



A second order fitted operator finite difference scheme for a modified Burgers equation

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Abstract

In this paper, a one-dimensional modified Burgers' equation is considered for different Reynolds numbers. For very high Reynolds numbers, the solution possesses a multiscale character in some part of the independent domain and thus can be classified as a singularly perturbed problem. A numerical scheme that uses a fitted operator finite difference scheme to solve the spatial derivatives and the implicit Euler scheme for the time derivative is proposed to solve the modified Burgers' equation via Rothe's method. It is important to note that the proposed fitted operator finite difference scheme is based on the midpoint upwind scheme. The stability of the scheme is established and the error associated with each discretisation is estimated. Numerical simulations are carried out to validate the theoretical findings.

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1. Introduction

A one-dimensional modified Burgers' equation

$$\mathcal{L}_\varepsilon u(x, t) \equiv u_t(x, t) - \varepsilon u_{xx}(x, t) + u^2 u_x(x, t) = 0, \quad (x, t) \in Q, \quad (1.1)$$

subject to the initial and the boundary conditions

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \quad u(0, t) = 0, \quad u(1, t) = 0, \quad t \in (0, T], \quad (1.2)$$

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is considered in this study. Equation (1.1) comprises of an unsteady term u_t , a non-linear convection term $u^2 u_x(x, t)$, a viscous dissipation $u_{xx}(x, t)$ and a perturbation parameter $\varepsilon \in (0, 1]$, which is the dissipation coefficient and the inverse of an effective Reynolds number. Thus for large Reynolds numbers, the solution of Problem (1.1)–(1.2) possesses steep gradients. In this instance, classical numerical method can not serve as good approximates to the exact solution especially in the parts of the domain where the steep gradients occur [9, 15]. Burgers and modified Burgers' equations have been studied by many researchers in different fields, see for example the articles [1, 2, 3, 5, 6, 11, 16, 17] and the references therein.

From singular perturbation point of view, Kadalbajoo and Awasthi [7] designed a numerical scheme which was of almost first order accuracy in space and first order in time to solve Problem (1.1)–(1.2). Their scheme employed the upwind finite difference scheme on a piecewise uniform Shishkin mesh to solve the spatial derivatives and the backward Euler finite difference scheme was used for the time derivatives.

Gupta and Kadalbajoo [8] constructed a numerical scheme to solve Problem (1.1)–(1.2) for different Reynolds numbers. Their scheme was a combination of the implicit Euler and a hybrid finite difference scheme on a piecewise uniform Shishkin mesh for the time and spatial discretisations, respectively. These authors established the asymptotic bounds of the solution by using singular perturbation analysis and analysed their scheme for convergence. Their analysis led to a first and second order accuracies in time and space, respectively, except for a logarithmic factor in space. Notice that their hybrid scheme employed the central difference scheme in layer region and the midpoint scheme in the non-layer region.

Liu et. al [12] considered a first order non-linear singularly perturbed problem with integral boundary condition. The proposed scheme uses the backward Euler on an equidistributing monitor function based on arc-length. They analysed their scheme for convergence and obtained a first order accuracy independent of the perturbation parameter in the maximum norm.

Ravi Kanth and Murali Mohan Kumar [14] considered a stationary non-linear reaction diffusion problem with delay. The authors first converted the non-linear problems into sequence of linear problems and then designed an exponentially fitted spline method to solve it. They analysed their method for convergence and obtained an almost second order accuracy.

Erdogan and Sakar [4] presented a quasilinearization technique to solve a singularly perturbed delay differential equation. Their scheme employed the implicit finite difference scheme on piecewise-uniform S-meshes. Their analysis resulted in a first order uniformly convergent scheme with respect to the perturbation parameter.

In literature, the work done on the general numerical solution of Burgers equation and modified Burgers equation is very huge. However, the same cannot be said for the modified Burgers equation in the context of singular perturbation. Thus to fill this gap, we propose a fitted operator finite difference scheme to solve Problem (1.1)–(1.2). The ideas of the non-standard finite difference scheme [10] are employed to design the scheme. Thus the denominator function in the classical finite difference scheme is replaced with a new positive function which reflects the analytical properties of the problem under study.

