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Symmetry reductions and exact solutions of a two-wave mode Korteweg-de Vrie equation

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Abstract

A two-wave mode Korteweg-de Vries equation is investigated by using Lie symmetry analysis. The similarity reductions and new exact solutions are obtained via the simplest equation method. Exact solutions including solitons are shown. In addition, the conservation laws are derived using the multiplier approach.

Keywords: Two-wave mode Korteweg-de Vries equation, Lie symmetry method, Simplest equation method
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1. Introduction

Nonlinear partial differential equations (NLPDEs) depict a range of physical occurrence in the area such as physics, chemistry, biology and fluid dynamics. Thus computing solutions of such NLPDEs is unavoidable task. To obtain the exact solutions of NLPDEs, a number of methods have been suggested in the literature. Some of the renowned approaches include the inverse scattering transform method, Hirota's bilinear method, homogeneous balance methods, etc. [16, 10, 26, 29, 25, 13, 9, 20, 21, 27, 12, 22, 11, 15, 2, 5, 3, 4, 14].

Conservation laws [24, 23, 6, 19] play an important role in reducing and finding the solution of differential equations. Computing conservation laws of different equations is the first step in

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determining exact solutions. The number of conservation laws determine how close one is to the complete solution. Thus the more conservation laws one finds, the closer one gets to the solution.

The Lie symmetry method [8, 7] is employed to construct a symmetry reduction, and exact solutions. We then derive the conservation laws using the multiplier approach for a two-wave mode Korteweg-de Vrie equation (TKdV) [28, 18]

$$u_{tt} - s^2 u_{xx} + u_t u_x + u u_{tx} - \alpha s \left(u u_{xx} + {u_x}^2 \right) + u_{txxx} - \beta s u_{xxxx} = 0. \tag{1.1}$$

The above equation was proposed as a nonlinear multi-mode dispersive wave equation that describes the wave propagation in one space dimension in a weakly nonlinear and weakly dispersive system.

2. Symmetry reductions and exact solutions of (1.1)

The vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.1}$$

is a Lie point symmetry of (1.1) if

$$X^{[4]}\left\{ u_{tt} - s^2 u_{xx} + u_t u_x + u u_{tx} - \alpha s \left(u u_{xx} + {u_x}^2 \right) + u_{txxx} - \beta s u_{xxxx} \right\}\Big|_{(1.1)} = 0,$$

where $X^{[4]}$ is the fourth prolongation of (2.1). Expanding the above equation and splitting on the derivatives of u leads to the following overdetermined system of linear partial differential equations:

$$\xi^1_x = 0, \tag{2.2}$$
$$\xi^2_u = 0, \tag{2.3}$$
$$\xi^1_u = 0, \tag{2.4}$$
$$\eta_{uu} = 0, \tag{2.5}$$
$$\eta_{xu} - \xi^2_{xx} = 0, \tag{2.6}$$
$$2\,\beta\, s\xi^1_t - 4\,\beta\, s\xi^2_x + \xi^2_t = 0, \tag{2.7}$$
$$\beta\, s\xi^1_t - \beta\, s\xi^2_x + \xi^2_t = 0, \tag{2.8}$$
$$-\alpha\,\beta\, s\eta_u - 2\,\alpha\,\beta\, s\xi^2_x + \alpha\xi^2_t - \beta\xi^2_t = 0, \tag{2.9}$$
$$4\,\beta\, s\eta_{xu} - 6\,\beta\, s\xi^2_{xx} - \eta_{tu} + 3\,\xi^2_{tx} = 0, \tag{2.10}$$
$$-\beta\, s\eta_u + \beta\, s\xi^1_t - 3\,\beta\, s\xi^2_x + \xi^2_t = 0, \tag{2.11}$$
$$u\eta_{xu} + \eta_x + 2\,\eta_{tu} + \eta_{xxxu} - \xi^1_{tt} = 0, \tag{2.12}$$
$$-\alpha\, su\eta_{xx} - \beta\, s\eta_{xxxx} - s^2\eta_{xx} + u\eta_{tx} + \eta_{tt} + \eta_{txxx} = 0, \tag{2.13}$$
$$u\beta\, s\xi^1_t - 3\,u\beta\, s\xi^2_x - \beta\, s\eta_1 - 3\,\beta\, s\eta_{xxu} + 2\,\beta\, s\xi^2_t + \beta\, s\xi^2_{xxx} + u\xi^2_t = 0, \tag{2.14}$$
$$\begin{aligned}&-2\,\alpha\,\beta\, su\xi^2_x - \alpha\,\beta\, s\eta_1 - 6\,\beta^2 s\eta_{xxu} + 4\,\beta^2 s\xi^2_{xxx} - 2\,\beta\, s^2\xi^2_x + \alpha\, u\xi^2_t - \beta\, u\xi^2_t \\ &+3\,\beta\eta_{txu} - 3\,\beta\xi^2_{txx} + s\xi^2_t = 0,\end{aligned} \tag{2.15}$$
$$\begin{aligned}&2\,\alpha\, su\eta_{xu} - \alpha\, su\xi^2_{xx} + 2\,\alpha\, s\eta_x + 4\,\beta\, s\eta_{xxxu} - \beta\, s\xi^2_{xxxx} + 2\, s^2\eta_{xu} - s^2\xi^2_{xx} \\ &-u\eta_{tu} + u\xi^2_{tx} - \eta_t - 3\,\eta_{txxu} + \xi^2_{tt} + \xi^2_{txxx} = 0.\end{aligned} \tag{2.16}$$

