Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2517-2529
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23099.2477

On the existence of solutions for the variational inequality problem $VI(A, \psi, \varphi, g, K)$

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we are concerned with the existence of a solution $u \in K$ for the variational inequality problem $VI(A, \psi, \varphi, g, K)$. Furthermore, we propose some conditions that ensure the *well-posedness* of this problem. We study an operator type $g - ql$ which extends the linear problem. Finally, we investigate the existence of solutions for general vector variational inequalities in the inclusion form.

Keywords: Approximating sequence, Convex hull, GKKM mapping, Type g-ql, Variational inequality and Well-posedness .
2010 MSC: Primary 90C33; Secondary 26B25.

1. Introduction

The theory of variational inequalities can be used to describe the principles of virtual work and power which was initially proposed by Fourier in 1823. The prototypes, which lead to a class of variational inequalities, are the problems of Signorini Fichera and frictional contact in elasticity. However, the first complete proof of unique solvability to Signorini's Problem was provided by student of Signorini Fichera in 1964. The solution of the Signorini's Problem coincides with the birth of the field of variational inequalities. As the generalization of variational inequalities, the theory of hemivariational inequalities was first introduced and studied by Panagiotopoulos in [18]. The mathematical theory of hemivariational inequalities has been of great interest recently which is due to the intensive development of applications of hemivariational inequalities in contact mechanics, control theory, games and so forth. Some comprehensive references are [2,3,9,12,16].

The theory of variational inequalities has begun with the results of Stampacchia [22]. This theory has shown to be very useful in the study of some problems arising in mechanics, optimization,

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Received: April 2021 *Accepted:* August,2021

transportation, economics equilibrium, contact problems in elasticity, and some other problems of practical interest.

The variational inequality $VI(A,K)$ considered by Stampacchia in [22] was concerned by finding an element u of K such that for each $v \in K$,

$$\langle A(u), v-u\rangle \geq 0.$$

In recent years, many generalizations of this problem have been considered, studied and applied to various fields. See for example [8], [9], [13] and for general variational inequalities were introduced.

Recently, Laszlo [14], motivated by the work of Noor [17], studied the so-called *general variational inequality of Stampacchia type*, $VI(A,a,K)$. This problem is to find an element u of K for which the inequality

$$\langle A(u), a(v)-a(u)\rangle \geq 0 \tag{1.1}$$

holds for any $v \in K$, where $a : K \longrightarrow X$ is a given map with some properties.

Harandi and Laszlo [11] extended (1.1) by replacing $a(v)-a(u)$ with an operator $\psi(u,v)$. They extended the variational inequality problem (1.1) to $VI(A,\psi,K)$, which was introduced in [12]. Their problem is to find $u \in K$ such that

$$\langle A(u), \psi(u,v)\rangle \geq 0$$

for every $v \in K$, where $\psi(u,v) : K \times K \longrightarrow X$ is given map with suitable properties.
Harandi and Lazlo [11] introduced a new class of operators of type $g-ql$, that contains in particular the set of operators of type ql and they extend some results already established in [14] for general variational inequalities of Stampacchia type involving operators of type ql.

Chadli and Gwinner [4] found the existence of a solution for semi-coercive variational inequalities $VI(A,\varphi,g,K)$ in a reflexive Banach space. They studied the existence of $u \in K$ for which

$$\langle A(u), v-u\rangle + \varphi(u,v) \geq \langle g, v-u\rangle \tag{1.2}$$

holds for any $v \in K$, where $A : X \longrightarrow X^*$ is a semi-coercive map, $\varphi : K \times K \longrightarrow \mathbb{R}$ is a pseudo-monotone function such that $\varphi(u,.)$ is convex and $\varphi(.,v)$ is upper semi-continuous and $g \in X^*$. They used theorem of Browder [2]. Chadli and Gwinner [4] for problem (1.2) used theorem 3.9 of Gwinner thesis and as the fact that the sum of a pseudomonotone bifunction and a monotone bifunction is pseudomonotone showed if there is a constant $R > 0$ in which

$$-\langle A(v), v\rangle + \varphi(v,0) + \langle g, v\rangle < 0 \quad \forall v \in K \quad with \quad \| v \| = R,$$

then (1.2) has a solution.
This paper deals with an extension of problem (1.2), namely $VI(A,\psi,\varphi,g,K)$:
Find an element u of K such that for every $v \in K$,

$$\langle A(u), \psi(u,v)\rangle + \varphi(u,v) \geq \langle g, \psi(u,v)\rangle. \tag{1.3}$$

We will use *"finite intersection property"* to prove that problem (1.3) has a solution. More precisely, we use *GKKM mappings* and the *finite intersection theorem* to solve (1.3). We consider the class of all operators of type $g-ql$, which contains in particular the set of operators of type ql, which in turn provide an extension of linear and monotone operators used in problem (1.3).

Theory of vector variational inequalities was proposed by Giannessi [10]. It is very useful in the study of some problems arising in pure and applied sciences, engineering and technology, financial mathematics, transportation, and other problems of practical interest, for example See [1,5,15,16,21].

