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A study on Langevin equation with three different fractional orders

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Abstract

Using a novel norm that is comfy for fractional and singular differential equations the existence and uniqueness of IVP for new type nonlinear Langevin equations involving three fractional orders are discussed. This norm is a tool to measure how far a numerical solution is from the exact one. New results are based on the contraction mapping principle. Lemma 2.2 has a prominent role in proving the main theorem. The fractional derivatives are described in Caputo sense. Two examples are presented to illustrate the theory.

Keywords: Fractional Langevin equation; Fixed point theorem; Existence results.
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1. Introduction

In the early 19-th century, the Langevin equation was formulated and presented by French scientist in physics "Paul Langevin" [1]. He exhibited a precise description of Brown's motion helping of the Langevin equation.

In fact, the use of the Langevin differential equation is one of the most commonly used methods of all description of the time evolution of the velocity of the Brownian motion [3, 2, 4].

Some of the substantial applications of this equation are analyzing the stock market [5], self-organization in complex systems [6], deuteron cluster dynamics [7], and anomalous transport [8].

One of the issues that can be lovable is the generalized Langevin equation (GLE)[9]. The motion of the particle is described by the generalized Langevin equation if the random fluctuation force is not white noise [10].

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A special case of the GLE is the fractional Langevin equation (FLE). In the late 19-th century, the Langevin-Qarray equation was introduced by Mainardi and colleagues [12, 13].

There exist many papers including a number of fractional derivatives have been considered by researchers in the fractional Langevin equation. Lutz [14] used the fractional Langevin equation involving one fractional derivative to survey fractional Brownian motion. Linear and nonlinear FLE, ?including two different fractional derivatives, have been investigated in [15, 17, 18, 19, 20, 21, 22, 23, 24].

In the mentioned articles, the existence and uniqueness of the solution for the linear and nonlinear of fractional Langevin equation are investigated. Banach contraction principle, Schauder fixed point, Krasnosel'skii fixed point and Leray-Schauder nonlinear alternative theorems are main tools to obtain results.

This work is concerned with the following IVP for nonlinear fractional Langevin equation:

$$\begin{cases} {}_0^c\mathcal{D}_s^{\mu}\big({}_0^c\mathcal{D}_s^{\nu} + \gamma\big)u(s) = \hbar\big(s,\ {}_0^c\mathcal{D}_s^{\theta}u(s)\big), & 0 < s < 1, \\ u^{(k)}(0) = a_k, & 0 \le k < l, \\ u^{(\nu+k)}(0) = b_k, & 0 \le k < n, \end{cases} \tag{1.1}$$

where $\gamma \in \mathbb{R}$, $m-1 < \nu \le m$, $n-1 < \mu \le n$, $0 < \theta \le 1$, $l = \max\{m,n\}$, $m,n \in \mathbb{N}$, ${}_0^c\mathcal{D}_s^{\mu}$, ${}_0^c\mathcal{D}_s^{\nu}$ and ${}_0^c\mathcal{D}_s^{\theta}$ are the Caputo fractional derivatives and $\hbar : [0,1] \to \mathbb{R}\times\mathbb{R}$ is a Lebesgue measurable function.

This article was motivated by the works of Baghani [15] and Tao Yu et.al [17]. In those papers fractional Langevin equation involving two fractional orders

$$\begin{cases} \mathcal{D}^{\beta}\big(\mathcal{D}^{\alpha} + \gamma\big)x(t) = f\big(t,\ x(t)\big), & 0 < t < 1, \\ x^{(k)}(0) = \mu_k, & 0 \le k < l, \\ x^{(\alpha+k)}(0) = \nu_k, & 0 \le k < n, \end{cases} \tag{1.2}$$

where $\gamma \in \mathbb{R}$, $m-1 < \alpha \le m$, $n-1 < \beta \le n$, $l = \max\{m,n\}$, $m,n \in \mathbb{N}$, $\mathcal{D}^{\alpha}$ is the Caputo fractional derivative and $f : [0,1] \to \mathbb{R}\times\mathbb{R}$ is a Lebesgue measurable function are studied while here fractional Langevin equation involving three fractional orders is the subject of discussion.

