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New Fractional Operators Theory and Applications

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Abstract

In this article, we present a new fractional integral with a non-singular kernel and by using Laplace transform, we derived the corresponding fractional derivative. By composition between our fractional integration operator with classical Caputo and Riemann-Liouville fractional operators, we establish a new fractional derivative which is interpolated between the generalized fractional derivatives in a sense Riemann-Liouville and Caputo-Fabrizio with non-singular kernels. Additionally, we introduce the fundamental properties of these fractional operators with applications and simulations. Finally, a model of Coronavirus (COVID-19) transmission is presented as an application.

Keywords: Fractional integral; fractional derivative; non-singular kernels; Mittag-Leffler function; Coronavirus (COVID-19).

1. Introduction

Fractional calculus has become a popular and significant area of study. It is due mainly to the widespread use of fractional differential equations in various technical and scientific fields, including physics, biology, chemistry, control theory, economics, signal and image processing, blood flow phenomena, biophysics, aerodynamics and data fitting[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Fractional derivatives are also well-suited to describing the memory and heredity characteristics of different materials and processes. Because of these properties of fractional derivatives, fractional-order models are seen to be more realistic and practical than integer-order models, which ignore such effects. There are many kinds of fractional derivatives, like Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov, and Hilfer, for more details; see [11, 12, 13, 14].

By developing the kernel of the fractional derivative (integral) operator, the researchers tried to reach the best description of the mathematical models for many real-world problems. Lately,

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many researchers have been interested in developing new types of fractional operators without singular kernels. Caputo and Fabrizio, in [15], introduced a new fractional derivative without singular (exponential) kernel, and in [16], they presented some of its applications. Atangana and Baleanu in [17] suggested a new definition of a fractional derivative without singular (Mittag-Leffler) kernel for Riemann–Liouville and Caputo sense. Al-Refai in [18] presented the definition of the weighted Atangana–Baleanu fractional derivative in a Caputo sense. In 2020, Hattaf [19] presented a generalization definition of the fractional derivative given in [18]; also, he introduced the weighted Atangana–Baleanu fractional derivative in a Riemann–Liouville sense with related fractional integral. The fractional derivatives without singular kernels gave adequately described for models of dissipative phenomena where the classical fractional operators cannot give it, see [20, 21, 21].

This work aims to establish and investigate properties of a new definition of fractional derivative, as an interpolate fractional derivative between generalized fractional derivatives Caputo-Fabrizio and Riemann-Liouville with Mittag- Leffler kernels. The rest of the article is structured as follows. In Sec. 2, we mention some important results and definitions related to the fractional calculus. In Sec. 3, we present a new fractional integral with a non-singular kernel and establish some properties and application with simulation. In Sec. 4, we derive a fractional derivative associated with the fractional integral in Sec. 3 and its properties with simulation. As for Sec. 5, it represents our main purpose in this article, where we introduce a new fractional derivative with fractional integral related to it and establish their fundamental properties with applications and simulations. In Sec. 6, an application is presented.

2. Basics and Preliminaries

In this section, we recall some definitions, lemmas and notations, which help us later to establish our main results. For more details, we refer to the references [13, 14].

Definition 2.1. *The Mittag-Leffler function of one parameter is defined as*

$$E_\rho(\theta t^\rho) = \sum_{j=0}^{\infty} \theta^j \frac{t^{\rho j}}{\Gamma(\rho j + 1)}, \quad (0 \neq \theta \in \mathbb{R}, t \in \mathbb{C}, \rho > 0), \tag{2.1}$$

and the generalization of the Mittag-Leffler function with two parameters ρ and σ is given by

$$E_{\rho,\sigma}(\theta t^\rho) = \sum_{j=0}^{\infty} \theta^j \frac{t^{\rho j}}{\Gamma(\rho j + \sigma)}, \quad (0 \neq \theta \in \mathbb{R}, t, \sigma \in \mathbb{C}, \rho > 0), \tag{2.2}$$

where $E_{\rho,1}(\theta t^\rho) = E_\rho(\theta t^\rho)$.