Solving the above system of equations with the aid of Maple leads to the two dimensional Lie algebra spanned by the following linearly independent operators:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial x}.
\end{aligned}
$$

The linear combination of the translation symmetries $\omega X_1 + X_2$ gives rise to the group-invariant solution

$$u(x,t) = F(z), \tag{2.17}$$

where $z = x - \omega t$ is an invariant of the symmetry $\omega X_1 + X_2$. Substitution of (2.17) into (1.1) results in

$$
\begin{aligned}
&s\beta F_{zzzz} + \omega F_{zzzz} + s^2 F_{zz} - \omega^2 F_{zz} + \omega F F_{zz} - \omega F_z^2 \\
&+ s\alpha \left(F F_{zz} + F_z^2\right) = 0,
\end{aligned} \tag{2.18}
$$

whose solution is given by

$$F(z) = \frac{3\left(\omega^2 - s^2\right) \operatorname{sech}^2\left(\frac{z\sqrt{\omega^2 - s^2}}{2\sqrt{s\beta + \omega}}\right)}{s\alpha + \omega}. \tag{2.19}$$

Consequently the required group invariant solution is completed by (2.17).

2.1. Exact solutions using the simplest equation method

We now use the simplest equation method [25], to solve (2.18) and as a result we obtain the exact solutions system (1.1). The simplest equations that will be used are the Bernoulli and Riccati equations.

We briefly recall the simplest equation method here. Let us consider the solutions of (1.1) in the form

$$F(z) = \sum_{i=0}^{M} A_i (H(z))^i, \tag{2.20}$$

where $H(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined. We note that the Bernoulli and Riccati equations are well-known nonlinear equations whose solutions can be expressed in terms of elementary functions.

We consider the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \tag{2.21}$$

where a, b and c are constants, we shall use the solutions

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left[\frac{1}{2}\theta(z + C)\right] \tag{2.22}$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}, \tag{2.23}$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.
For Bernoulli equation

$$H'(z) = eH(z) + dH^2(z), \tag{2.24}$$

where a and b are constants. The solution of this equation is given by

$$H(z) = e\left\{\frac{\cosh[e(z+C)] + \sinh[e(z+C)]}{1 - d\cosh[e(z+C)] - d\sinh[e(z+C)]}\right\}. \tag{2.25}$$

2.1.1. Application of the simplest equation method: Riccati equation

The balancing procedure yields $M = 2$ so the solutions of (2.18) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2. \tag{2.26}$$

Substituting equation (2.26) into equation (2.18) and making use of equation (2.21) and then equating all coefficients of the functions H^i to zero,with the aid of Maple we obtain the following overdetermined system of algebraic equations in terms of A_0, A_1, A_2:

$$\begin{aligned}
&-120\,a^4\beta\,sA_2{k_1}^4 + 120\,a^4A_2{k_1}^3k_2 - 10\,a^2\alpha\,s{A_2}^2{k_1}^2 + 10\,a^2{A_2}^2k_1k_2 = 0,\\
&-24\,a^4\beta\,sA_1{k_1}^4 - 336\,a^3b\beta\,sA_2{k_1}^4 + 24\,a^4A_1{k_1}^3k_2 + 336\,a^3bA_2{k_1}^3k_2\\
&-12\,a^2\alpha\,sA_1A_2{k_1}^2 - 18\,a\alpha\,bs{A_2}^2{k_1}^2 + 12\,a^2A_1A_2k_1k_2 + 18\,ab{A_2}^2k_1k_2 = 0,\\
&-8\,ab\beta\,c^2sA_1{k_1}^4 - 16\,a\beta\,c^3sA_2{k_1}^4 - b^3\beta\,csA_1{k_1}^4 - 14\,b^2\beta\,c^2sA_2{k_1}^4\\
&+8\,abc^2A_1{k_1}^3k_2 + 16\,ac^3A_2{k_1}^3k_2 + b^3cA_1{k_1}^3k_2 + 14\,b^2c^2A_2{k_1}^3k_2 - \alpha\,bcsA_0A_1{k_1}^2\\
&-2\,\alpha\,c^2sA_0A_2{k_1}^2 - \alpha\,c^2s{A_1}^2{k_1}^2 - bcs^2A_1{k_1}^2 - 2\,c^2s^2A_2{k_1}^2 + bcA_0A_1k_1k_2\\
&+2\,c^2A_0A_2k_1k_2 + c^2{A_1}^2k_1k_2 + bcA_1{k_2}^2 + 2\,c^2A_2{k_2}^2 = 0,\\
&-60\,a^3b\beta\,sA_1{k_1}^4 - 240\,a^3\beta\,csA_2{k_1}^4 - 330\,a^2b^2\beta\,sA_2{k_1}^4 + 60\,a^3bA_1{k_1}^3k_2\\
&+240\,a^3cA_2{k_1}^3k_2 + 330\,a^2b^2A_2{k_1}^3k_2 - 6\,a^2\alpha\,sA_0A_2{k_1}^2 - 3\,a^2\alpha\,s{A_1}^2{k_1}^2\\
&-21\,a\alpha\,bsA_1A_2{k_1}^2 - 16\,a\alpha\,cs{A_2}^2{k_1}^2 - 8\,\alpha\,b^2s{A_2}^2{k_1}^2 - 6\,a^2s^2A_2{k_1}^2\\
&+6\,a^2A_0A_2k_1k_2 + 3\,a^2{A_1}^2k_1k_2 + 21\,abA_1A_2k_1k_2 + 16\,ac{A_2}^2k_1k_2 + 8\,b^2{A_2}^2k_1k_2\\
&+6\,a^2A_2{k_2}^2 = 0,\\
&-40\,a^3\beta\,csA_1{k_1}^4 - 50\,a^2b^2\beta\,sA_1{k_1}^4 - 440\,a^2b\beta\,csA_2{k_1}^4 - 130\,ab^3\beta\,sA_2{k_1}^4\\
&+40\,a^3cA_1{k_1}^3k_2 + 50\,a^2b^2A_1{k_1}^3k_2 + 440\,a^2bcA_2{k_1}^3k_2 + 130\,ab^3A_2{k_1}^3k_2\\
&-2\,a^2\alpha\,sA_0A_1{k_1}^2 - 10\,a\alpha\,bsA_0A_2{k_1}^2 - 5\,a\alpha\,bs{A_1}^2{k_1}^2 - 18\,a\alpha\,csA_1A_2{k_1}^2\\
&-9\,\alpha\,b^2sA_1A_2{k_1}^2 - 14\,\alpha\,bcs{A_2}^2{k_1}^2 - 2\,a^2s^2A_1{k_1}^2 - 10\,abs^2A_2{k_1}^2\\
&+2\,a^2A_0A_1k_1k_2 + 10\,abA_0A_2k_1k_2 + 5\,ab{A_1}^2k_1k_2 + 18\,acA_1A_2k_1k_2 + 9\,b^2A_1A_2k_1k_2\\
&+14\,bc{A_2}^2k_1k_2 + 2\,a^2A_1{k_2}^2 + 10\,abA_2{k_2}^2 = 0,
\end{aligned}$$