General variational inequality problems provide us with a unified, simple, innovative, and natural framework for the study of a wide class of problems involving unilateral, moving boundary, obstacle, free boundary and equilibrium problems. Knowing about the existence of solutions for such problems is helpful, because one may have the chance to be sure that a solution exists, even before finding some plausible algorithms for the solution. While existence results concerning the solutions of Stampacchia variational inequalities were abundant in the last years, this is not the case for general variational inequalities of Stampacchia type [22].

In [20] Salahuddin and Verma considered general vector variational like inequality $GVVLI(A, \eta, K)$ to find $x \in K$ such that

$$\langle A(x), \eta(y, x)\rangle \notin -intC(x) \quad \forall y \in K,$$

where $\eta : K \times K \longrightarrow X$ is an operator. In a special case $\eta(y, x) = a(y) - a(x)$ where $a : K \longrightarrow X$ is an operator is called general vector varitional inequality. In [20], they used the KKM theorem to find the solution. Xiao and hung established the well-posedness of the hemi-variational inequality $HVI(A, g, j)$ in [18].

In this research, using some results of [4], [11], [14] and [18] together with what we will obtain in Section 3, we show that problem (1.3) has a solution.

In Section 4, we find the well-posedness of the variational inequality. In Section 5, we show that problem (1.3) for the inclusion case has a solution. We extend the recent works [5, 7, 20] by using the class of $g - ql$ type operators. The problem that we shall study in Section 4 is the so-called *general vector variational-like inequalities* ($GVVLI(A, \psi, \varphi, K)$) concerning an element $x \in K$ such that

$$\langle A(x), \psi(y, x)\rangle + \varphi(x, y) \notin -intH \tag{1.4}$$

for all $y \in K$, where H is a convex cone in Y.

2. Preliminaries

Let X and Y be real linear spaces, X^* denote the topological dual of X, and $K \subseteq X$ be a non-void closed convex set. We are concerned with a monotone operator $A : X \longrightarrow X^*$ which is continuous from finite-dimensional subspaces of X to X^* endowed with the weak topology. We do not assume that A is coercive; instead, we assume that A is *semi-coercive* in the following sense:

There exists an increasing function $G : [0, \infty) \longrightarrow [0, \infty)$ satisfying $G(0) = 0$ and $G(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, such that for all $v, w \in X$

$$\langle A(v) - A(w), v - w\rangle \geq |v - w|G(|v - w|),$$

where $|.|$ is a lower semi-continuous semi-norm on X. See [2].

The operator $A : X \longrightarrow X^*$ is said to be *monotone* if $\langle A(v) - A(w), v - w\rangle \geq 0$ for all $v, w \in X$ and for a function $\psi : X \times X \longrightarrow X$, A is called $\psi-monotone$ whenever $\langle A(v) - A(w), \psi(w, v)\rangle \geq 0$ for all $v, w \in X$.

Definition 2.1. *A mapping $A : X \longrightarrow X^*$ is said to be hemi-continuous if for any $v, w \in X$, the function $t \mapsto \langle A(v + t(w - v)), v - w\rangle$ defined on $[0, 1]$ is continuous at 0^+.*

Let $A : K \subset X \longrightarrow L(X, Y)$ be a given operator, where $L(X, Y)$ is the vector space of all continuous linear mappings from X into Y. Suppose that $H \in 2^Y$ is a closed convex cone with non-empty interior such that

(i) $\lambda H \subset H$ for $\lambda > 0$,
(ii) $H + H \subset H$, and
(iii) $H \cap (-H) = \{0\}$.

We define a partial ordering by writing $x <_{intH} y$ if and only if $y - x \in intH$.

For $x, y \in X$, we set $[x, y] := \{(1-t)x + ty : t \in [0,1]\}$, which is *closed line segment* with end points x and y. An operator $B : D \subseteq X \longrightarrow Y$ is said to be of *type ql* if

$$B([x, y] \cap D) \subseteq [B(x), B(y)]$$

for all $x, y \in D$. See [12].

Proposition 2.2. *([14]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is of type ql if and only if it is monotone.*

Proposition 2.3. *([12]) Any linear operator $B : X \longrightarrow Y$ is of type ql.*

Before stating the following proposition, we first introduce the concept of convex hull. A set of points in a Euclidean space is defined to be convex if it contains the line segments connecting each pair of its points. The convex hull of a given set X be defined as The set of all convex combinations of points in X.

Proposition 2.4. *([14]) Let $D \subseteq X$ be a convex set, and $B : D \longrightarrow Y$ be an operator of type ql. Then, for every $n \in \mathbb{N}$ and any $x_1, ..., x_n \in D$,*

$$B(co\{x_1, ..., x_n\}) \subseteq co\{B(x_1), ..., B(x_n)\},$$

where $co(V)$ denotes the convex hull of the set $V \subseteq Y$.