We recall that the Laplace transform of the Mittag-Leffler functions (2.1) and (2.2)

$$\mathcal{L}\left\{E_\rho(\theta t^\rho)\right\}(\lambda) = \frac{\lambda^{\rho-1}}{\lambda^\rho - \theta}, \qquad \mathcal{L}\left\{t^{\sigma-1} E_{\rho,\sigma}(\theta t^\rho)\right\}(\lambda) = \frac{\lambda^{\rho-\sigma}}{\lambda^\rho - \theta}, \quad \text{respectively.} \tag{2.3}$$

It should be noted, the Mittag-Leffler function is a generalization function of the exponential function, where $E_1(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = exp(z)$.

Let $-\infty < \hat{a} < a < \infty$. Let $\mathrm{C}_\delta[\hat{a}, a]$ be the weighted space of continuous functions $\mathrm{C}_\delta[\hat{a}, a] = \{x : (\hat{a}, a] \to \mathbb{R} : \ (t-\hat{a})^\delta x(t) \in \mathrm{C}[\hat{a}, a]\}, \ \delta \in [0, 1)$, with the norm

$$\|x\|_{\mathrm{C}_\delta} = \left\|(t-\hat{a})^\delta x(t)\right\|_{\mathrm{C}},$$

and for $n \in \mathbb{N}$, let $C_{\delta}^{n}[\hat{a},\ a]$ be the space of continuously differentiable on $[\hat{a},\ a]$ up to order $n-1$ such that $D^{n}x(t) \in C_{\delta}[\hat{a}, a]$ $\left(D^{n} = \frac{d^{n}}{dt^{n}}\right)$, with the norm

$$\|x\|_{C_{\delta}^{n}} = \sum_{k=0}^{n-1} \left\|D^{k}x\right\|_{C} + \|D^{n}x\|_{C_{\delta}}.$$

Remark 2.2. *The above spaces have the following properties*

i. $C_{0}[\hat{a},\ a] = C[\hat{a},\ a]$,
ii. $C_{\delta}^{0}[\hat{a},\ a] = C_{\delta}[\hat{a},\ a]$,
iii. *if* $0 \leq \delta_{1} < \delta_{2} < 1$, *then* $C_{\delta_{1}}[\hat{a},\ a] \subset C_{\delta_{2}}[\hat{a},\ a]$.

Next, we recall the following definitions related to fractional calculus.

Definition 2.3. *The fractional integral of order $q > 0$ with the lower limit $\hat{a}$ for a function $\zeta \in C_{\delta}[\hat{a},\ a]$ and defined by*

$${}^{RL}I_{\hat{a}}^{q}\zeta(\epsilon) = \frac{1}{\Gamma(q)} \int_{\hat{a}}^{\epsilon} (\epsilon - s)^{q-1} \zeta(s)\, ds, \qquad \epsilon \in (\hat{a},\ a],\ q > 0,$$

is called Riemann-Liouville, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.4. *For a function $\zeta \in C_{\delta}[\hat{a},\ a]$ the expression*

$${}^{RL}D_{\hat{a}}^{q}\zeta(\epsilon) = \frac{1}{\Gamma(1-q)} \frac{d}{d\epsilon} \int_{\hat{a}}^{\epsilon} (\epsilon - s)^{-q} \zeta(s)\, ds = DI_{\hat{a}}^{1-q}\zeta(\epsilon), \quad \epsilon \in (\hat{a},\ a] \qquad 0 < q < 1,$$

is called Riemann-Liouville derivative of order q.

Definition 2.5. *For a function $\zeta(\epsilon) \in C_{\delta}^{1}[\hat{a},\ a]$ the expression*

$${}^{C}D_{\hat{a}}^{q}\zeta(\epsilon) = \frac{1}{\Gamma(1-q)} \int_{\hat{a}}^{\epsilon} (\epsilon - s)^{-q} \dot{\zeta}(s)\, ds = I_{\hat{a}}^{1-q}D\zeta(\epsilon), \qquad \epsilon \in (\hat{a},\ a] \qquad 0 < q < 1,$$

is called the Caputo derivative of order q.

