

4.3. Convergence

Our objective is to show that (u_n, v_n, w_n) converges to some (u, v, w) solution of our problem. According to Lemma 4.2, (u_n, v_n, w_n) is bounded in $\left(L^2(0,T;\mathcal{V})\right)^3$ and $\left(\frac{\partial u_n}{\partial t}, \frac{\partial v_n}{\partial t}, \frac{\partial w_n}{\partial t}\right)$ is bounded in $\left(L^2(0,T;\mathcal{V}') + L^1(Q_T)\right)^3$. Therefore, by Aubin-Simon, (u_n, v_n, w_n) is relatively compact in $\left(L^2(Q_T)\right)^3$, see Simon [25], then we can extract a subsequence also noted (u_n, v_n, w_n) in $\left(L^2(Q_T)\right)^3$, such that :

- $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ in $L^2(Q_T)$ and a.e. in Q_T,
- $\nabla G_\sigma * u_n \rightharpoonup \nabla G_\sigma * u$ and $\nabla G_\sigma * v_n \rightharpoonup \nabla G_\sigma * v$ in $L^2(Q_T)$ and a.e. in Q_T,
- $A\left(|\nabla u_{n\sigma}|\right) \longrightarrow A\left(|\nabla u_\sigma|\right)$ and $B\left(|\nabla v_{n\sigma}|\right) \to B\left(|\nabla v_\sigma|\right)$ in $L^2(Q_T)$,
- $f_n\left(t,x,u_n,v_n,w_n\right) \to f\left(t,x,u,v,w\right)$ for a.e in Q_T,
- $g_n\left(t,x,u_n,v_n,w_n\right) \to g\left(t,x,u,v,w\right)$ for a.e in Q_T,
- $h_n\left(t,x,u_n,v_n,w_n\right) \to h\left(t,x,u,v,w\right)$ for a.e in Q_T.

This is not sufficient to ensure that (u_n, v_n, w_n) is a weak solution of our problem. In fact, we have to prove that the previous convergences are in $L^1(Q_T)$. In view of the Vitali Theorem, to show that $f_n \to f$, $g_n \to g$ and $h_n \to h$ strongly in $L^1(Q_T)$, is equivalent to proving that f_n, g_n and h_n are equi-integrable in $L^1(Q_T)$. This is confirmed by the following Lemma :

Lemma 4.3. *Under the additional assumption that, for $R > 0$,*

$$\sup_{|r|+|s|+|q|\leq R} (|f(t,x,r,s,q)| + |g(t,x,r,s,q)| + |h(t,x,r,s,q)|) \in L^1(Q_T)$$

(i) There exists $C > 0$, such that

$$\int_{Q_T} (u_n + 2v_n + 3w_n)(|f_n| + |g_n| + |h_n|) \leq C$$

(ii) f_n, g_n and h_n are equi-integrable in $L^1(Q_T)$.

Proof . (i) Let

$$\begin{aligned} R_n &= L_1(1 + u_n + v_n + w_n) - f_n - g_n - h_n \\ S_n &= L_1(1 + u_n + v_n + w_n) - g_n - h_n \\ P_n &= L_1(1 + u_n + v_n + w_n) - h_n \end{aligned}$$

and

$$\theta_n = u_n + 2v_n + 3w_n \quad \text{and} \quad E_n = u_n + v_n + w_n$$

we have by hypothesis (2.2)

$$R_n \geq 0 \ , \ S_n \geq 0 \ , \ P_n \geq 0$$

Combining the equations of system (4.1), we have

$$\begin{aligned} \frac{\partial \theta_n}{\partial t} - \xi_n &= f_n + 2g_n + 3h_n \\ &= -R_n + L_1(1 + u_n + v_n + w_n) \\ &\quad -S_n + L_2(1 + u_n + v_n + w_n) \\ &\quad -P_n + L_3(1 + u_n + v_n + w_n) \end{aligned}$$

where

$$\xi_n = \text{div}\ (A(|\nabla u_{n\sigma}|)\nabla u_n) + 2\text{div}\ (B(|\nabla v_{n\sigma}|)\nabla v_n) + 3d_w \Delta w_n$$

Multiplying by $u_n + 2v_n + 3w_n$ and integrating over Q_T, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega \theta_n^2(T) + \int_{Q_T} \nabla\xi_n.\nabla\theta_n + \int_{Q_T} (R_n + S_n + P_n)\theta_n \\ = \ &\frac{1}{2}\int_\Omega \theta_n^2(0) + (L_1 + L_2 + L_3)\int_{Q_T} (1 + u_n + v_n + w_n)\theta_n \end{aligned}$$

which implies

$$\begin{aligned} \int_{Q_T} (R_n + S_n + P_n)\theta_n \ \leq \ &\int_{Q_T} |\nabla\xi_n|.|\nabla\theta_n| + \frac{1}{2}\int_\Omega \theta_n^2(0) \\ &+ (L_1 + L_2 + L_3)\int_{Q_T} (1 + u_n + v_n + w_n)\theta_n \end{aligned}$$

Using Young's inequality, we conclude that

$$\begin{aligned}\int_{Q_T}(R_n+S_n+P_n)\,\theta_n &\leq \frac{1}{2}\int_{Q_T}\left[|\nabla\xi_n|^2+|\nabla\theta_n|^2\right]+\frac{1}{2}\int_{\Omega}\theta_n^2(0)\\ &\quad+(L_1+L_2+L_3)\int_{Q_T}(1+u_n+v_n+w_n)\,\theta_n\\ &\leq C\end{aligned}$$

By the previous lemmas, we obtain the desired result

(ii) We know that f_n, g_n and h_n converge almost everywhere toward f, g and h. We will show that f_n, g_n and h_n are equi-integrable in $L^1(Q_T)$. The proof will be given for f_n, however the same result holds for g_n and h_n. For this, we let $\varepsilon>0$ and prove that there exists $\delta>0$ such that $|Y|<\delta$ implies that $\int_Y f_n<\varepsilon$. We have

$$\begin{aligned}\int_A|f_n(t,x,u_n,v_n,w_n)| &= \int\limits_{A\cap[E_n>k]}|f_n|+\int\limits_{A\cap[E_n\leq k]}|f_n|\\ &\leq \int\limits_{A\cap[\theta_n>k]}|f_n|+\int\limits_{A\cap[E_n\leq k]}|f_n|\\ &\leq \frac{1}{k}\int_A(u_n+2v_n+3w_n)\,.\,|f_n|+|A|\sup_{|u_n|+|v_n|+|w_n|\leq k}|f(t,x,u_n,v_n,w_n)|\end{aligned}$$

We can choose δ small enough and a large k such that $\int_Y f_n<\varepsilon$.

In the same way, we treat g_n and h_n. $\square$

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