$$-16\,a^2\beta\,c^2sA_1{k_1}^4-22\,ab^2\beta\,csA_1{k_1}^4-120\,ab\beta\,c^2sA_2{k_1}^4-b^4\beta\,sA_1{k_1}^4$$
$$-30\,b^3\beta\,csA_2{k_1}^4+16\,a^2c^2A_1{k_1}^3k_2+22\,ab^2cA_1{k_1}^3k_2+120\,abc^2A_2{k_1}^3k_2+b^4A_1{k_1}^3k_2$$
$$+30\,b^3cA_2{k_1}^3k_2-2\,a\alpha\,csA_0A_1{k_1}^2-\alpha\,b^2sA_0A_1{k_1}^2-6\,\alpha\,bcsA_0A_2{k_1}^2-3\,\alpha\,bcs{A_1}^2{k_1}^2$$
$$-6\,\alpha\,c^2sA_1A_2{k_1}^2-2\,acs^2A_1{k_1}^2-b^2s^2A_1{k_1}^2-6\,bcs^2A_2{k_1}^2+2\,acA_0A_1k_1k_2$$
$$+b^2A_0A_1k_1k_2+6\,bcA_0A_2k_1k_2+3\,bc{A_1}^2k_1k_2+6\,c^2A_1A_2k_1k_2+2\,acA_1{k_2}^2+b^2A_1{k_2}^2$$
$$+6\,bcA_2{k_2}^2=0,$$
$$-60\,a^2b\beta\,csA_1{k_1}^4-136\,a^2\beta\,c^2sA_2{k_1}^4-15\,ab^3\beta\,sA_1{k_1}^4-232\,ab^2\beta\,csA_2{k_1}^4$$
$$-16\,b^4\beta\,sA_2{k_1}^4+60\,a^2bcA_1{k_1}^3k_2+136\,a^2c^2A_2{k_1}^3k_2+15\,ab^3A_1{k_1}^3k_2+232\,ab^2cA_2{k_1}^3k_2$$
$$+16\,b^4A_2{k_1}^3k_2-3\,a\alpha\,bsA_0A_1{k_1}^2-8\,a\alpha\,csA_0A_2{k_1}^2-4\,a\alpha\,cs{A_1}^2{k_1}^2-4\,\alpha\,b^2sA_0A_2{k_1}^2$$
$$-2\,\alpha\,b^2s{A_1}^2{k_1}^2-15\,\alpha\,bcsA_1A_2{k_1}^2-6\,\alpha\,c^2s{A_2}^2{k_1}^2-3\,abs^2A_1{k_1}^2-8\,acs^2A_2{k_1}^2$$
$$-4\,b^2s^2A_2{k_1}^2+3\,abA_0A_1k_1k_2+8\,acA_0A_2k_1k_2+4\,ac{A_1}^2k_1k_2+4\,b^2A_0A_2k_1k_2$$
$$+2\,b^2{A_1}^2k_1k_2+15\,bcA_1A_2k_1k_2+6\,c^2{A_2}^2k_1k_2+3\,abA_1{k_2}^2+8\,acA_2{k_2}^2$$
$$+4\,b^2A_2{k_2}^2=0.$$