Lemma 2.6. *For $t > \hat{a}$ we have*

i. $\left[{}^{RL}I_{\hat{a}}^{q}(\tau - \hat{a})^{p-1}\right](t) = \frac{\Gamma(p)}{\Gamma(p+q)}(t - \hat{a})^{p+q-1}$ $\qquad q \geq 0, p > 0,$
ii. $\left[{}^{RL}D_{\hat{a}}^{q}(\tau - \hat{a})^{q-1}\right](t) = 0$ $\qquad 0 < q < 1.$

Lemma 2.7. *For $q > 0$, $I_{\hat{a}}^{q}$ maps $C[\hat{a},\ a]$ into $C[\hat{a},\ a]$.*

Lemma 2.8. *Let $q > 0$ and $\delta \in [0, 1)$. Then ${}^{RL}I_{\hat{a}}^{q}$ is bounded from $C_{\delta}[\hat{a},\ a]$ into $C_{\delta}[\hat{a},\ a]$:*

$$\left\|{}^{RL}I_{\hat{a}}^{q}x\right\|_{C_{\delta}} \leq (a - \hat{a})^{q} \frac{\Gamma(1-\delta)}{\Gamma(q+1-\delta)} \|x\|_{C_{\delta}}$$

Lemma 2.9. *Let $q > 0$ and $\delta \in [0, 1)$. If $\delta \leq q$, then ${}^{RL}I_{\hat{a}}^{q}$ is bounded from $C_{\delta}[\hat{a},\ a]$ into $C[\hat{a},\ a]$.*

Lemma 2.10 (semigroup property). *Let* $q > 0, \delta \in [0,1)$, $p \geq 0$ *and* $\zeta \in C_\delta[\hat{a},\ a]$. *Then*

$$^{RL}I_{\hat{a}}^{p}\left(^{RL}I_{\hat{a}}^{q}\zeta\right)(t) = \left(^{RL}I_{\hat{a}}^{p+q}\zeta\right)(t), \qquad t \in (\hat{a},\ a].$$

Lemma 2.11. *Let* $0 < q < 1$, $\delta \in [0,1)$ *and* $\zeta \in C_\delta[\hat{a},\ a]$. *Then*

$$^{RL}D_{\hat{a}}^{q}\left(^{RL}I_{\hat{a}}^{q}\zeta\right)(t) = \zeta(t) \qquad \text{for all} \ \ t \in (\hat{a}, a]$$

Lemma 2.12. *Let* $0 < q < 1$, $\delta \in [0,1)$. *If* $\zeta \in C_\delta[\hat{a},\ a]$ *and* $^{RL}I_{\hat{a}}^{q-1}\zeta \in C_\delta^1[\hat{a},\ a]$, *then*

$$^{RL}I_{\hat{a}}^{q}\left(^{RL}D_{\hat{a}}^{q}\zeta\right)(t) = \zeta(t) - \frac{\left(^{RL}I_{\hat{a}}^{1-q}\zeta\right)(\hat{a})}{\Gamma(q)}(t-\hat{a})^{q-1} \qquad \text{for all } t \in (\hat{a},\ a].$$

Lemma 2.13. *If* $0 < q < 1$ *and* $\zeta \in C_\delta[\hat{a},\ a]$, *then*

$$\left(^{RL}I_{\hat{a}}^{q\,C}D_{\hat{a}}^{q}\zeta\right)(t) = \zeta(t) - \zeta(\hat{a}) \qquad \text{for all } t \in (\hat{a},\ a].$$