We use of a new norm that is comfy for singular and fractional differential equations. The Lemma 2.2 plays important role in proving the theorem 4.1.

The Lipschitz coefficient in [17], is assumed to be continuous, while there is the weaker condition of measurement and makes the novel results to coverage the case of one-sided Lipschitz continuity. We have stronger results than those obtained. Also, in [17] the condition $B > 0$ can be omitted since the equality $B = 0$ means that a_1 and ϕ are tantamount to 0. In addition, since the function ϕ is trivially bounded, it can be omitted in this inequality.

The contents of this paper as follows:

✓ Introduction.
✓ Preliminaries.
✓ $L_{p,\nu}$ spaces.
✓ Main results.
✓ Test examples.
✓ Conclusion.
✓ References.

2. Preliminaries

We tender some symbols, definitions, and lemmas that we need in continuance.

Definition 2.1. *[25, 26](chapter2) The Riemann-Liouville fractional integral of order $\alpha > 0$, of a function $u(s) \in C[0,b]$ is defined by*

$$\mathcal{I}^{\nu}u(s) = \int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)}u(\tau)d\tau, \quad n-1<\nu<n, \quad n\in\mathbb{N}.$$

Lemma 2.2. *[26](chapter3). Let $\nu \in \mathbb{R}^+$ and $\hbar : (0,a] \to \mathbb{C}$ be a function. If $\hbar \in \mathrm{L}^1(0,a)$ then $\mathcal{I}^{\nu}\hbar \in \mathrm{L}^1(0,a)$*

$$\|\mathcal{I}^{\nu}\hbar\|_1 \leq \frac{a^{\nu}}{\Gamma(\nu+1)}\|\hbar\|_1. \tag{2.1}$$

Definition 2.3. *[25, 26](chapter2). The Caputo fractional derivative of order $\nu > 0$, of a function $u(s) \in C^n[0,b]$, $n \in \mathbb{N}$, is defined by*

$${}_0^c\mathcal{D}_s^{\nu}u(s) = \mathcal{I}^{n-\nu}u^{(n)}(s) = \int_0^s \frac{(s-\tau)^{n-\nu-1}}{\Gamma(n-\nu)}u^{(n)}(\tau)d\tau, \quad n-1<\nu<n, \quad n\in\mathbb{N}.$$

Theorem 2.4. *[27](chapter2). Let $f \in C^m[0,1]$ and $\alpha \in (m-1,m), m \in \mathbb{N}$ and $g \in C^1[0,1]$. Then for $s \in [0,1]$,*

i) ${}_0^c\mathcal{D}_s^{\nu}\ \mathcal{I}^{\nu}g(s) = g(s)$;

ii) $\mathcal{I}^{\nu}\ {}_0^c\mathcal{D}_s^{\nu}f(s) = f(s) - \sum\limits_{k=0}^{m-1}\frac{s^k}{k!}f^{(k)}(0)$;

iii) $\lim\limits_{t\to0^+}\ {}_0^c\mathcal{D}_s^{\nu}f(s) = \lim\limits_{s\to0^+}\mathcal{I}^{\nu}f(s)$;

iv) if $\nu_i \in (0,1], i = 1,\cdots,n$, with $\nu = \sum\limits_{i=1}^{n}\nu_i$, are such that, for each $k = 1,\cdots,m-1$, there exist $i_k < n$ with $\sum\limits_{j=1}^{k}\nu_j = k$, then the following composition formula holds:
${}_0^c\mathcal{D}_s^{\nu}f(t) = {}_0^c\mathcal{D}_s^{\nu_n}\cdots {}_0^c\mathcal{D}_s^{\nu_2}\ {}_0^c\mathcal{D}_s^{\nu_1}f(t)$.