Definition 2.5. *Let $D \subseteq X$ be a convex set. An operator $B : D \longrightarrow Y$ is said to be of type $g - ql$ if for every $n \in \mathbb{N}$ and any $x_1, x_2, ..., x_n \in D$, there exist $y_1, y_2, ..., y_n \in D$, which are not necessarily distinct, such that for any subset $\{y_{i_1}, ..., y_{i_k}\}$ of $\{y_1, ..., y_n\}$, $1 \leq k \leq n$,*

$$B(co\{y_{i_1}, ..., y_{i_k}\}) \subseteq co\{B(x_{i_1}), ..., B(x_{i_k})\}.$$

According to Proposition (2.4), any operator of type ql is of type $g - ql$.

Definition 2.6. *Let $D \subseteq X$ be a convex set. An operator $\psi : D \times D \longrightarrow Y$ is said to be of type $g-ql$ if for every $n \in \mathbb{N}$ and any $x_1, x_2, ..., x_n \in D$ there exist $y_1, y_2, ..., y_n \in D$ which are not necessarily distinct such that for any subset $\{y_{i_1}, ..., y_{i_k}\}$ of $\{y_1, ..., y_n\}$, $1 \leq k \leq n$, and any $y \in co\{y_{i_1}, ..., y_{i_k}\}$,*

$$0 \in co\{\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})\}.$$

Given any $x \in K$

$$A^+(x) := \{y \in X : \langle A(x), y\rangle \geq 0\}$$

and

$$A^+_\varphi(x) := \{y \in X : \langle A(x), y\rangle + \varphi(x, y) \geq 0\}.$$

It is obvious that $A^+(x) \subseteq A^+_\varphi(x)$ whenever $\varphi(x, y) \geq 0$ on $K \times K$.

Definition 2.7. *([11]) Let $D \subseteq X$ be a convex set and let $A : X \longrightarrow X^*$ be an operator. An operator $\psi : D \times D \longrightarrow X$ is said to be of type $g - ql$ with respect to A if for every $n \in \mathbb{N}$ and any $x_1, x_2, ..., x_n \in D$ there exist $y_1, y_2, ..., y_n \in D$, which are not necessarily distinct such that for any subset $\{y_{i_1}, ..., y_{i_k}\}$ of $\{y_1, ..., y_n\}$, $1 \leq k \leq n$ and for every $y \in co\{y_{i_1}, ..., y_{i_k}\}$,*

$$co\{\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})\} \cap A^+(y) \neq \emptyset.$$

Obviously, if $\varphi(., y) \geq 0$, since $0 \in A^+(y)$ for each $y \in D$ so any operator $\psi : D \times D \longrightarrow X$ of type $g - ql$ is of type $g - ql$ with respect to A.

Theorem 2.8. *([11]) Let $I \subseteq \mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a function. If there exists an interval $\tilde{I} \subseteq I$ such that $f(\tilde{I}) = f(I)$ and the restriction of $f_{|\tilde{I}}$ is monotone then f is of type g-ql.*

From Proposition 2.4 we obtain immediately, that every operator of type ql is of type $g - ql$. In the following example, using Theorem 2.8 and Proposition 2.2, show that an operator may be of type $g - ql$ but not of type ql.

Example 2.9. *Let $I = \mathbb{R}$ and consider the function $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ and*

$$g(x) = \begin{cases} 1 & if\ x = 0, \\ \\ 0 & if\ x \neq 0. \end{cases}$$

According to Proposition 2.2 f and g are not of the type ql, but by selecting $\tilde{I} = [0, \infty]$ and apply theorem 2.8, we see that f and g are of type $g - ql$.

Lemma 2.10. *([11]) Let X and Y be two real linear spaces, $D \subseteq X$ be a convex set and let $B : D \longrightarrow Y$ be of type $g - ql$. Then the operator $\psi : D \times D \longrightarrow Y$ which is defined by $\psi(x, y) = B(y) - B(x)$ is of type $g - ql$.*

Definition 2.11. *An operator $h : X \longrightarrow X$ is an affine operator if for erery $\{\lambda_i\}_{i\in I}$ such that $\Sigma_{i\in I}\lambda_i = 1$, we have*

$$h(\Sigma_{i\in I}\lambda_i a_i) = \Sigma_{i\in I}\lambda_i h(a_i).$$

3. On the existence of solutions for variational inequalities

In this section, we establish the existence of solutions for the extended general variational inequalities. We need the following definition of a generalized KKM mapping, and the finite intersection property theorem.

Definition 3.1. *([1]) Let E be a linear space and consider some $Y \subseteq E$. Also, suppose that X is an arbitrary non-empty set. A set-valued map $G : X \longrightarrow Y$ is called a $GKKM$ mapping if for each finite subset $\{x_1, ..., x_n\}$ of X, there exists $\{y_1, ..., y_n\} \subseteq Y$ such that for any subset $\{y_{i_1}, ..., y_{i_k}\}$ of $\{y_1, ..., y_n\}$, $1 \leq k \leq n$,*

$$co\{y_{i_1}, ..., y_{i_k}\} \subseteq \bigcup_{1\leq j\leq k} G(x_{i_j}).$$

Theorem 3.2. [25] *Let X be an arbitrary non-empty set, and Y be a closed non-empty subset of a topological linear space E. Suppose that $G : X \longrightarrow Y$ is a $GKKM$ mapping which is non-empty and closed-valued, the family $\{G(x) : x \in X\}$ has the finite intersection property, that is, $\bigcap_{x\in S} G(x) \neq \emptyset$ for every finite subset S of X. Moreover, if there exists a finite subset S of X such that $\bigcap_{x\in S} G(x)$ is compact, then*

$$\bigcap_{x\in X} G(x) \neq \emptyset.$$

Lemma 3.3. ([11]) *For a bounded net $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$, assume that one of the following holds.*

(a) The net $\{x_i\}_{i\in I}$ converges to x in the weak topology of X and $\{x_i^*\}$ converges to x^* in the norm topology of X^*.