Notice that the fitted operator finite difference scheme is based on the midpoint upwind scheme. Using the Rothe's method or the transversal method of lines procedure, the backward Euler finite difference scheme is employed along with this fitted operator finite difference scheme to obtain the numerical solution.

The rest of the paper is organised as follows: In Section 2, we integrate the non-linear problem in time and then analyse it for convergence. The resulting systems of semi-discrete non-linear boundary value problems are linearised and analysed for convergence in Section 3. In Section 4, a priori estimate

of the solution of the semi-discrete boundary value problems and its derivatives are presented. The fitted operator finite difference scheme is designed in Section 5, whilst its stability is established in Section 6. The convergence analysis of the scheme is presented in Section 7. Numerical results are presented in Section 8 whilst a summary of the main result and future direction of this research is presented in the last section.

2. Time Discretisation

Below we transform the Problem (1.1)–(1.2) into semi-discrete boundary value problems via the discretisation of the time variable. At this stage the spatial domain is held continuous. Using the backward Euler finite difference scheme on a uniform mesh, we integrate Problem (1.1)–(1.2) in time to obtain the semi-discrete problem

$$\begin{aligned} \mathcal{L}_\varepsilon^{**} u^{k+1}(x) &\equiv \frac{u^{k+1} - u^k}{\Delta t} - \varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2(x) u_x^{k+1}(x) + b^{k+1}(x) u^{k+1}(x) \\ &= 0, \quad k = 0, 1, 2, \dots, m, \end{aligned} \tag{2.1}$$

along with the initial and boundary conditions,

$$u^0 = u(x, 0) = \varphi(x), \quad u^{k+1}(0) = 0, \quad u^{k+1}(1) = 0. \tag{2.2}$$

Here m is the number of sub-intervals. The scheme (2.1)–(2.2) is rewritten as

$$\mathcal{L}_\varepsilon^m u^{k+1}(x) \equiv -\varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2 u_x^{k+1}(x) + d^{k+1}(x) u^{k+1}(x) = \frac{u^k(x)}{\Delta t}, \tag{2.3}$$

$$u(x, 0) = \varphi(x), \quad u^{k+1}(0) = 0, \quad u^{k+1}(1) = 0, \tag{2.4}$$

where

$$d^{k+1}(x) = 1/\Delta t + b^{k+1}(x), \quad d^{k+1}(x) \geq \gamma,$$

or

$$\mathcal{L}_\varepsilon^* u^{k+1}(x) \equiv \Delta t(-\varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2 u_x^{k+1}(x) + d^{k+1}(x) u^{k+1}(x)) = u^k(x), \tag{2.5}$$

$$u(x, 0) = \varphi(x), \quad u^{k+1}(0) = 0, \quad u^{k+1}(1) = 0, \tag{2.6}$$

where

$$d^{k+1}(x) = 1/\Delta t + b^{k+1}(x), \quad d^{k+1} \geq \gamma.$$

The operator $\mathcal{L}_\varepsilon^*$ defined by the scheme (2.5)–(2.6) satisfies a discrete maximum principle which ensures the stability of the temporal semi-discretisation process.

Lemma 2.1. *The local truncation error of the temporal semi-discretisation process satisfies*

$$\|e^{k+1}\|_\infty \leq C^*(\Delta t)^2. \tag{2.7}$$

Proof . The local truncation error is defined as

$$e^{k+1} = u^{k+1}(x) - \bar{u}^{k+1}(x), \tag{2.8}$$

where $u^{k+1}(x)$ is the exact solution of (2.5)–(2.6) and $\bar{u}^{k+1}(x)$ is the computed solution of

$$\mathcal{L}_\varepsilon^* \bar{u}^{k+1}(x) \equiv \Delta t(-\varepsilon \bar{u}_{xx}^{k+1}(x) + (\bar{u}^{k+1})^2 \bar{u}_x^{k+1}(x) + d^{k+1}(x) \bar{u}^{k+1}(x)) = \bar{u}^k(x), \tag{2.9}$$

$$\bar{u}(x, 0) = \varphi(x), \quad \bar{u}^{k+1}(0) = 0, \quad \bar{u}^{k+1}(1) = 0. \tag{2.10}$$