Lemma 2.5. *[17](page1663). $u(s)$ is a solution of the initial problem (1.1) if and only if $u(s)$ is a solution of the integral equation*

$$u(s) = \int_0^s \frac{(s-\tau)^{\nu+\mu-1}}{\Gamma(\nu+\mu)}\hbar\big(\tau,\ {}_0^c\mathcal{D}_{\tau}^{\theta}u(\tau)\big)d\tau - \lambda\int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)}u(\tau)\,d\tau + H(s), \tag{2.2}$$

where $H(s)$ is introduced by

$$H(s) = \sum_{i=0}^{n-1}\frac{\gamma\mu_i+\nu_i}{\Gamma(\nu+i+1)}s^{\nu+i} + \sum_{j=0}^{m-1}\frac{\mu_i}{\Gamma(j+1)}s^j. \tag{2.3}$$

3. $L_{p,\nu}$ spaces

Definition 3.1. *[15](page676). Assume that $0<\nu<1$ and $1 \leq p < \infty$. We say that a measurable function $u : [0,1] \to \mathbb{R}^n$ belongs to $L_{p,\nu}([0,1],\mathbb{R}^n)$ if and only if the following quantity is finite:*

$$\|u\|_{p,\nu} := \sup_{0\leq s\leq1}\Big(\int_0^s \frac{|u(\tau)|^p}{(s-\tau)^{\nu}}d\tau\Big)^{\frac{1}{p}}.$$

In the subsequent sections, we use the symbol $L_{p,\nu}$ to display instead of $L_{p,\nu}([0,1],\mathbb{R}^n)$. It is easily checked that $\|.\|_{p,\nu}$ is really a norm, when two elements of $L_{p,\nu}$ are equaled almost everywhere (a.e.). The proof of the triangular inequality is similar to that of classical L_p spaces [16]. In the other words, the $L_{p,\nu}$ norm is a compound of two classical norms L_p (p-norm) on the interval $(0,s)$ with measure $d\mu = \frac{d\tau}{(s-\tau)^\nu}$ and L_∞ (sup-norm).

Lemma 3.2. *Suppose $0 < \mu, \nu < 1$ and $1 \le p < \infty$. If $\mu < \nu$, then $\|u\|_{p,\mu} \le \|u\|_{p,\nu}$, and clearly $L_{p,\nu} \subseteq L_{p,\mu}$.*

Proof . Since $(s-\tau) \le 1$ we have $(s-\tau)^\nu \le (s-\tau)^\mu$. So, it follows immediately from the following obvious inequalities,

$$\left(\int_0^s \frac{|u(\tau)|^p}{(s-\tau)^\mu} d\tau\right)^{\frac{1}{p}} \le \left(\int_0^s \frac{|u(\tau)|^p}{(s-\tau)^\nu} d\tau\right)^{\frac{1}{p}},$$

□

Lemma 3.3. *If $0 < \nu < 1$ and $1 \le p < \infty$, $\|u\|_p \le \|u\|_{p,\nu}$, and clearly $L_{p,\nu} \subseteq L_p$.*

Proof . The proof is evident because of $(s-\tau)^\nu \le 1$. □

Lemma 3.4. *[15](page677). For $0 < \nu < 1$ and $1 \le p < \infty$, $L_{p,\alpha}$ is a complete metric space.*

4. Main results

In this section, the existence and uniqueness of the solution for problem (1.1) in $L_p[0,1]$ are proved by using the contraction mapping principle.

Theorem 4.1. *Suppose that Let $\frac{1}{\nu} < q \le \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\mu+\nu \in (0,1)$. If the following conditions hold:*

i) $\hbar(s,0) \in L_{1,1-(\mu+\nu)}$,

ii) *there exists $\lambda \in L^q[0,1]$ such that $|\hbar(s,\tau_2) - \hbar(s,\tau_1)| \le \lambda(s)|\tau_2 - \tau_1|$, for each $s \in [0,1]$ and $\tau_1, \tau_2 \in \mathbb{R}^n$,*

iii) $0 < \zeta < 1$, *where* $\zeta := \left(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)} + \frac{|\gamma|}{\Gamma(\nu)}\right)\left(\frac{1}{1-p(1-\nu)}\right)^{\frac{1}{p}}$,

iv) $\exists\ 0 < \delta < 1$, *such that* $\|u'\| \le \delta\|u\|$.