On solving the resultant system of algebraic equations, we obtain

$$\beta=\frac{1}{2}\,\frac{12\,a^2{k_1}^2k_2-\alpha\,sA_2k_1+A_2k_2}{a^2s{k_1}^3},$$
$$A_0=-\frac{1}{12}\,\frac{-8\,a\alpha\,csA_2{k_1}^2-\alpha\,b^2sA_2{k_1}^2+12\,a^2s^2{k_1}^2+8\,acA_2k_1k_2+b^2A_2k_1k_2-12\,a^2{k_2}^2}{a^2k_1\left(\alpha\,sk_1-k_2\right)},$$
$$A_1=\frac{bA_2}{a}.$$

Consequently a solutions of equation (2.18) are

$$\begin{aligned}u(x,t) &= A_0+A_1\Big\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\Big(\frac{1}{2}\theta(z+2)\Big)\Big\}\\ &\quad A_2\Big\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\Big(\frac{1}{2}\theta(z+2)\Big)\Big\}^2\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}u(x,t) &= A_0+A_1\Big\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\Big(\frac{1}{2}\theta z\Big)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\Big\}\\ &\quad A_2\Big\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\Big(\frac{1}{2}\theta z\Big)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\Big\}^2,\end{aligned}\tag{2.28}$$

where $z=k_1x+k_2t+k_3$ and $\theta^2=b^2-4ac>0$.

2.1.2. Application of the simplest equation method: Bernoulli equation

The balancing procedure yields $M=2$ so the solutions of (2.18) are of the form

$$F(z)=A_0+A_1H+A_2H^2.\tag{2.29}$$

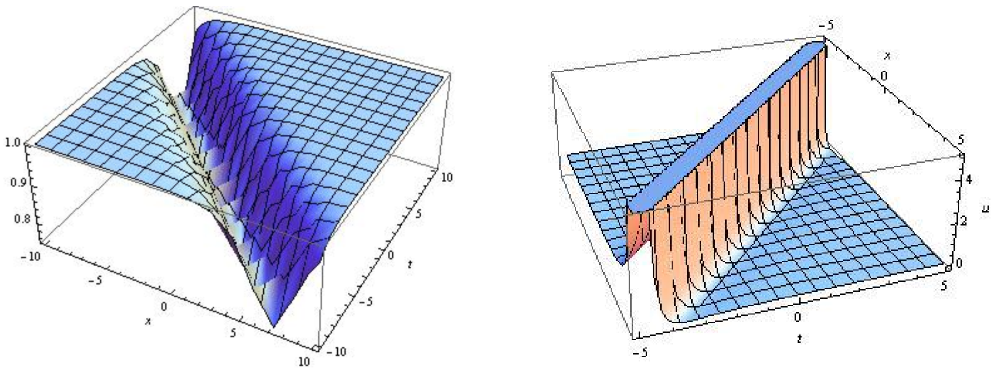

Figure 1: Profiles of solutions (2.27) and (2.28)

Substituting equation (2.29) into equation (2.18) and making use of equation (2.24) and then equating all coefficients of the functions H^i to zero, using Maple, we obtain the following overdetermined system of algebraic equations in terms of A_0, A_1, A_2:

$$
\begin{aligned}
&-120\,\beta\, e^4 sA_2{k_1}^4 + 120\, e^4 A_2{k_1}^3 k_2 - 10\,\alpha\, e^2 s{A_2}^2{k_1}^2 + 10\, e^2{A_2}^2 k_1 k_2 = 0,\\
&-\beta\, d^4 sA_1{k_1}^4 + d^4 A_1{k_1}^3 k_2 - \alpha\, d^2 sA_0A_1{k_1}^2 - d^2 s^2 A_1{k_1}^2 + d^2 A_0 A_1 k_1 k_2\\
&+d^2 A_1{k_2}^2 = 0,\\
&-336\,\beta\, de^3 sA_2{k_1}^4 - 24\,\beta\, e^4 sA_1{k_1}^4 + 336\, de^3 A_2{k_1}^3 k_2 + 24\, e^4 A_1{k_1}^3 k_2\\
&-18\,\alpha\, des{A_2}^2{k_1}^2 - 12\,\alpha\, e^2 sA_1A_2{k_1}^2 + 18\, de{A_2}^2 k_1 k_2 + 12\, e^2 A_1 A_2 k_1 k_2 = 0,\\
&-330\,\beta\, d^2e^2 sA_2{k_1}^4 - 60\,\beta\, de^3 sA_1{k_1}^4 + 330\, d^2e^2 A_2{k_1}^3 k_2 + 60\, de^3 A_1{k_1}^3 k_2\\
&-8\,\alpha\, d^2 s{A_2}^2{k_1}^2 - 21\,\alpha\, desA_1A_2{k_1}^2 - 6\,\alpha\, e^2 sA_0A_2{k_1}^2 - 3\,\alpha\, e^2 s{A_1}^2{k_1}^2\\
&-6\, e^2 s^2 A_2{k_1}^2 + 8\, d^2{A_2}^2 k_1 k_2 + 21\, deA_1A_2k_1k_2 + 6\, e^2 A_0A_2k_1k_2 + 3\, e^2{A_1}^2 k_1 k_2\\
&+6\, e^2 A_2{k_2}^2 = 0,\\
&-16\,\beta\, d^4 sA_2{k_1}^4 - 15\,\beta\, d^3 esA_1{k_1}^4 + 16\, d^4 A_2{k_1}^3 k_2 + 15\, d^3 eA_1{k_1}^3 k_2 - 4\,\alpha\, d^2 sA_0A_2{k_1}^2\\
&-2\,\alpha\, d^2 s{A_1}^2{k_1}^2 - 3\,\alpha\, desA_0A_1{k_1}^2 - 4\, d^2 s^2 A_2{k_1}^2 - 3\, des^2 A_1{k_1}^2 + 4\, d^2 A_0A_2k_1k_2\\
&+2\, d^2{A_1}^2 k_1 k_2 + 3\, deA_0A_1k_1k_2 + 4\, d^2 A_2{k_2}^2 + 3\, deA_1{k_2}^2 = 0,\\
&-130\,\beta\, d^3 esA_2{k_1}^4 - 50\,\beta\, d^2e^2 sA_1{k_1}^4 + 130\, d^3 eA_2{k_1}^3 k_2 + 50\, d^2e^2 A_1{k_1}^3 k_2\\
&-9\,\alpha\, d^2 sA_1A_2{k_1}^2 - 10\,\alpha\, desA_0A_2{k_1}^2 - 5\,\alpha\, des{A_1}^2{k_1}^2 - 2\,\alpha\, e^2 sA_0A_1{k_1}^2\\
&-10\, des^2 A_2{k_1}^2 - 2\, e^2 s^2 A_1{k_1}^2 + 9\, d^2 A_1A_2k_1k_2 + 10\, deA_0A_2k_1k_2 + 5\, de{A_1}^2 k_1 k_2\\
&+2\, e^2 A_0A_1k_1k_2 + 10\, deA_2{k_2}^2 + 2\, e^2 A_1{k_2}^2 = 0.
\end{aligned}
$$

On solving the resultant system of algebraic equations, we obtain

$$
\begin{aligned}
&\beta = -\frac{1}{12}\,\frac{-12\, e^2{k_1}^2 k_2 + \alpha\, sA_2k_1 - A_2k_2}{{k_1}^3 se^2},\\
&A_0 = \frac{1}{2}\,\frac{\alpha\, d^2 sA_2{k_1}^2 - 12\, e^2 s^2{k_1}^2 - d^2 A_2k_1k_2 + 12\, e^2{k_2}^2}{e^2 k_1\left(\alpha\, sk_1 - k_2\right)},\\
&A_1 = \frac{dA_2}{e}.
\end{aligned}
$$

Consequently a solution of equation (2.18) is

$$u(x,t) = A_0 + A_1 e\left\{\frac{\cosh[e(z+C)]+\sinh[e(z+C)]}{1-d\cosh[e(z+C)]-d\sinh[e(z+C)]}\right\}. \\ A_2\left\{e\frac{\cosh[e(z+C)]+\sinh[e(z+C)]}{1-d\cosh[e(z+C)]-d\sinh[e(z+C)]}\right\}^2, \tag{2.30}$$

where $z = k_1x + k_2t + k_3$ and C is an arbitrary constant of integration.

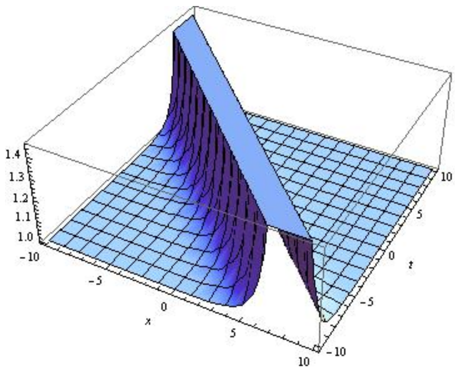

Figure 2: Profile of solution (2.30)

Hereby, we give some figures to describe our solutions.

2.2. Conservation laws of (1.1)

A local conservation law for equation (1.1)

$$G = u_{tt} - s^2u_{xx} + u_tu_x + uu_{tx} - \alpha s\left(uu_{xx} + {u_x}^2\right) + u_{txxx} - \beta s u_{xxxx} \tag{2.31}$$

is a space-time divergence

$$D_tT^t + D_xT^x|_{(1.1)} = 0, \tag{2.32}$$

which holds for all formal solutions $u(t,x)$ of equation (1.1), where the conserved density T^t and the spatial fluxe T^x are functions of t, x, u and derivatives of u.