Remark 2.14. *For* $q \in (0,1), \lambda > 0$ *the Laplace transform of operators* $^{RL}I_0^q, {}^{RL}D_0^q$*and* $^{c}D_0^q$ *are given by*

$$\begin{aligned}
&\mathcal{L}\left\{\left(^{RL}I_0^q\zeta\right)(t)\right\}(\lambda) = \lambda^{-q}\mathcal{L}\{\zeta(t)\}(\lambda),\\
&\mathcal{L}\left\{\left(^{RL}D_0^q\zeta\right)(t)\right\}(\lambda) = \lambda^{q}\mathcal{L}\{\zeta(t)\}(\lambda) - \left(^{RL}I_{\hat{a}}^{1-q}\zeta\right)(0),\\
&\mathcal{L}\left\{\left(^{C}D_0^q\zeta\right)(t)\right\}(\lambda) = \lambda^{q}\mathcal{L}\{\zeta(t)\}(\lambda) - \lambda^{q-1}(\zeta)(0),
\end{aligned}$$

respectively.

3. Generalized Fractional Integral

In this section, we introduce a new fractional integral definition by generalizing the Caputo-Fabrizio fractional integral[16], where we replace the exponential kernel with the Mittag-Leffler kernel. Additionally, we establish properties for this integral.

Definition 3.1. *Let* $0<\alpha< 1$,$\beta> 0$ *and* $x \in C_\delta[\hat{a},\ a]$. *The new fractional integral of order* α,β *of the function* $x(t)$ *is defined as follows*

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t) = \frac{1}{\alpha}\int_{\hat{a}}^{t} E_\beta\left[-\mathcal{C}_\alpha(t-s)^\beta\right]x(s)\ ds, \tag{3.1}$$

where $\mathcal{C}_\alpha = \frac{1-\alpha}{\alpha}$ *and* $E_\beta(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\beta j+1)}$ *is Mittag-Leffler function.*

To show that the new fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ is actuality a generalization of the Caputo-Fabrizio fractional integral [16]. Taking β= 1 leads to

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,1}x\right)(t) = \frac{1}{\alpha}\int_{\hat{a}}^{t} E_1\left[-\mathcal{C}_\alpha(t-s)^1\right]x(s)\ ds = \frac{1}{\alpha}\int_{\hat{a}}^{t} exp\left[-\mathcal{C}_\alpha(t-s)\right]x(s)\ ds,$$

Below we establish some properties related to the fractional integral operator $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$.

Theorem 3.2. *For any α, β , x satisfying the conditions from Definition 3.1, the fractional integral operator $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ can be written as*

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x\right)(t) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(-\mathcal{C}_\alpha\right)^n \left({}^{RL}I_{\hat{a}}^{n\beta+1} x\right)(t). \tag{3.2}$$

Proof . According to Definition 2.1 and Definition 2.3, we have

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x\right)(t) = \frac{1}{\alpha} \int_{\hat{a}}^{t} x(s) \sum_{n=0}^{\infty} \frac{\left(-\mathcal{C}_\alpha\right)^n}{\Gamma(n\beta+1)} (t-s)^{n\beta} ds$$

$$= \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{\left(-\mathcal{C}_\alpha\right)^n}{\Gamma(n\beta+1)} \int_{\hat{a}}^{t} x(s)(t-s)^{n\beta} ds = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(-\mathcal{C}_\alpha\right)^n \left({}^{RL}I_{\hat{a}}^{n\beta+1} x\right)(t),$$

where ${}^{RL}I_{\hat{a}}^{\gamma}$ is the standard Riemann-Liouville fractional integral of order γ. □

Dealing with convergent series is essential in fractional calculus, so the series which is given in Theorem 3.2 will be helpful to study the properties of fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$.

Remark 3.3. *By linearity of Riemann-Liouville fractional integral, the fractional $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ is linear operator, that is*

$$\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left[ux(t) + vy(t)\right] = u\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) + v\mathcal{K}_{\hat{a}}^{\alpha,\beta} y(t),$$

holds for all scalars u, v and $x, y \in \mathrm{C}_\delta[\hat{a},\ a]$.