A truncated Taylor series expansion of $u(x, t_k)$ takes the form

$$u^k(x) = u^{k+1}(x) - \Delta t u_t^{k+1}(x) + \frac{\Delta t^2}{2} u_{tt}^{k+1}(x) + \mathcal{O}(\Delta t)^3. \tag{2.11}$$

From Equation (2.11), we have

$$\begin{aligned} \frac{u^{k+1}(x) - u^k(x)}{\Delta t} &= u_t^{k+1}(x) + \mathcal{O}(\Delta t), \\ \mathcal{L}_\varepsilon^* u^{k+1}(x) &\equiv \Delta t \left(-\varepsilon u_{xx}^{k+1}(x) + (u^{k+1})^2 u_x^{k+1}(x) + d^{k+1}(x) u^{k+1}(x) \right) + \mathcal{O}(\Delta t)^2. \end{aligned} \tag{2.12}$$

Using Equations (2.9)–(2.10) and (2.12), the local truncation error satisfies

$$\mathcal{L}_\varepsilon^* e^{k+1} \equiv -\varepsilon e_{xx}^{k+1}(x) + (e^{k+1})^2 e_x^{k+1}(x) + d^{k+1}(x) e^{k+1} = \mathcal{O}((\Delta t)^2), \tag{2.13}$$

$$e^{k+1}(0) = e^{k+1}(1) = 0. \tag{2.14}$$

Since the operator is stable, the result follows. \square

The global error of the time-discretisation satisfies the result below.

Lemma 2.2. *The global error \mathcal{E}^m satisfies*

$$\|\mathcal{E}^m\|_\infty \leq C(\Delta t). \tag{2.15}$$

Proof .

$$\|\mathcal{E}^m\|_\infty \leq \left| \sum_{k=1}^m e^{k+1}(x) \right| \leq C^* m (\Delta t)^2 = C \Delta t. \tag{2.16}$$

\square

3. Quasilinearisation

In this section, the semi-discrete non-linear equation (2.1) is transformed into a sequence of linear convection diffusion problems by the quasilinearisation technique in [4]. The linearisation of the non-linear term $(u^{k+1}(x))^2$ in Equation (2.1) is done by choosing $u_0^{k+1}(x)$ to be the initial approximation of the function $u^{k+1}(x)$ in the $(u^{k+1}(x))^2$. Now we expand $(u^{k+1}(x))^2$ around $u_0^{k+1}(x)$ in Taylor series to obtain

$$[u_1^{k+1}(x)]^2 = [u_0^{k+1}(x)]^2 + 2[u_0^{k+1}(x)][u_1^{k+1}(x) - u_0^{k+1}(x)] + \dots \tag{3.1}$$

Using $j = 0, 1, 2, \dots$, as the iteration index, Equation (3.1) can be written as

$$[u_{j+1}^{k+1}(x)]^2 = [u_j^{k+1}(x)]^2 + 2[u_j^{k+1}(x)][u_{j+1}^{k+1}(x) - u_j^{k+1}(x)] + \dots \tag{3.2}$$

Truncating Equation (3.2) and utilizing it in (2.3) yields

$$-\varepsilon \frac{\partial^2}{\partial x^2} u_{j+1}^{k+1}(x) + (u_j^{k+1}(x))^2 + 2[u_j^{k+1}(x)][u_{j+1}^{k+1}(x) - u_j^{k+1}(x)] \frac{\partial}{\partial x} u_{j+1}^{k+1}(x) + d^{k+1}(x) u_{j+1}^{k+1}(x) = \frac{u_{j+1}^{k+1}}{\Delta t}.$$