We obtain a multiplier Λ, that is given by

$$\Lambda(t,x,u) = 2\,s^2\alpha^3A_2xt + A_2s^3\alpha^2t^2 + sA_2\alpha^2x^2 - 2\,s^2\alpha\,A_2xt - A_2s^3t^2 + sA_4\alpha t \\ -sA_2x^2 - 2\,\alpha\,A_2 + 2\,\beta\,A_2 + A_1x + A_4x + A_3 - \left\{A_6\cos\left(\frac{\alpha\,s^{3/2}\sqrt{\alpha^2-1}t}{\sqrt{\alpha-\beta}}\right)\right. \\ \left.-A_5\sin\left(\frac{\alpha\,s^{3/2}\sqrt{\alpha^2-1}t}{\sqrt{\alpha-\beta}}\right)\right\}\cos\left(\frac{\sqrt{s}\sqrt{\alpha^2-1}x}{\sqrt{\alpha-\beta}}\right) + \sin\left(\frac{\sqrt{s}\sqrt{\alpha^2-1}x}{\sqrt{\alpha-\beta}}\right) \\ \left\{A_5\cos\left(\frac{\alpha\,s^{3/2}\sqrt{\alpha^2-1}t}{\sqrt{\alpha-\beta}}\right) + A_6\sin\left(\frac{\alpha\,s^{3/2}\sqrt{\alpha^2-1}t}{\sqrt{\alpha-\beta}}\right)\right\}.$$

Thus, corresponding to the above multiplier we have the following conservation laws:

$$D_tT_1^t + D_xT_1^x|_{(1.1)} = 0,$$

$$
\begin{aligned}
T_1^t &= \frac{1}{4}\left(2xu_x u - u^2 + 4xu_t - u_{xx} + xu_{xxx}\right),\\
T_1^x &= \frac{1}{4}\big(- 4sx\alpha u_x u + 2xu_t u + 4s^2 u + 2s\alpha u^2 - 4s^2 x u_x + 4s\beta u_{xx} - 4sx\beta u_{xxx}\\
&-2u_{tx} + 3xu_{txx}\big);
\end{aligned}
$$

$$D_t T_2^t + D_x T_2^x|_{(1.1)} = 0,$$

$$
\begin{aligned}
T_2^t = \; & \frac{1}{4}\big(- 8t\alpha^2 us^3 + 8tus^3 - 2t^2 uu_x s^3 + 2t^2\alpha^2 uu_x s^3 - t^2 u_{xxx} s^3 + t^2\alpha^2 u_{xxx} s^3\\
& -4t^2 u_t s^3 + 4t^2\alpha^2 u_t s^3 - 2t\alpha^3 u^2 s^2 + 2t\alpha u^2 s^2 - 8x\alpha^3 us^2 + 8x\alpha us^2\\
& +4tx\alpha^3 uu_x s^2 - 4tx\alpha uu_x s^2 - 2t\alpha^3 u_{xx} s^2 + 2t\alpha u_{xx} s^2 + 2tx\alpha^3 u_{xxx} s^2\\
& -2tx\alpha u_{xxx} s^2 + 8tx\alpha^3 u_t s^2 - 8tx\alpha u_t s^2 - 2x\alpha^2 u^2 s + 2xu^2 s + 2\alpha^2 u_x s\\
& -2x^2 uu_x s + 2x^2\alpha^2 uu_x s - 2u_x s - 2x\alpha^2 u_{xx} s + 2xu_{xx} s - x^2 u_{xxx} s\\
& +x^2\alpha^2 u_{xxx} s - 4x^2 u_t s + 4x^2\alpha^2 u_t s - 4\alpha uu_x + 4\beta uu_x - 2\alpha u_{xxx} + 2\beta u_{xxx}\\
& -8\alpha u_t + 8\beta u_t\big),\\
T_2^x = \; & \frac{1}{4}(4t^2 u_x s^5 - 4t^2\alpha^2 u_x s^5 + 8t\alpha^3 us^4 - 8t\alpha us^4 - 8tx\alpha^3 u_x s^4 + 8tx\alpha u_x s^4\\
& -4t^2\alpha^3 uu_x s^4 + 4t^2\alpha uu_x s^4 + 4t^2\beta u_{xxx} s^4 - 4t^2\alpha^2\beta u_{xxx} s^4 + 4t\alpha^4 u^2 s^3\\
& -6t\alpha^2 u^2 s^3 + 2tu^2 s^3 + 8x\alpha^2 us^3 - 8xus^3 + 4x^2 u_x s^3 - 4x^2\alpha^2 u_x s^3\\
& -8tx\alpha^4 uu_x s^3 + 8tx\alpha^2 uu_x s^3 - 2t\alpha^2 u_{xx} s^3 + 2tu_{xx} s^3 + 8t\alpha^3\beta u_{xx} s^3\\
& -8t\alpha\beta u_{xx} s^3 - 8tx\alpha^3\beta u_{xxx} s^3 + 8tx\alpha\beta u_{xxx} s^3 - 2t^2 uu_t s^3 + 2t^2\alpha^2 uu_t s^3\\
& -3t^2 u_{txx} s^3 + 3t^2\alpha^2 u_{txx} s^3 + 2x\alpha^3 u^2 s^2 - 2x\alpha u^2 s^2 + 4\alpha^3 u_x s^2 + 4\alpha u_x s^2\\
& -8\alpha^2\beta u_x s^2 - 4x^2\alpha^3 uu_x s^2 + 4x^2\alpha uu_x s^2 - 2x\alpha^3 u_{xx} s^2 + 2x\alpha u_{xx} s^2\\
& +8x\alpha^2\beta u_{xx} s^2 - 8x\beta u_{xx} s^2 + 4x^2\beta u_{xxx} s^2 - 4x^2\alpha^2\beta u_{xxx} s^2 + 4tx\alpha^3 uu_t s^2\\
& -4tx\alpha uu_t s^2 - 4t\alpha^3 u_{tx} s^2 + 4t\alpha u_{tx} s^2 + 6tx\alpha^3 u_{txx} s^2 - 6tx\alpha u_{txx} s^2 + 8\alpha^2 uu_x s\\
& -8\alpha\beta uu_x s - 8\beta^2 u_{xxx} s + 8\alpha\beta u_{xxx} s + 2\alpha^2 u_t s - 2x^2 uu_t s + 2x^2\alpha^2 uu_t s - 2u_t s\\
& -4x\alpha^2 u_{tx} s + 4xu_{tx} s - 3x^2 u_{txx} s + 3x^2\alpha^2 u_{txx} s - 4\alpha uu_t + 4\beta uu_t - 6\alpha u_{txx}\\
& +6\beta u_{txx});
\end{aligned}
$$