(b) The net $\{x_i\}_{i\in I}$ converges to x in the norm topology of X and $\{x_i^*\}$ converges to x^* in the weak* topology of X^*.

Then, $\langle x_i^*, x_i\rangle \longrightarrow \langle x^*, x\rangle$.

For $A : X \longrightarrow X^*$, $\psi : K \times K \longrightarrow X$ and $\varphi : K \times K \longrightarrow \mathbb{R}$ which are given operators, where $K \subseteq X$ is a non-void convex set such that $0 \in K$, and $g \in X^*$. If in (1.3), $B(u) := A(u) - g$, then problem (1.3) reduces to $VI(B, \psi, \varphi, K)$:

$$\langle B(u), \psi(u, v)\rangle + \varphi(u, v) \geq 0,\ \forall v \in K. \tag{3.1}$$

Theorem 3.4. *Let K be a weakly compact, convex set, and ψ be of type $g - ql$ with respect to B. Suppose that φ is a bifunction such that*

i) $\varphi(., y)$ is weakly upper semi-continuous,

ii) $\varphi(x, y)$ is positive on $K \times K$,

iii) $\varphi(x, y) \geq \varphi(x, \psi(x, y))$ for $x, y \in K$ and

iv) one of the following two conditions holds.

(a) The operator A is weak-to-norm sequentially continuous and $\psi(., y)$ is weak-to-weak sequentially continuous for each $y \in K$.

(b) The operator A is weak-to-weak sequentially continuous and $\psi(., y)$ is weak-to-norm sequentially continuous for each $y \in K$.

Then, $VI(B, \psi, \varphi, K)$ has a solution.

Proof . Define $G : K \longrightarrow K$ by

$$G(y) := \{x \in K :\ \langle B(x), \psi(x, y)\rangle + \varphi(x, y) \geq 0\},$$

for every $y \in K$. It is easy to see that the existence of the solution of $VI(B, \psi, \varphi, K)$ is equivalent to

$$\bigcap_{y\in X} G(y) \neq \emptyset.$$

Therefore it is enough to show that the assumptions of Theorem 3.2 valid. Using the fact that ψ is of type $g - ql$ with respect to B, we show that $G(y) \neq \emptyset$. Otherwise there exists $y \in K$ such that for any $x \in K$, $\ x \notin G(y)$ that is

$$\langle B(x), \psi(x,y)\rangle + \varphi(x,y) < 0. \tag{3.2}$$

According to the Definition 2.6, for every $n \in \mathbb{N}$ and for arbitrary, (not necessarily distinct) elements $y_1, ..., y_n$ of K, there exist $x_1, ..., x_n \in K$ such that for any $\{x_{i_1}, ..., x_{i_k}\} \subseteq \{x_1, ..., x_n\}$, $1 \leq k \leq n$ and $x \in co\{x_{i_1}, ..., x_{i_k}\}$,

$$co\{\psi(x, y_{i_1}), ..., \psi(x, y_{i_k})\} \cap B^+(x) \neq \emptyset.$$

Since $B^+(x) \subseteq B^+_\varphi(x)$ and $y_1, ..., y_n \in K$, assuming $y_i = y$, there exists $z \in co\{\psi(x,y), ..., \psi(x,y)\} \cap B^+_\varphi(x)$ for which

$$\langle B(x), z\rangle + \varphi(x,z) \geq 0.$$

Equivalently,

$$\langle B(x), \psi(x,y)\rangle + \varphi(x, \psi(x,y)) \geq 0.$$

But, by the assumption $\varphi(x,y) \geq \varphi(x, \psi(x,y))$,

$$\langle B(x), \psi(x,y)\rangle + \varphi(x,y) \geq 0, \tag{3.3}$$

which is a contradiction, So $G(y) \neq \emptyset$.

Now, we show that for each $y \in K$, $G(y)$ is weakly compact. Since K is weakly compact, it is enough to show that $G(y)$ is weakly closed. To show this, for given $y \in K$ if $z \in K$ an weak accumulation point of $G(y)$, so there is a sequence $\{x_k\} \subseteq G(y)$ converging weakly to x.

Since $\varphi(.,y)$ is weakly upper semi-continuous, $\limsup_{k\longrightarrow\infty} \varphi(x_k, y) \leq \varphi(x,y)$.