Further simplification results in

$$\begin{aligned} &-\varepsilon \frac{\partial^2}{\partial x^2} u_{j+1}^{k+1}(x) + (u_j^{k+1}(x))^2 \frac{\partial}{\partial x} u_{j+1}^{k+1}(x) + \left(2u_j^{k+1}(x) \frac{\partial}{\partial x} u_{j+1}^{k+1}(x) + d^{k+1}(x) \right) u_{j+1}^{k+1} \\ &= \frac{u_{j+1}^k}{\Delta t} + 2 (u_j^{k+1})^2 \frac{\partial}{\partial x} u_{j+1}^{k+1}(x), \quad x \in \Omega, \quad k \geq 0, \quad j = 0, 1, \dots \end{aligned} \tag{3.3}$$

along with the initial and boundary conditions

$$u_{j+1}^0(x) = \varphi(x), \quad u_{j+1}^{k+1}(0) = 0, \quad u_{j+1}^{k+1}(1) = 0. \tag{3.4}$$

Rescaling (3.3) and using Green’s function, this equation is transformed into the integral equation

$$\begin{aligned} \varepsilon (u_{j+1}^{k+1} - u_j^{k+1})(x) &= \int_0^1 \mathcal{G}(x, s) \left[G(u_j^{k+1}) - G(u_{j-1}^{k+1}) - (u_j^{k+1} - u_{j-1}^{k+1}) \frac{\partial G}{\partial u_{j-1}^{k+1}} u_{j-1}^{k+1} \right. \\ &\quad \left. + (u_{j+1}^{k+1} - u_j^{k+1}) \frac{\partial G}{\partial u_j^{k+1}} u_j^{k+1} \right] ds. \end{aligned} \tag{3.5}$$

Here $\mathcal{G}(x, s)$ is the Greens function of the form

$$\mathcal{G}(x, s) = \begin{cases} (x - 1)s, & 0 \leq s \leq x \leq 1, \\ x(s - 1), & 0 \leq x \leq s \leq 1, \end{cases}$$

and satisfies $|\mathcal{G}(x, s)| \leq \frac{1}{4}, \forall x, s \in [0, 1]$ and $G(u^k) = \frac{\partial u^k}{\partial x^2}(x), x \in \Omega, k \geq 0$.

From the mean value theorem, we have

$$G(u_j^{k+1}) - G(u_{j-1}^{k+1}) = (u_j^{k+1} - u_{j-1}^{k+1}) \frac{\partial G}{\partial u_{j-1}^{k+1}} u_{j-1}^{k+1} + \frac{(u_j^{k+1} - u_{j-1}^{k+1})^2}{2} \frac{\partial^2 G(\theta)}{\partial (u^{k+1})^2}, \tag{3.6}$$

where $u_{j-1}^{k+1} \leq \theta \leq u_j^{k+1}$. Now substituting (3.6) into (3.5) yields

$$\varepsilon (u_{j+1}^{k+1} - u_j^{k+1})(x) = \int_0^1 \mathcal{G}(x, s) \left[(u_{j+1}^{k+1} - u_j^{k+1}) \frac{\partial G}{\partial u_j^{k+1}} u_j^{k+1} + \frac{(u_j^{k+1} - u_{j-1}^{k+1})^2}{2} \frac{\partial^2 G(\theta)}{\partial (u^{k+1})^2} \right] ds. \tag{3.7}$$

Let

$$\max_{|u| \leq 1} \left| \frac{\partial G}{\partial u^{k+1}} u^{k+1} \right| = s_1, \quad \max_{|u| \leq 1} \left| \frac{\partial^2 G}{\partial (u^{k+1})^2} u^{k+1} \right| = s_2. \tag{3.8}$$

Equation (3.7) yields

$$\left| (u_{j+1}^{k+1} - u_j^{k+1}) \right| \leq \frac{s_2}{(8\varepsilon - 2s_1)} \left| u_j^{k+1} - u_{j-1}^{k+1} \right|^2.$$

Therefore the sequence of (u_j^{k+1}) converges quadratically.