$$D_t T_3^t + D_x T_3^x|_{(1.1)} = 0,$$

$$
\begin{aligned}
T_3^t = \; & \frac{1}{4}\left(2u_x u + 4u_t + u_{xxx}\right),\\
T_3^x = \; & \frac{1}{4}\left(-4s\alpha u_x u + 2u_t u - 4s^2 u_x - 4s\beta u_{xxx} + 3u_{txx}\right);
\end{aligned}
$$

$$D_t T_4^t + D_x T_4^x|_{(1.1)} = 0,$$

$$\begin{aligned}
T_4^t =& \frac{1}{4}\big(2st\alpha u_x u + 2xu_x u - 4s\alpha u - u^2 + st\alpha u_{xxx} + 4st\alpha u_t + 4xu_t - u_{xx} + xu_{xxx}\big),\\
T_4^x =& \frac{1}{4}\big(- 4s^2t\alpha^2 u_x u - 4sx\alpha u_x u + 2st\alpha u_t u + 2xu_t u + 4s^2 u + s\alpha u^2 - 4s^3t\alpha u_x\\
&-4s^2t\alpha\beta u_{xxx} - 4s^2xu_x + 3st\alpha u_{txx} - s\alpha u_{xx} + 4s\beta u_{xx} - 4sx\beta u_{xxx} - 2u_{tx}\\
&+3xu_{txx}\big);
\end{aligned}$$

$$D_tT_5^t + D_xT_5^x|_{(1.1)} = 0,$$
$$\Gamma = \frac{\sqrt{\alpha^2-1}\sqrt{s}(\alpha st + x)}{\sqrt{\alpha-\beta}},$$

$$\begin{aligned}
T_5^t =& \frac{1}{4(\alpha-\beta)^{3/2}}\Big\{ - 3s^{3/2}\sqrt{\alpha^2-1}\cos(\Gamma)u\alpha^2 - s\sqrt{\alpha-\beta}\sin(\Gamma)u_x\alpha^2\\
&-\sqrt{s}\sqrt{\alpha^2-1}\cos(\Gamma)u^2\alpha + 4s^{3/2}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u\alpha + 2\sqrt{\alpha-\beta}\sin(\Gamma)uu_x\alpha\\
&-\sqrt{s}\sqrt{\alpha^2-1}\cos(\Gamma)u_{xx}\alpha + \sqrt{\alpha-\beta}\sin(\Gamma)u_{xxx}\alpha + 4\sqrt{\alpha-\beta}\sin(\Gamma)u_t\alpha\\
&+\sqrt{s}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u^2 - s^{3/2}\sqrt{\alpha^2-1}\cos(\Gamma)u + s\sqrt{\alpha-\beta}\sin(\Gamma)u_x\\
&-2\sqrt{\alpha-\beta}\beta\sin(\Gamma)uu_x + \sqrt{s}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u_{xx} - \sqrt{\alpha-\beta}\beta\sin(\Gamma)u_{xxx}\\
&-4\sqrt{\alpha-\beta}\beta\sin(\Gamma)u_t\Big\},\\
T_5^x =& \frac{1}{4(\alpha-\beta)^{3/2}}\Big\{3s^{5/2}\sqrt{\alpha^2-1}\cos(\Gamma)u\alpha^3 - 2s^2\sqrt{\alpha-\beta}\sin(\Gamma)u_x\alpha^3\\
&+s^{3/2}\sqrt{\alpha^2-1}\cos(\Gamma)u^2\alpha^2 - 4s^{5/2}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u\alpha^2 + 4s^2\sqrt{\alpha-\beta}\beta\sin(\Gamma)u_x\alpha^2\\
&-4s\sqrt{\alpha-\beta}\sin(\Gamma)uu_x\alpha^2 - s^{3/2}\sqrt{\alpha^2-1}\cos(\Gamma)u_{xx}\alpha^2 - s\sqrt{\alpha-\beta}\sin(\Gamma)u_t\alpha^2\\
&-s^{3/2}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u^2\alpha + s^{5/2}\sqrt{\alpha^2-1}\cos(\Gamma)u\alpha - 2s^2\sqrt{\alpha-\beta}\sin(\Gamma)u_x\alpha\\
&+4s\sqrt{\alpha-\beta}\beta\sin(\Gamma)uu_x\alpha + 5s^{3/2}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u_{xx}\alpha - 4s\sqrt{\alpha-\beta}\beta\sin(\Gamma)u_{xxx}\alpha\\
&+2\sqrt{\alpha-\beta}\sin(\Gamma)uu_t\alpha - 2\sqrt{s}\sqrt{\alpha^2-1}\cos(\Gamma)u_{tx}\alpha + 3\sqrt{\alpha-\beta}\sin(\Gamma)u_{txx}\alpha\\
&-4s^{3/2}\sqrt{\alpha^2-1}\beta^2\cos(\Gamma)u_{xx} + 4s\sqrt{\alpha-\beta}\beta^2\sin(\Gamma)u_{xxx} + s\sqrt{\alpha-\beta}\sin(\Gamma)u_t\\
&-2\sqrt{\alpha-\beta}\beta\sin(\Gamma)uu_t + 2\sqrt{s}\sqrt{\alpha^2-1}\beta\cos(\Gamma)u_{tx} - 3\sqrt{\alpha-\beta}\beta\sin(\Gamma)u_{txx}\Big\};
\end{aligned}$$