Then, the integral Eq.(2.2) has a unique fixed point in $L_p[0,1]$.

Proof . *The proof is straightforward when $q = \infty$. Let $1/\nu < q < \infty$. We describe the operator $\mathfrak{A}$ as:*

$$\mathfrak{A}u(s) = \int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)} \hbar\left(\tau,\ {}^c_0\mathcal{D}^\theta_\tau u(\tau)\right) d\tau - \gamma \int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)} u(\tau) d\tau + H(s). \tag{4.1}$$

For each $u \in L_{p,p-p\nu}$, we gain

$$\begin{aligned}
|\mathfrak{A}u(s)| &\leq \left|\int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\hbar(\tau,\ {}_0^c\mathcal{D}_\tau^\theta u(\tau))d\tau\right| + |\gamma|\int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)}|u(\tau)|d\tau + |H(s)| \\
&\leq \int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}|\hbar(\tau,0)|d\tau + \int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}|\hbar(\tau,\ {}_0^c\mathcal{D}_\tau^\theta u(\tau)) - \hbar(\tau,0)|d\tau \\
&\quad + |\gamma|\int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)}|u(\tau)|d\tau + |H(s)| \\
&\leq A_1 + \int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\lambda(\tau)|{}_0^c\mathcal{D}_\tau^\theta u(\tau)|d\tau + |\gamma|\int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)}|u(\tau)|d\tau + A_2 \\
&\leq A_1 + \int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\lambda(\tau)|\mathcal{I}^{1-\theta}u'(\tau)|d\tau + |\gamma|\int_0^s \frac{(s-\tau)^{\nu-1}}{\Gamma(\nu)}|u(\tau)|d\tau + A_2 \\
&\leq A_1 + \frac{1}{\Gamma(\mu+\nu)}\Big(\int_0^s \lambda^q(\tau)d\tau\Big)^{\frac{1}{q}}\frac{\|u'\|_1}{\Gamma(2-\theta)}\Big(\int_0^s \frac{1^p}{(s-\tau)^{p-p(\mu+\nu)}}d\tau\Big)^{\frac{1}{p}} \\
&\quad + \frac{|\gamma|}{\Gamma(\nu)}\Big(\int_0^s \frac{|u(\tau)|^p}{(s-\tau)^{p-p\nu}}d\tau\Big)^{\frac{1}{p}} + A_2 \\
&\leq A_1 + \frac{\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)}\|u'\|_{p,p-p\nu} + \frac{|\gamma|}{\Gamma(\nu)}\|u\|_{p,p-p\nu} + A_2 \\
&\leq A_1 + \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)} + \frac{|\gamma|}{\Gamma(\nu)}\Big)\|u\|_{p,p-p\nu} + A_2
\end{aligned}$$

where $A_1 = \int_0^s \frac{(s-\tau)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}|\hbar(\tau,0)|d\tau$ and $A_2 = \sup_{0\leq s\leq 1} H(s)$.
Letting $\Upsilon := A_1 + \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)} + \frac{|\gamma|}{\Gamma(\nu)}\Big)\|u\|_{p,p-p\nu} + A_2$, we obtain

$$\begin{aligned}
\left(\int_0^s \frac{|\mathfrak{A}u(\tau)|^p}{(s-\tau)^{p-p\nu}}d\tau\right)^{\frac{1}{p}} &\leq \left(\int_0^s \frac{\Upsilon^p}{(s-\tau)^{p-p\nu}}d\tau\right)^{\frac{1}{p}} \\
&\leq \Upsilon\left(\int_0^s \frac{1}{(s-\tau)^{p-p\nu}}d\tau\right)^{\frac{1}{p}} \\
&\leq \Upsilon\left(\frac{1}{1-p(1-\nu)}\right)^{\frac{1}{p}} < \infty,
\end{aligned}$$

which means that $\mathfrak{A} : L_{p,p-p\nu} \to L_{p,p-p\nu}$.