Assuming (a) and using Lemma 3.3 (a) and the fact that A and so B are weak-to-norm sequentially continuous,

$$\lim_{k\longrightarrow\infty} \langle B(x_k), \psi(x_k,y)\rangle = \langle B(x), \psi(x,y)\rangle.$$

Therefore,

$$0 \leq \limsup_{k\longrightarrow\infty}(\langle B(x_k), \psi(x_k,y)\rangle + \varphi(x_k,y)) \leq \langle B(x), \psi(x,y)\rangle + \varphi(x,y).$$

Thus $x \in G(y)$ and $G(y)$ is weakly sequentially closed.

Case (b) is similar to that one of (a).

To use Theorem 3.2, it is sufficient to show that G is $GKKM$. Otherwise, $\{x_1, ..., x_n\} \subseteq X$ can be found such that for any $\{y_1, ..., y_n\} \subseteq Y$, there is a subseqence $\{y_{i_{i_1}}, ..., y_{i_k}\} \subseteq \{y_1, ..., y_n\}$, $1 \leq k \leq n$, for which

$$co\{y_{i_1}, ..., y_{i_k}\} \subseteq \bigcup_{1\leq j\leq k} G(x_{i_j}).$$

If there exists $y \in co\{y_{i_{i_1}}, ..., y_{i_k}\}$ such that $y \notin \bigcup_{1\leq j\leq k} G(x_{i_j})$, then $y \notin G(x_{i_j})$
Therefore

$$\langle B(y), \psi(y, x_{i_j})\rangle + \varphi(y, x_{i_j}) < 0, \tag{3.4}$$

for every $1 \leq j \leq k$. Since ψ is of type $g - ql$ with respect to B,

$$\exists z \in co(\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})) \cap \ B^+(y).$$

Moreover, $B^+(y) \subseteq B^+_\varphi(y)$. So,

$$z \in co(\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})) \cap \ B^+_\varphi(y).$$

But there is a presentation $z = \sum_{1\le j\le k} \lambda_{i_j}\psi(y, x_{i_j})$, where $\sum_{1\le j\le k} \lambda_{i_j} = 1$. Thus

$$\langle B(y), z\rangle + \varphi(y, z) \ge 0,$$

or

$$\langle B(y), \sum_{1\le j\le k} \lambda_{i_j}\psi(y, x_{i_j})\rangle + \varphi(y, \sum_{1\le j\le k} \lambda_{i_j}\psi(y, x_{i_j})) \ge 0.$$

Since $\varphi(y, .)$ is convex and the inner product is bilinear,

$$\sum_{1\le j\le k} \lambda_{i_j}\langle B(y), \psi(y, x_{i_j})\rangle + \sum_{1\le j\le k} \lambda_{i_j}\varphi(y, \psi(y, x_{i_j})) \ge 0.$$

From the assumption $\varphi(x, y) \ge \varphi(x, \psi(x, y))$,

$$\sum_{1\le j\le k} \lambda_{i_j}\langle B(y), \psi(y, x_{i_j})\rangle + \sum_{1\le j\le k} \lambda_{i_j}\varphi(y, x_{i_j}) \ge 0,$$

which contradicts (3.4). Therefore, G is a $GKKM$ mapping. Finally, Theorem 3.2 allows us to conclude that problem $VI(B, \psi, \varphi, K)$ has a solution. □

Remark 3.5. *Instead of assuming that ψ is of type $g - ql$ with respect to B, one can assume that ψ is of type $g - ql$ with respect to A and that g is negative on K.*

Example 3.6. *Let function $\psi : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ by $\psi(u, v) = u^3 - v^3$. Consider it holds in the case of the Theorem* 3.4.
According to Propositions 2.2 *and* 2.4*, since the two function u^3, v^3 are monotone, they are type ql and as a result are type $g - ql$. According to Lemma* 2.10*, operator ψ is type $g - ql$, also based on Definition* 2.7 *and being positive ϕ, the condition that ψ is type $g - ql$ with to $B(u)$ is that $0 \in B^+(u)$. Therefore for any function $B(u)$ that is $0 \in B^+(u)$, the operator $\psi(u, v) = u^3 - v^3$ is type $g - ql$ with to $B(u)$.*
Obviously u^3 is continuous, so ψ is continuous in $\mathbb{R}$ and valid in condition iv) of Theorem 3.4.

4. The well-posedness of $VI(A, \psi, \varphi, g)$

In this section we provide some conditions that ensure the strong (weak) well-posedness of the variational inequality.

Given any $\epsilon > 0$, we define the following set.

$$\Omega(\epsilon) = \{u \in X : \langle Au - g, \psi(u, v)\rangle + \varphi(u, v) \ge -\epsilon\|\psi(u, v)\|_X, \quad \forall v \in X\}.$$

Definition 4.1. *A sequence $\{u_n\} \subset V$ is said to be an approximating sequence for $VI(A, \psi, \varphi, g)$ if there exists a non-negative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \longrightarrow \infty$ such that*

$$\langle Au_n - g, \psi(u_n, v)\rangle + \varphi(u_n, v) \ge -\epsilon_n\|\psi(u_n, v)\|_X \qquad v \in X.$$

Definition 4.2. *We say that $VI(A,\psi,\varphi,g)$ is strongly (respectively, weakly) well-posed if it has a unique solution in X and every approximating sequence converges strongly (respectively, weakly) to the unique solution.*

It is clear that strong well-posedness implies weak well-posedness; but its converse is not true in general (See [22,23]).