According to the analytically of the Mittag-Leffler function $E_\beta\left[-\mathcal{C}_\alpha(t-s)^\beta\right]$ at any point s in the interval $[\hat{a}, t)$. Also, we know that $\lim_{s\to^- t} E_\beta\left[-\mathcal{C}_\alpha(t-s)^\beta\right] = 1$, and by continuity of $x(t)$ the fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ is well-defined.

Lemma 3.4. *For $0 \le \delta < 1$ and $\alpha \in (0,1)$, the fractional integration operator $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ is bounded from $\mathrm{C}_\delta[\hat{a},\ a]$ into space $\mathrm{C}[\hat{a},\ a]$:*

$$\left\|\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x\right)(t)\right\|_C \le M\|x\|_{\mathrm{C}_\delta}, \tag{3.3}$$

where

$$M = \frac{\Gamma(1-\delta)}{\alpha} (a-\hat{a})^{1-\delta} E_{\beta,2-\delta}\left[\mathcal{C}_\alpha(a-\hat{a})^\beta\right]$$

Proof . Note that by the definition of the space $\mathrm{C}_\delta[\hat{a},\ a]$ and Lemma 2.9, we have $(t-\hat{a})^\delta x(t) \in \mathrm{C}[\hat{a},\ a]$ and

$$\left\|\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x\right)(t)\right\|_C = \left\|\frac{1}{\alpha} \sum_{n=0}^{\infty} \left(-\mathcal{C}_\alpha\right)^n \left({}^{RL}I_{\hat{a}}^{n\beta+1} x\right)(t).\right\|_C \le \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(\mathcal{C}_\alpha\right)^n \left\|\left({}^{RL}I_{\hat{a}}^{n\beta+1} x\right)(t)\right\|_C,$$

Lemma 2.6 (i), gives

$$\left\|\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x\right)(t)\right\|_C \le \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(\mathcal{C}_\alpha\right)^n \frac{\Gamma(1-\delta)}{\Gamma(k\beta+2-\delta)} (t-\hat{a})^{k\beta+1-\delta} \left\|x(t)\right\|_{\mathrm{C}_\delta} \le M\|x\|_{\mathrm{C}_\delta}.$$

□

Lemma 3.5. *Let* $x(t) \in \mathrm{C}_\delta[\hat{a}, a]$. *Then* $\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) \in \mathrm{C}_\delta^1[\hat{a}, a]$.

Proof . By the definition of the space $\mathrm{C}_\delta^1[\hat{a}, a]$, it suffices to show that $D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) \in \mathrm{C}_\delta[\hat{a}, a]$. For this, let $\{x_k\}_{k=1}^{\infty}$ be any convergent sequence in the space $\mathrm{C}_\delta[\hat{a}, a]$, i.e.
there exists $x \in \mathrm{C}_\delta[\hat{a}, a]$ such that $x_k \to x$, as $k \to \infty$. Now we need to prove that
$D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x_k(t) \to D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t)$, as $k \to \infty$, using Remark 3.3, yields

$$\left\| D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x_k(t) - D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) \right\|_{\mathrm{C}_\delta} = \left\| D\mathcal{K}_{\hat{a}}^{\alpha,\beta}[x_k(t) - x(t)] \right\|_{\mathrm{C}_\delta} = \left\| \frac{1}{\alpha} \frac{d}{dt} \int_{\hat{a}}^{t} E_\beta \left[-\mathcal{C}_\alpha (t-s)^\beta \right] [x_k(s) - x(s)] \, ds \right\|_{\mathrm{C}_\delta}$$