Next, assume that u and v are two elements in $L_{p,p-p\nu}$. So, similar to above discussion we have

$$
\begin{aligned}
|\mathfrak{A}u(s)-\mathfrak{A}v(s)| \leq & \frac{1}{\Gamma(\mu+\nu)}\int_0^s \frac{\left|\hbar\left(\tau,\ {}_0^c\mathcal{D}_\tau^\theta u(\tau)\right)-\hbar\left(\tau,\ {}_0^c\mathcal{D}_\tau^\theta v(\tau)\right)\right|}{(s-\tau)^{1-(\mu+\nu)}}ds+\frac{|\gamma|}{\Gamma(\nu)}\int_0^s \frac{|u(\tau)-v(\tau)|}{(s-\tau)^{1-\nu}}d\tau \\
\leq & \frac{1}{\Gamma(\mu+\nu)}\int_0^s \frac{\lambda(\tau)\left|\ {}_0^c\mathcal{D}_\tau^\theta u(\tau)-\ {}_0^c\mathcal{D}_\tau^\theta v(\tau)\right|}{(s-\tau)^{1-(\mu+\nu)}}d\tau+\frac{|\gamma|}{\Gamma(\nu)}\int_0^s \frac{|u(\tau)-v(\tau)|}{(s-\tau)^{1-\nu}}ds \\
\leq & \frac{1}{\Gamma(\mu+\nu)}\int_0^s \frac{\lambda(\tau)\left|\mathcal{I}_s^{1-\theta}\big(u'(\tau)-v'(\tau)\big)\right|}{(s-\tau)^{1-(\mu+\nu)}}d\tau+\frac{|\gamma|}{\Gamma(\nu)}\int_0^s \frac{|u(\tau)-v(\tau)|}{(s-\tau)^{1-\nu}}d\tau \\
\leq & \frac{1}{\Gamma(\mu+\nu)}\left(\int_0^s \lambda^q(\tau)d\tau\right)^{\frac{1}{p}}\frac{\|u'-v'\|_1}{\Gamma(2-\theta)}\left(\int_0^s \frac{1^p}{(s-\tau)^{p-p(\mu+\nu)}}d\tau\right)^{\frac{1}{p}} \\
& +\frac{|\gamma|}{\Gamma(\nu)}\left(\int_0^s \frac{|u(\tau)-v(\tau)|^p}{(s-\tau)^{p-p\,\nu}}d\tau\right)^{\frac{1}{p}} \\
\leq & \frac{\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)}\|(u-v)'\|_{p,p-p\nu}+\frac{|\gamma|}{\Gamma(\nu)}\|u-v\|_{p,p-p\nu} \\
\leq & \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)}+\frac{|\gamma|}{\Gamma(\nu)}\Big)\|u-v\|_{p,p-p\nu}
\end{aligned}
$$

which mentions that

$$
|\mathfrak{A}u(s)-\mathfrak{A}v(s)| \leq \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)}+\frac{|\gamma|}{\Gamma(\nu)}\Big)^p\|u-v\|_{p,p-p\nu}^p.
$$

So,

$$
\left(\int_0^s \frac{|\mathfrak{A}u(\tau)-\mathfrak{A}v(\tau)|^p}{(s-\tau)^{p-p\nu}}d\tau\right)^{\frac{1}{p}} \leq \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)}+\frac{|\gamma|}{\Gamma(\nu)}\Big)\|u-v\|_{p,p-p\nu}\left(\int_0^s \frac{1}{(s-\tau)^{p-p\nu}}d\tau\right)^{\frac{1}{p}}.
$$