Theorem 4.3. *Suppose that $A : X \longrightarrow X^*$ is a $\psi-$monotone and hemi-continuous mapping. Furthermore, assume that*

i) $\psi(u,u) = 0$,

ii) $\psi(.,v)$ *is norm-to-weak sequentially continuous,* $\psi(u,.)$ *is an affine mapping,*

iii) $\varphi(u,.)$ *is a convex function,* $\varphi(.,v)$ *is upper semi-continuous and* $\varphi(u,u) = 0$.

Then, $VI(A,\psi,\varphi,g)$ is strongly well-posed if and only if

$$\Omega(\epsilon) \neq \emptyset \textit{ for every } \epsilon > 0 \quad \textit{and} \quad diam(\Omega(\epsilon)) \longrightarrow 0 \quad \textit{as} \quad \epsilon \longrightarrow 0. \tag{4.1}$$

Proof . Suppose that $VI(A,\psi,\varphi,g)$ is strongly well-posed. Then, $VI(A,\psi,\varphi,g)$ has a solution which belongs to $\Omega(\epsilon)$ and so $\Omega(\epsilon) \neq \emptyset$ for every $\epsilon > 0$.

If $\lim_{\epsilon\longrightarrow 0} diam(\Omega(\epsilon)) \neq 0$, then there exist a constant $l > 0$, a non-negative sequence $\{\epsilon_n\}$ in which $\epsilon_n \longrightarrow 0$ and $u_n, v_n \in \Omega(\epsilon_n)$ such that

$$\| u_n - v_n\|_X > l, \ \forall n \in \mathbb{N}. \tag{4.2}$$

Then, $\{u_n\}$ and $\{v_n\}$ are approximating sequences for $VI(A,\psi,\varphi,g)$. Since $VI(A,\psi,\varphi,g)$ is well-posed, both $\{u_n\}$ and $\{v_n\}$ converge strongly to the unique solution of $VI(A,\psi,\varphi,g)$ which it contradicts (4.2).

Let $\{u_n\} \subset X$ be an approximating sequence for $VI(A,\psi,\varphi,g)$. Then, there exists a non-negative sequence $\{\epsilon_n\}$ with $\epsilon_n \longrightarrow 0$ such that

$$\langle Au_n - g, \psi(u_n,v)\rangle + \varphi(u_n,v) \geq -\epsilon_n\|\psi(u_n,v)\|_X, \quad \forall v \in X, n \in \mathbb{N}. \tag{4.3}$$

Therefore, $u_n \in \Omega(\epsilon_n)$. Since $\lim_{\epsilon\longrightarrow 0} diam(\Omega(\epsilon)) = 0$, so $\{u_n\}$ is a Cauchy sequence. But X is a Banach space. Therefore, (u_n) converges strongly to some point $u \in X$. Since $\varphi(.,v)$ is upper semi-continuous, A is $\psi-$monotone and $\psi(.,v)$ is norm-to-weak continuous. By (4.3),

$$\begin{aligned}
\langle Av - g, \psi(u,v)\rangle + \varphi(u,v) &\geq \limsup \langle Av - g, \psi(u_n,v)\rangle + \varphi(u_n,v)\\
&\geq \limsup \langle Au_n - g, \psi(u_n,v)\rangle + \varphi(u_n,v)\\
&\geq \limsup -\epsilon_n\|\psi(u_n,v)\|_X\\
&= 0,
\end{aligned}$$

for every $v \in X$. Given $w \in X$ and $t \in [0,1]$, if $v := tw + (1-t)u$, then for each $v \in X$,

$$\langle Av - g, \psi(u,v)\rangle + \varphi(u,v) \geq 0. \tag{4.4}$$

Since $\psi(u,.)$ is affine, $\varphi(u,.)$ is convex and $\psi(u,u) = 0 = \varphi(u,u)$, we deduce from (4.4) that

$$\begin{aligned}
0 &\leq \langle Av - g, \psi(u, t(w-u)+u)\rangle + \varphi(u, t(w-u)+u)\\
0 &\leq t\langle Av - g, \psi(u,w)\rangle + t\varphi(u,w)\\
0 &\leq \langle A(tw + (1-t)u) - g, \psi(u,w)\rangle + \varphi(u,w).
\end{aligned}$$

But A is hemi-continuous, So if $t \longrightarrow 0^+$ for any $w \in X$,

$$\langle Au - g, \psi(u, w)\rangle + \varphi(u, w) \geq 0.$$

To prove the uniqueness, assume that there are two distinct solutions u_1 and u_2 for $VI(A, \psi, \varphi, g)$. Then, $u_1, u_2 \in \Omega(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \| u_1 - u_2 \| \leq diam(\Omega(\epsilon)), \ \forall \epsilon > 0. \tag{4.5}$$