In the next section, we estimate the solution of the semi-discrete problem and its derivatives.

4. A Priori Estimate of the Linear Semi-discrete Problem

In this section bounds of the solution of the semi-discrete problem and it derivatives are presented.

Lemma 4.1. *Maximum principle*

Suppose $\tilde{\Psi}^{k+1}(x)$ is a smooth function satisfying $\tilde{\Psi}^{(k+1)}(x) \geq 0, \forall x \in \partial\Omega$. Then $\mathcal{L}_\varepsilon^{m,l} \tilde{\Psi}^{(k+1)} x \geq 0, \forall x \in \Omega$, implies that $\tilde{\Psi}^{(k+1)} x \geq 0, \forall x \in \bar{Q}$.

Proof . The proof of this Lemma is similar to Lemma 3.1 of [7]. \square

Lemma 4.2. *Stability estimate*

Let $\tilde{u}^{k+1}(x)$ be the solution of the linear problem (3.3)–(3.4). Then we have

$$|\tilde{u}^{k+1}(x)| \leq c_0^{-1} \|f\|, \quad k = 0, 1, 2, \dots, m. \tag{4.1}$$

Proof . The proof follows the same lines as in the proof of Lemma 2.2 of [8]. \square

Lemma 4.3. $|\tilde{u}_x^{k+1}(x)| \leq C(1 + \varepsilon^{-1} \exp(-\alpha(1 - x)/\varepsilon)), \quad x \in \bar{\Omega}.$

Proof . Equation (3.3)–(3.4) is written as

$$-\varepsilon \tilde{u}_{xx}^{k+1} + \tilde{a}(x) \tilde{u}_x^{k+1} = h_1(x), \tag{4.2}$$

where $h_1(x) = \tilde{f}(x) - \tilde{c}(x) \tilde{u}^{k+1}(x)$ and for notational simplicity we let $u_{j+1}^{k+1}(x) = \tilde{u}^{k+1}(x)$. Using the integration factor techniques yields

$$\tilde{u}_x^{k+1}(x) = \tilde{u}_x^{k+1}(1) \exp(-\varepsilon^{-1}(A(1) - A(x))) + z_1(x), \tag{4.3}$$

where $z_1(x)$ is given by

$$z_1(x) = \varepsilon^{-1} \int_x^1 h_1(\gamma) \exp(-\varepsilon^{-1}A(\gamma) - A(x)) d\gamma.$$

To derive the bound of $u_x^{k+1}(1)$ in (4.3) we integrate (4.3) from x to 1, to obtain

$$\tilde{u}^{k+1}(1) - \tilde{u}^{k+1}(x) = \tilde{u}_x^{k+1}(1) \int_x^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds + \int_x^1 z_1(s) ds. \tag{4.4}$$

Evaluating (4.4) at $x = 0$ yields

$$\tilde{u}_x^{k+1}(1) = - \frac{\int_0^1 z_1(s) ds}{\int_0^1 \exp(-\varepsilon^{-1}(A(1) - A(s))) ds}, \tag{4.5}$$

since $\tilde{u}^{k+1}(1) = \tilde{u}^{k+1}(0) = 0$. Substituting (4.5) into Equation (4.3) and taking the norm yield

$$|\tilde{u}_x^{k+1}(x)| \leq C \left(1 + \frac{|\exp(\varepsilon^{-1}\bar{\alpha}x)|}{|\int_0^1 \exp(\varepsilon^{-1}\alpha s) ds|} \right), \tag{4.6}$$

where $\bar{\alpha}$ is an upper bound of $a(x)$ over $[0, 1]$ and C is also an upper bound of $|z_1(x)|$ over $\bar{\Omega}$. Further simplification of (4.6) results in

$$|\tilde{u}_x^{k+1}(x)| \leq C \left(1 + \frac{\alpha \varepsilon^{-1} \exp(\bar{\alpha} \varepsilon^{-1} x) \exp(-\bar{\alpha} \varepsilon^{-1})}{\exp(\alpha \varepsilon^{-1}) - 1} \right). \tag{4.7}$$