Leibniz integral rule gives us

$$\left\| D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x_k(t) - D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) \right\|_{\mathrm{C}_\delta} = \left\| \frac{1}{\alpha} [x_k(t) - x(t)] + \frac{1}{\alpha} \int_{\hat{a}}^{t} [x_k(s) - x(s)] \left(\frac{\partial}{\partial t} E_{\beta} \left[-\mathcal{C}_\alpha (t-s)^\beta \right] \right) ds \right\|_{\mathrm{C}_\delta}$$
$$= \left\| \frac{1}{\alpha} [x_k(t) - x(t)] + \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-\mathcal{C}_\alpha)^n}{\Gamma(n\beta)} \int_{\hat{a}}^{t} [x_k(s) - x(s)] (t-s)^{n\beta - 1} ds \right\|_{\mathrm{C}_\delta}$$

Since the Mittag-Leffler function and its t-derivative are analytic functions at every point s in $[\hat{a}, t)$ and the Riemann fractional integrals ${}^{RL}I_{\hat{a}}^{n\beta}$ $(n = 1, 2, ..)$ is well-defined for any function in the space $\mathrm{C}_\delta[\hat{a}, a]$, then the integrals

$$\int_{\hat{a}}^{t} [x_k(s) - x(s)] (t-s)^{n\beta - 1} ds, \quad n = 1, 2, ..$$

are converges. Hence

$$\left\| D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x_k(t) - D\mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) \right\|_{\mathrm{C}_\delta} \leq \frac{1}{\alpha} \| x_k(t) - x(t) \|_{\mathrm{C}_\delta} + \frac{1}{\alpha} \sum_{n=0}^{\infty} (-\mathcal{C}_\alpha)^{n} {}^{RL}I_{\hat{a}}^{n\beta} \| x_k(t) - x(t) \|_{\mathrm{C}_\delta}$$

By our assumption, the right-hand side leads to zero whenever $k \to \infty$. □

Lemma 3.6. *Let* $\alpha \in (0, 1)$, $0 \leq \delta < 1$ *and* $x \in \mathrm{C}_\delta[\hat{a}, a]$. *Then the fractional integration operator* $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ *is bounded in* $\mathrm{C}_\delta[\hat{a}, a]$ *:*

$$\left\| \left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x \right)(t) \right\|_{\mathrm{C}_\delta} \leq \overline{M} \| x \|_{\mathrm{C}_\delta},$$

Where $\overline{M} = \frac{\Gamma(1-\delta)}{\alpha} (a - \hat{a}) E_{\beta, 2-\delta} \left[\mathcal{C}_\alpha (a - \hat{a})^\beta \right]$.
Moreover

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x \right)(a) := \lim_{t \to a} \mathcal{K}_{\hat{a}}^{\alpha,\beta} x(t) = 0.$$

Proof . Since $x \in \mathrm{C}_\delta[\hat{a}, a]$ then $(t - \hat{a})^\delta x(t) \in \mathrm{C}[\hat{a}, a]$ and thus

$$\| x(t) \|_{\mathrm{C}_\delta} = \left\| (t - \hat{a})^\delta x(t) \right\|_{\mathrm{C}}.$$

Using Theorem 3.2, we get that

$$\left\| \left(\mathcal{K}_{\hat{a}}^{\alpha,\beta} x \right)(t) \right\|_{\mathrm{C}_\delta} = \left\| \frac{1}{\alpha} \sum_{k=0}^{\infty} (-\mathcal{C}_\alpha)^k \left({}^{RL}I_{\hat{a}}^{k\beta+1} x \right)(t) \right\|_{\mathrm{C}_\delta} \leq \frac{1}{\alpha} \sum_{k=0}^{\infty} (\mathcal{C}_\alpha)^k \left\| \left({}^{RL}I_{\hat{a}}^{k\beta+1} x \right)(t) \right\|_{\mathrm{C}_\delta},$$

on the other hand, we have

$$\left\|\left({}^{RL}I_{\hat{a}}^{k\beta+1}x\right)(t)\right\|_{C_\delta} \le \frac{(t-\hat{a})^{\delta}}{\Gamma(k\beta+1)}\int_{\hat{a}}^{t}(t-s)^{k\beta}(s-\hat{a})^{-\delta}\|x(s)\|_{C_\delta}\, ds$$

let $s=\hat{a}+\gamma(t-\hat{a})$. Then

$$\int_{\hat{a}}^{t}(t-s)^{k\beta}(s-\hat{a})^{-\delta}\, ds=\int_{0}^{1}(t-\tau)^{k\beta}(1-\hat{a})^{k\beta}(t-\hat{a})^{-\delta}\gamma^{-\delta}(t-\hat{a})\, d\gamma$$