As a result,

$$
\begin{aligned}
\|\mathfrak{A}u-\mathfrak{A}v\|_{p-p\nu} \leq & \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)}+\frac{|\gamma|}{\Gamma(\nu)}\Big)\|u-v\|_{p,p-p\nu}\left(\frac{1}{1-p(1-\nu)}\right)^{\frac{1}{p}} \\
\leq & \zeta\|u-v\|_{p,p-p\nu}.
\end{aligned}
$$

Since $0<\zeta<1$, the operator $\mathfrak{A}$ is a contractive mapping with connectivity constant ζ. So, As a result of contraction mapping principle, there exist a unique solution $\overline{\chi}\in L_{p,p-p\nu}$ such that $\mathfrak{A}\overline{\chi}=\overline{\chi}$. Since $p-p\nu\in(0,1)$, Lemma 3.3 concludes $L_{p,p-p\nu}\subseteq L_p$. Thus, $\overline{\chi}\in L_p[0,1]$ and the proof is complete. □

5. Test examples

Example 5.1. *Consider the following fractional Langevin problem*

$$
\begin{cases}
{}_0^c\mathcal{D}_s^\mu\big(\ {}_0^c\mathcal{D}_s^\nu+\gamma\big)u(s)=\hbar\big(s,\ {}_0^c\mathcal{D}_s^\theta u(s)\big), \quad 0<s<1, \\
u(0)=a_0, \\
u^{(\nu)}(0)=b_0,
\end{cases} \tag{5.1}
$$

where $\mu=\frac{1}{6},\ \nu=\frac{1}{3},\ \gamma=\frac{1}{2},\ a_0=\frac{1}{4},\ b_0=3,\ \hbar\big(s,\ \mathcal{D}^\theta u(s)\big)=\frac{s-u'(s)}{12.5(1+t)^2}$ and $\delta=\frac{1}{2}$. We show that the conditions in Theorem 4.1 are established.

Selecting $q = \infty$ *and* $\lambda(s) = \frac{1}{12.5(1+t)^2}$, *it is easy to see that* $\|\lambda\|_\infty \leq \frac{2}{25}$.
Since $\hbar(s,0) = \frac{t}{12.5(1+t)^2}$ *and* $p = 1$ *and, we get*

$$\int_0^s \frac{|\hbar(s,0)|}{(s-\tau)^{\frac{1}{2}}} d\tau = \frac{2}{25} \int_0^s \frac{s}{(1+s)^2(s-\tau)^{\frac{1}{2}}} d\tau \leq \frac{2}{25} \int_0^s \frac{1}{(s-\tau)^{\frac{1}{2}}} d\tau < \infty. \tag{5.2}$$

So, $\hbar(s,0) \in L_{1,1-(\frac{1}{6}+\frac{1}{3})} = L_{1,\frac{1}{2}}$.
For $u_1, u_2 \in [0,s]$ *and* $s \in [0,1]$ *we attain to*

$$|\hbar(s,u_1) - \hbar(s,u_2)| \leq \frac{2}{25(1+s)^2}|u_1' - u_2'| \leq \frac{1}{25(1+s)^2}|u_1 - u_2|. \tag{5.3}$$

Finally, we cheque that $\zeta < 1$

$$\zeta = \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)} + \frac{|\gamma|}{\Gamma(\nu)}\Big)\big(\frac{1}{1-p(1-\nu)}\big)^{\frac{1}{p}} = 3\big(\frac{\frac{1}{2}\times\frac{2}{25}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} + \frac{\frac{1}{2}}{\Gamma(\frac{1}{3})}\big) \simeq 0.5973 < 1. \tag{5.4}$$

Thus, Theorem 4.1 yields, the problem (5.1) has a unique solution in $L_1[0,1]$.