Since $0 < \varepsilon \ll 1$, we can find ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and hence $\exp(\bar{\alpha} \varepsilon^{-1}) < \exp(\bar{\alpha} \varepsilon_0^{-1})$. Thus the estimate (4.7) can be written as

$$|\tilde{u}_x^{k+1}(x)| \leq C \exp(\bar{\alpha} \varepsilon_0^{-1}) \left(1 + \frac{\alpha \varepsilon^{-1} \exp(-\bar{\alpha}(1 - x)\varepsilon^{-1})}{\exp(\alpha \varepsilon^{-1}) - 1} \right). \tag{4.8}$$

Let f be a real valued function defined by $f(\alpha) = \exp(\alpha \varepsilon^{-1})$ over the interval $[0, 1]$. Now expanding $f(\alpha)$ in Maclaurin series yields $\alpha \leq (\exp(\alpha \varepsilon^{-1}) - 1)$ and then the estimate (4.8) reduces to

$$|\tilde{u}_x^{k+1}(x)| \leq C_5 (1 + \varepsilon^{-1} \exp(-\bar{\alpha}(1 - x)/\varepsilon)), \tag{4.9}$$

where $C_5 = C \exp(\bar{\alpha} \varepsilon_0^{-1})$. \square

The proof the higher order derivatives can be derived in a similar manner.

Theorem 4.4. *The solution of the sequence of linear problems (3.3)–(3.4) and its derivatives satisfy*

$$\left| \frac{\partial^i}{\partial x^i} \tilde{u}^{k+1}(x) \right| \leq C(1 + \varepsilon^{-i} \exp(-\alpha(1-x)/\varepsilon)), \quad x \in \Omega,$$

where i is a non-negative integer which satisfies $0 \leq i \leq 6$.

Proof . The proof of this Theorem for $0 \leq i \leq 1$, follows from the lemma 4.3. The proof of the higher order derivatives, that is $1 < i \leq 6$, can be obtained analogously. \square

5. Spatial Discretization

In this section we construct a fitted operator finite difference scheme to solve the sequence of linear problems. We adapt the notation $U_i \equiv U(x_i)$ as the numerical solution of $\tilde{u}^{k+1}(x_i)$. Notice that we have dropped the superscript index for notational simplicity. Now we perform the discretisation as follows:

$$\mathcal{L}_\varepsilon^{m,l,n} U_i \equiv -\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi_i^2} + \hat{a}_i \frac{U_i - U_{i-1}}{h} + \hat{c}_i U_i = \hat{f}_i, \quad i = 1, 2, \dots, n-1, \tag{5.1}$$

$$U(0) = 0, \quad U(1) = 0, \tag{5.2}$$

where n is the number of sub-intervals. The denominator function, the coefficient functions and the source term are given by

$$\phi_i^2(h, \varepsilon) = \frac{\varepsilon h}{\hat{a}_i} \left(\exp\left(\frac{\hat{a}_i h}{\varepsilon}\right) - 1 \right), \quad \hat{a}_i = \frac{a_i + a_{i-1}}{2}, \quad \hat{c}_i = \frac{c_i + c_{i-1}}{2}, \quad \hat{f}_i = \frac{f_i + f_{i-1}}{2},$$

respectively.

In matrix notation, the scheme (5.1)–(5.2) is a triadiagonal systems of linear equation

$$AU = F,$$

where A is a $((n-1) \times (n-1))$ square matrix whilst U and F are vectors of size $(n-1)$. Their entries are given by

$$\begin{aligned} A_{ij} &= r_i^-, \quad i = 2, 3, \dots, n-1, \quad j = i-1, \\ A_{ij} &= r_i^c, \quad i = 1, 2, \dots, n-1, \quad j = i, \\ A_{ij} &= r_i^+, \quad i = 1, 2, \dots, n-2, \quad j = i+1, \\ F_i &= \hat{f}_i, \quad i = 1, 2, \dots, n-1, \end{aligned}$$

with

$$r_i^- = \frac{-\varepsilon}{\phi_i^2} - \frac{\hat{a}_i}{h}, \quad r_i^c = \frac{2\varepsilon}{\phi_i^2} + \frac{\hat{a}_i}{h} + \hat{c}_i, \quad r_i^+ = \frac{-\varepsilon}{\phi_i^2}.$$

6. Stability of the Scheme

Here we establish the stability of the scheme (5.1)–(5.2).