Letting $\epsilon \longrightarrow 0$, lim $diam(\Omega(\epsilon)) = 0$, which is a contradiction. Therefore, $VI(A, \psi, \varphi, g)$ has a unique solution. □

Example 4.4. *Consider*

$$\psi(u, v) = \| u \| (v - u).$$

Then $\psi(u, u) = 0$. *Moreover, for any* $\{u_n\} \subseteq X$, *in which* $u_n \longrightarrow u$ *then* $u_n \rightharpoonup u$ *meaning that for all* $y^* \in X^*$, $y^* u_n \longrightarrow y^* u$ *and* $y^*(v - u_n) \longrightarrow y^*(v - u)$. $\| u_n \| \rightarrow \| u \|$. *So* $\psi(., v)$ *is norm-to-weak sequentially continuous.*
On the other hand, suppose that $\sum_1^\infty \lambda_i = 1$ *and* $(v_i) \subseteq X$. *Then*

$$\psi(u, \Sigma_i \lambda_i v_i) = \| u \| (\sum_i \lambda_i (v_i - u)) = \Sigma_i \lambda_i \psi(u, v_i).$$

5. On the existence of solutions for general vector variational inequalities

In this section we extend Definition (2.5) in Banach spaces with respect to given operator A. For $x \in K$

$$A^+(x) := \{y \in X : \langle A(x), y\rangle \notin -intH\}$$

and

$$A^+_\varphi(x) := \{y \in X : \langle A(x), y\rangle + \varphi(x, y) \notin -intH\}.$$

Definition 5.1. *Let* $D \subseteq X$ *be a convex set, and* Y *be a Banach space. Moreover, assume that* $A : X \longrightarrow L(X, Y)$ *is an operator. An operator* $\psi : D \times D \longrightarrow X$ *is said to be of type* $g - ql$ *with respect to* A *if for every* $n \in \mathbb{N}$ *and any* $x_1, x_2, ..., x_n \in D$, *there exist* $y_1, y_2, ..., y_n \in D$, *which are not necessarily distinct, such that for any subset* $\{y_{i_1}, ..., y_{i_k}\}$ *of* $\{y_1, ...y_n\}$, $1 \leq k \leq n$, *and for every* $y \in co\{y_{i_1}, ..., y_{i_k}\}$,

$$co\{\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})\} \cap A^+(y) \neq \emptyset.$$

Theorem 5.2. *([20]) Let* Y *be a topological vector space with a closed, convex and pointed cone* H *such that* $intH \neq \emptyset$. *Then, the following hold for any* $x, y, z \in Y$.

i) If $x - y \in intH$ *and* $x \notin -intH$, *then* $y \notin -intH$.
ii) If $x + y \in -H$ *and* $x + z \notin -intH$, *then* $z - y \notin -intH$.

Obviously, it follows from Theorem (5.2 i) that if $-\varphi(x, y) \notin -intH$ on $K \times K$, then $A^+(x) \subseteq A^+_\varphi(x)$.

Theorem 5.3. *Let* $K \subseteq X$ *be a weakly compact and convex set and* ψ *be of type* $g - ql$ *with respect to A. Suppose that* φ *is a bifunction such that*

i) $\varphi(.,y)$ *is continuous,*
ii) $-\varphi(x,y) \in -intH$ *for any* $x, y \in K$*,*
iii) $-\varphi(x,y) + \varphi(x,\psi(x,y)) \in -intH$*,*
iv) *one of the following two conditions is satisfied.*
  (a) The operator A *is weak-to-norm sequentially continuous, and* $\psi(.,y)$ *is weak-to-weak sequentially continuous for each* $y \in K$*.*
  (b) The operator A *is weak-to-weak sequentially continuous and* $\psi(.,y)$ *is weak-to-norm sequentially continuous for each* $y \in K$*.*

Then, $GVVLI(A,\psi,\varphi,K)$ *has a solution.*

Proof . Define the set-valued mapping

$$G(y) := \{x \in K : \langle A(x), \psi(x,y)\rangle + \varphi(x,y) \notin -intH\},$$

for all $y \in K$.
It is easy to see that the existence of the solution of $GVVLI(A,\psi,\varphi,K)$ is equivalent to $\bigcap_{y\in X} G(y) \neq \emptyset$.

We show that the assumptions of Theorem 3.2 valid. We claim $G(y) \neq \emptyset$ for any $y \in K$. Otherwise, there exists $y \in K$ such that for all $x \in K$, $x \notin G(y)$ that is,

$$\langle A(x), \psi(x,y)\rangle + \varphi(x,y) \in -intH. \tag{5.1}$$

Since ψ is of type $g-ql$ with respect to A, by Definition 2.6, for every $n \in \mathbb{N}$ and any $y_1,...,y_n \in K$, there exist $x_1,...,x_n \in K$ such that for every $\{x_{i_1},...,x_{i_k}\} \subseteq \{x_1,...,x_n\}$, $1 \leq k \leq n$ and $x \in co\{x_{i_1},...,x_{i_k}\}$,