$$D_tT_6^t + D_xT_6^x|_{(1.1)} = 0,$$
$$\Gamma = \frac{\sqrt{\alpha^2-1}\sqrt{s}(\alpha st + x)}{\sqrt{\alpha-\beta}},$$

$$
\begin{aligned}
T_6^t = \ & \frac{1}{4(\alpha-\beta)^{3/2}}\Big\{ -3s^{3/2}\sqrt{\alpha^2-1}\sin(\Gamma)u\alpha^2 + s\sqrt{\alpha-\beta}\cos(\Gamma)u_x\alpha^2 \\
& -\sqrt{s}\sqrt{\alpha^2-1}\sin(\Gamma)u^2\alpha + 4s^{3/2}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u\alpha - 2\sqrt{\alpha-\beta}\cos(\Gamma)uu_x\alpha \\
& -\sqrt{s}\sqrt{\alpha^2-1}\sin(\Gamma)u_{xx}\alpha - \sqrt{\alpha-\beta}\cos(\Gamma)u_{xxx}\alpha - 4\sqrt{\alpha-\beta}\cos(\Gamma)u_t\alpha \\
& +\sqrt{s}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u^2 - s^{3/2}\sqrt{\alpha^2-1}\sin(\Gamma)u - s\sqrt{\alpha-\beta}\cos(\Gamma)u_x \\
& +2\sqrt{\alpha-\beta}\beta\cos(\Gamma)uu_x + \sqrt{s}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u_{xx} + \sqrt{\alpha-\beta}\beta\cos(\Gamma)u_{xxx} \\
& +4\sqrt{\alpha-\beta}\beta\cos(\Gamma)u_t\Big\}, \\
T_6^x = \ & \frac{1}{4(\alpha-\beta)^{3/2}}\Big\{3s^{5/2}\sqrt{\alpha^2-1}\sin(\Gamma)u\alpha^3 + 2s^2\sqrt{\alpha-\beta}\cos(\Gamma)u_x\alpha^3 \\
& +s^{3/2}\sqrt{\alpha^2-1}\sin(\Gamma)u^2\alpha^2 - 4s^{5/2}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u\alpha^2 - 4s^2\sqrt{\alpha-\beta}\beta\cos(\Gamma)u_x\alpha^2 \\
& +4s\sqrt{\alpha-\beta}\cos(\Gamma)uu_x\alpha^2 - s^{3/2}\sqrt{\alpha^2-1}\sin(\Gamma)u_{xx}\alpha^2 + s\sqrt{\alpha-\beta}\cos(\Gamma)u_t\alpha^2 \\
& -s^{3/2}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u^2\alpha + s^{5/2}\sqrt{\alpha^2-1}\sin(\Gamma)u\alpha + 2s^2\sqrt{\alpha-\beta}\cos(\Gamma)u_x\alpha \\
& -4s\sqrt{\alpha-\beta}\beta\cos(\Gamma)uu_x\alpha + 5s^{3/2}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u_{xx}\alpha + 4s\sqrt{\alpha-\beta}\beta\cos(\Gamma)u_{xxx}\alpha \\
& -2\sqrt{\alpha-\beta}\cos(\Gamma)uu_t\alpha - 2\sqrt{s}\sqrt{\alpha^2-1}\sin(\Gamma)u_{tx}\alpha - 3\sqrt{\alpha-\beta}\cos(\Gamma)u_{txx}\alpha \\
& -4s^{3/2}\sqrt{\alpha^2-1}\beta^2\sin(\Gamma)u_{xx} - 4s\sqrt{\alpha-\beta}\beta^2\cos(\Gamma)u_{xxx} - s\sqrt{\alpha-\beta}\cos(\Gamma)u_t \\
& +2\sqrt{\alpha-\beta}\beta\cos(\Gamma)uu_t + 2\sqrt{s}\sqrt{\alpha^2-1}\beta\sin(\Gamma)u_{tx} + 3\sqrt{\alpha-\beta}\beta\cos(\Gamma)u_{txx}\Big\}.
\end{aligned}
$$

3. Concluding remarks

In this work we studied the two-wave Korteweg-de Vries equation via Lie symmetry method and applied the simplest equation method to generate travelling wave solutions. Finally, derived conservation laws using multiplier approach.

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