Lemma 6.1. *Discrete maximum principle*

The operator $\mathcal{L}_\varepsilon^{m,l,n}$ defined by the scheme (5.1)–(5.2) satisfies a discrete maximum principle. That is if $\xi(x_i)$ is a mesh function which satisfies $\xi(x_i) \geq 0, \forall x_i \in \partial\Omega^n$ and $\mathcal{L}_\varepsilon^{m,l,n}\xi(x_i) > 0, \forall x_i \in \Omega^n$. Then $\xi(x_i) \geq 0, \forall x_i \in \bar{\Omega}^n$.

Proof . Let s be an index such that $\xi_s = \min_{x_i \in \Omega^n} \xi_i$, holds and assume $\xi_s < 0$. Clearly $s \neq 0, s \neq n$. On the domain $x_i \in \Omega^n$, we have

$$\mathcal{L}_\varepsilon^{m,l,n} \xi_s \equiv -\frac{\varepsilon}{\phi_i^2}(\xi_{s+1} - 2\xi_s + \xi_{s-1}) + \frac{\hat{a}_s}{h}(\xi_s - \xi_{s-1}) + \hat{c}_s \xi_s \leq 0,$$

which is a contradiction. Therefore, $\xi_i \geq 0, \forall (x_i) \in \bar{\Omega}^n$. \square

Next we show that the scheme (5.1)–(5.2) satisfies a uniform stability estimate below.

Lemma 6.2. *Let u_i be the solution of the discrete problem (5.1)–(5.2). Then it satisfies*

$$|u_i| \leq c_0^{-1} \max_{x_i \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,l,n} u_i|.$$

Proof . Let $z = c_0^{-1} \max_{x_i \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,l,n} u_i|$, and define the mesh function Ψ_i^\pm by $\Psi_i^\pm = z \pm u_i$. At $i = 0$ and $i = n$, we have $\Psi_n^\pm = z \pm u_i \geq 0$. Further on the domain $x_i \in \Omega^n$ we obtain

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,l,n} \Psi_i^\pm &= \varepsilon \left(\frac{z + u_{i+1} - 2(z + u_i) + z \pm u_{i-1}}{\phi_i^2(h, \varepsilon)} \right) + \hat{a}_i \left(\frac{z \pm u_i - (z \pm u_{i-1})}{h} \right) + \hat{c}_i(z \pm u_i) \\ &= \hat{c}_i z \pm \mathcal{L}_\varepsilon^{m,l,n} u_i = \hat{c}_i \left(c_0^{-1} \max_{x_i \in \bar{\Omega}} |\mathcal{L}_\varepsilon^{m,l,n} u_i| \right) \pm \hat{f}_i \geq 0. \end{aligned}$$

From Lemma 6.1, $\Psi_i^\pm \geq 0, \forall x_i \in \bar{\Omega}^n$, and this completes the proof. \square

7. Error Estimate

The error associated with the spatial discretisation is estimated as follows:

$$\begin{aligned} \mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i)) &= \mathcal{L}_\varepsilon^{m,l,n} u(x_i) - \mathcal{L}_\varepsilon^{m,l,n} U(x_i) = \mathcal{L}_\varepsilon^{m,n} u(x_i) - \hat{f}_i \\ &= \mathcal{L}_\varepsilon^{m,l,n} u(x_i) - \mathcal{L}_\varepsilon^{m,l} u(x_{i-1/2}) \\ &= -\varepsilon (\delta^2 u_i - u''(x_{i-1/2})) + \hat{a}_i (D^- u_i - u'(x_{i-1/2})). \end{aligned}$$