$$co\{\psi(x,y_{i_1}),...,\psi(x,y_{i_k})\} \cap A^+(x) \neq \emptyset.$$

Since $A^+(x) \subseteq A^+_\varphi(x)$, and $y_1,...,y_n \in K$ are not necessarily distinct, we may let $y_i = y$. Therefore, there exists $z \in co\{\psi(x,y),...,\psi(x,y)\} \cap A^+_\varphi(x) = \{\psi(x,y)\}$ such that

$$\langle A(x), z\rangle + \varphi(x,z) \notin -intH,$$

or equivalently,

$$\langle A(x), \psi(x,y)\rangle + \varphi(x,\psi(x,y)) \notin -intH.$$

From the assumption $-\varphi(x,y) + \varphi(x,\psi(x,y)) \in intH$, so

$$\langle A(x), \psi(x,y)\rangle + \varphi(x,y) \notin -intH. \tag{5.2}$$

By (5.1), this is a contradiction. So $G(y) \neq \emptyset$.

Now, we show that for each $y \in K$ $G(y)$ is weakly compact. Since K is weakly compact, it is enough to show that $G(y)$ is weakly sequentially closed. To show this, for $y \in K$ consider a sequence $\{x_k\} \subseteq G(y)$ that converges weakly to some $x \in K$.

Since $\varphi(.,y)$ is a continuous function, $\lim_{k\longrightarrow\infty} \varphi(x_k,y) = \varphi(x,y)$.

Assuming (a), from Lemma 3.3 (a) and the fact that A is weak-to-norm sequentially continuous we deduce that

$$\lim_{k\longrightarrow\infty} \langle A(x_k), \psi(x_k,y)\rangle = \langle A(x), \psi(x,y)\rangle.$$

Therefore,

$$\langle A(x), \psi(x,y)\rangle + \varphi(x,y) = \lim_{k\longrightarrow\infty}(\langle A(x_k), \psi(x_k,y)\rangle + \varphi(x_k,y)) \notin -intH.$$

Thus $x \in G(y)$ and so, $G(y)$ is weakly sequentially closed.

If (b) is the case, then a reasoning similar to that for (a) works.

To use Theorem 3.2, it is enough to show that G is a $GKKM$ mapping. Note that otherwise, there exist $\{x_1, ..., x_n\} \subseteq X$ such that for any $\{y_1, ..., y_n\} \subseteq Y$, there exists a subset $\{y_{i_{i_1}}, ..., y_{i_k}\}$ of $\{y_1, ..., y_n\}$, $1 \leq k \leq n$, for which

$$co\{y_{i_1}, ..., y_{i_k}\} \subseteq \bigcup_{1\leq j\leq k} G(x_{i_j}).$$

If there exists $y \in co\{y_{i_{i_1}}, ..., y_{i_k}\}$ such that $y \notin \bigcup_{1\leq j\leq k} G(x_{i_j})$, then
$y \notin G(x_{i_j})$
and so

$$\langle A(y), \psi(y, x_{i_j})\rangle + \varphi(y, x_{i_j}) \in -intH, \tag{5.3}$$

for all $1 \leq j \leq k$. Since ψ is of type $g - ql$ with respect to A,

$$\exists z \in co(\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})) \cap \ A^+(y).$$

Moreover, $A^+(y) \subseteq A^+_\varphi(y)$ and so

$$z \in co(\psi(y, x_{i_1}), ..., \psi(y, x_{i_k})) \cap \ A^+_\varphi(y).$$

But there is representation $z = \sum_{1\leq j\leq k} \lambda_{i_j}\psi(y, x_{i_j})$, where $\sum_{1\leq j\leq k} \lambda_{i_j} = 1$. Hence,

$$\langle A(y), z\rangle + \varphi(y, z) \notin -intH$$

or

$$\langle A(y), \sum_{1\leq j\leq k} \lambda_{i_j}\psi(y, x_{i_j})\rangle + \varphi(y, \sum_{1\leq j\leq k} \lambda_{i_j}\psi(y, x_{i_j})) \notin -intH. \tag{5.4}$$

Since $\varphi(y, .)$ is convex and the inner product is bilinear,

$$\sum_{1\leq j\leq k} \lambda_{i_j}\langle B(y), \psi(y, x_{i_j})\rangle + \sum_{1\leq j\leq k} \lambda_{i_j}\varphi(y, \psi(y, x_{i_j})) \notin -intH.$$

By the assumption, $-\varphi(x, y) + \varphi(x, \psi(x, y)) \in intH$. Therefore,

$$\sum_{1\leq j\leq k} \lambda_{i_j}[\langle B(y), \psi(y, x_{i_j})\rangle + \varphi(y, x_{i_j})] \notin -intH.$$

Definition of a cone implies that sum of finitely many elements of $-intH$ lies in $-intH$ which is a contradiction. Therefore, G is a $GKKM$ map. Finally, Theorem 3.2 allows us to conclude that the problem $GVVLI(A, \psi, \varphi, K)$ has a solution. □

Acknowledgment

The authors would like to thank the referees for their careful reading of the manuscript and the helpful comments.

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