Example 5.2. *Consider the fractional Langevin problem (1.1) with* $\mu = \frac{3}{4}$, $\nu = \frac{1}{8}$, $\gamma = \frac{1}{3}$, $a_0 = 1$, $b_0 = \frac{1}{2}$, $\hbar\big(s,\ \mathcal{D}^\theta u(s)\big) = \frac{s - {}_0^c\mathcal{D}_s^{\frac{1}{2}} u(s)}{\Gamma(\frac{1}{4})e^t t^{\frac{1}{4}}}$ *and* $\delta = \frac{1}{2}$. *Obviously,* $\hbar$ *is discontinuous of infinite type at* $s = 0$. *Therefore, we can not use of the results in [17]. Now, we investigate the conditions in theorem 4.1 for this problem.*

$$\begin{aligned}\int_0^s \frac{|\hbar(s,0)|}{(s-\tau)^{\frac{1}{8}}} d\tau = \int_0^s \frac{\tau}{\Gamma(\frac{1}{4})e^\tau \tau^{\frac{1}{4}}(s-\tau)^{\frac{1}{8}}} d\tau &\leq \frac{1}{\Gamma(\frac{1}{4})} \int_0^s \frac{\tau}{\tau^{\frac{1}{4}}(s-\tau)^{\frac{1}{8}}} d\tau \\ &\leq \frac{1}{\Gamma(\frac{1}{4})} \int_0^s \frac{1}{(s-\tau)^{\frac{1}{8}}} d\tau < \infty.\end{aligned}$$

So, $\hbar(s,0) \in L_{1,1-(\frac{3}{4}+\frac{1}{8})} = L_{1,\frac{1}{8}}$.
For $u_1, u_2 \in [0,s]$ *and* $s \in [0,1]$, *using lemma 2.2 we attain to*

$$\begin{aligned}|\hbar(s,u_1) - \hbar(s,u_2)| = \frac{|\ {}_0^c\mathcal{D}_s^{\frac{1}{2}} u_1 -\ {}_0^c\mathcal{D}_s^{\frac{1}{2}} u_2|}{\Gamma(\frac{1}{4})e^t t^{\frac{1}{4}}} &\leq \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})e^e s^{\frac{1}{4}}}|\mathcal{I}^{\frac{1}{2}}(u_2 - u_1)'| \\ &\leq \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})e^e s^{\frac{1}{4}}}|u_2' - u_1'| \\ &\leq \frac{\frac{1}{2}}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})e^e s^{\frac{1}{4}}}|u_2 - u_1| \\ &= \lambda(s)|u_2 - u_1|,\end{aligned}$$

where $\lambda(s) = \frac{1}{\pi\Gamma(\frac{1}{4})e^e s^{\frac{1}{4}}}$.
Choosing $q = 2$ *and so* $p = 2$, *we have*

$$\|\lambda\|_q = \|\lambda\|_2 = \int_0^s \frac{d\tau}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})e^\tau \tau^{\frac{1}{4}}} d\tau \leq \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})} \leq \frac{(\frac{\pi}{2})^{\frac{1}{4}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}$$

Finally, we cheque that $\zeta < 1$:

$$\begin{aligned}\zeta = \Big(\frac{\delta\|\lambda\|_q}{\Gamma(\mu+\nu)\Gamma(2-\theta)} + \frac{|\gamma|}{\Gamma(\nu)}\Big)\big(\frac{1}{1-p(1-\nu)}\big)^{\frac{1}{p}} &= \sqrt{2}\big(\frac{1}{2^{\frac{1}{4}\pi^{\frac{3}{4}}}\Gamma(\frac{7}{8})\Gamma(\frac{1}{4})} + \frac{1}{\Gamma(\frac{3}{2})}\big)\\ &\simeq \sqrt{2}\big(\frac{1}{11.08556} + \frac{1}{3.7662}\big)\\ &\simeq 0.503 < 1.\end{aligned}$$

Therefore, the conditions (i)-(iii) theorem 4.1 are satisfied. Thus, the problem (5.1) has a unique solution in $L_2[0,1]$.

6. Conclusion

We have successfully applied a norm to obtain existence and uniqueness results for new form nonlinear Langevin equations involving three different fractional orders. This norm is a tool to measure how far a numerical solution from the exact one. ?We transformed problem (1.1) to a fixed point problem by using concepts in the fractional calculus. New results obtained by means of contraction mapping principle. Two examples illustrated the correctness of the results.

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