Using a truncated Taylor series expansion of the terms u_{i-1}, u_i and ϕ_i^2 and simplifying further gives

$$\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i)) = \left(-\frac{\varepsilon}{8.3!} u^{iv} + \frac{\hat{a}_i}{8.3!} u^{iii} \right) h^2 + \left(-\frac{\varepsilon}{32.5!} u^{vi} + \frac{\hat{a}_i}{32.5!} u^v \right) h^4 + \dots$$

From theorem 4.4 and noticing that the exponential terms vanish as $\varepsilon \rightarrow 0$ (see [13] for proof), yields the estimate

$$|\mathcal{L}_\varepsilon^{m,l,n}(u(x_i) - U(x_i))| \leq Ch^2.$$

Application of Lemma 6.1 leads to the following result.

Lemma 7.1. *The error associated with the fitted operator finite difference scheme based on the midpoint satisfies*

$$|u_i - U_i| \leq Ch^2.$$

From the lemmas 2.2 and 7.1, we obtain the main results in this paper.

Theorem 7.2. *(Uniform Convergence)*

Let u be the exact solution of the continuous Problem (1.1)–(1.2) and U be the approximate solution. The error associated with the proposed numerical scheme satisfies

$$\max_{0 \leq i \leq n; 0 \leq k \leq m} |u(x_i, t_k) - U(x_i, t_k)| \leq C(h^2 + \Delta t).$$

8. Numerical Results

In this section, a test problem is simulated numerically using the proposed scheme to demonstrate the method in practice. Both the maximum pointwise error and the rate of convergence will be computed. The exact solutions of the test problem is not available thus to compute the maximum pointwise errors, we use the formula

$$E_{\varepsilon,n} = \max_{0 \leq i \leq n; 0 \leq k \leq m} |U(x_i, t^k) - U^{2n}(x_i, t^k)|,$$

where $U^{2n}(x_i, t^k)$ is a computed solution with $2n$ and m mesh points.

Also, the numerical rates of convergence are calculated using the formula

$$r = \log_2 \left(\frac{E_{\varepsilon,n}}{E_{\varepsilon,2n}} \right). \quad (8.1)$$

To obtain the ε uniform maximum errors and the ε -uniform rates of convergence we use the formulae

$$E = \max_{0 < \varepsilon \leq 1} E_{\varepsilon,n} \quad \text{and} \quad R = \max_{0 < \varepsilon \leq 1} r,$$

respectively.

Example 8.1. [8] Consider the problem

$$\begin{aligned} u_t(x, t) - \varepsilon u_{xx}(x, t) + u^2 u_x(x, t) &= 0, \quad (x, t) \in Q, \\ u(x, 0) = \sin(\pi x), \quad u(0, t) = u(1, t) &= 0, \quad t \in [0, T]. \end{aligned}$$

9. Conclusion

A numerical scheme was proposed to solve a modified Burgers' equation in this paper. The scheme was combination of the implicit Euler finite difference scheme and a fitted operator finite difference scheme. More specifically, the implicit Euler finite difference scheme was first used to discretise the time variable. This resulted in a sequence of non-linear boundary value problems which was then quasilinearized to sequence of linear problems. A fitted operator finite difference scheme based on the midpoint upwind scheme was then designed to solve the boundary value problems. Convergence analysis lead to a first order accuracy in time and second order in space. To confirm the theoretical findings, numerical simulation were conducted and the results were in conformity with Theorem 7.2.

As future direction of this research, the scheme is being explored on non-linear problems with delays and non-smooth data.

Table 1: Maximum pointwise error and rate of convergence for Example 8.1.

ε	m=n=32	64	128	256	512	1024
10^{-4}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9924	1.9962	
10^{-6}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9923	1.9962	
10^{-8}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9923	1.9962	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
10^{-10}	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
	1.9262	1.9667	1.9842	1.9923	1.9962	
E	6.57E-03	1.73E-03	4.42E-04	1.12E-04	2.81E-05	7.04E-06
R	1.9262	1.9667	1.9842	1.9923	1.9962	

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