Hence

$$\left\|\left({}^{RL}I_{\hat{a}}^{k\beta+1}x\right)(t)\right\|_{C_\delta} \le \frac{\Gamma(1-\delta)}{\Gamma(k\beta+2-\delta)}(t-\hat{a})^{k\beta+1}\|x\|_{C_\delta},$$

and then

$$\begin{aligned}\left\|\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t)\right\|_{C_\delta} &\le \frac{1}{\alpha}\sum_{k=0}^{\infty}(\mathcal{C}_\alpha)^k\frac{\Gamma(1-\delta)}{\Gamma(k\beta+2-\delta)}(t-\hat{a})^{k\beta+1}\|x\|_{C_\delta}\\ &=\frac{\Gamma(1-\delta)}{\alpha}(t-\hat{a})E_{\beta,2-\delta}\left[\mathcal{C}_\alpha(t-\hat{a})^{\beta}\right]\|x\|_{C_\delta}\le \overline{M}\|x\|_{C_\delta}\end{aligned}$$

Directly we conclude that the right-hand side approach to zero as $t\to\hat{a}$. □

In the following result, we derive the formula for Laplace transform of $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$.

Lemma 3.7. *The Laplace transform of fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ is given by*

$$\mathcal{L}\left\{\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t)\right\}(\lambda)=\frac{\lambda^{\beta-1}}{\alpha\lambda^{\beta}+(1-\alpha)}\mathcal{L}\left\{x\left(t\right)\right\}(\lambda).$$

Proof . According to the Laplace convolution operator theorem, we have

$$\mathcal{L}\left\{\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t)\right\}(\lambda)=\frac{1}{\alpha}\mathcal{L}\left\{\ E_{\beta}\left[-\mathcal{C}_\alpha t^{\beta}\right]\right\}(\lambda)\,\mathcal{L}\left\{x\left(t\right)\right\}(\lambda)\ ,$$

and by identity (2.3)

$$\mathcal{L}\left\{\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t)\right\}(\lambda)=\frac{1}{\alpha}\frac{\lambda^{\beta-1}}{\lambda^{\beta}+\mathcal{C}_\alpha}\mathcal{L}\left\{x\left(t\right)\right\}(\lambda)=\frac{\lambda^{\beta-1}}{\alpha\lambda^{\beta}+(1-\alpha)}\mathcal{L}\left\{x\left(t\right)\right\}(\lambda).$$

□

Now, we apply the fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ of a power function $(t-\hat{a})^{p}, p>-1$, that is we derive $\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}(\tau-\hat{a})^{p}\right)(t)$ where we have use Theorem 3.2 as follows

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}(\tau-\hat{a})^{p}\right)(t)=\frac{1}{\alpha}\sum_{n=0}^{\infty}(-\mathcal{C}_\alpha)^{n}\left({}^{RL}I_{\hat{a}}^{n\beta+1}(t-\hat{a})^{p}\right)(t),$$

and from Lemma 2.6 (i), we get

$$\begin{aligned}\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}(\tau-\hat{a})^{p}\right)(t) &= \frac{1}{\alpha}\sum_{n=0}^{\infty}(-\mathcal{C}_\alpha)^{n}\frac{\Gamma(p+1)}{\Gamma(n\beta+p+2)}(t-\hat{a})^{n\beta+p+1}\\ &= \frac{\Gamma(p+1)}{\alpha}(t-\hat{a})^{p+1}E_{\beta,\mathrm{p}+2}\left[-\mathcal{C}_\alpha(t-\hat{a})^{\beta}\right]\end{aligned} \tag{3.4}$$

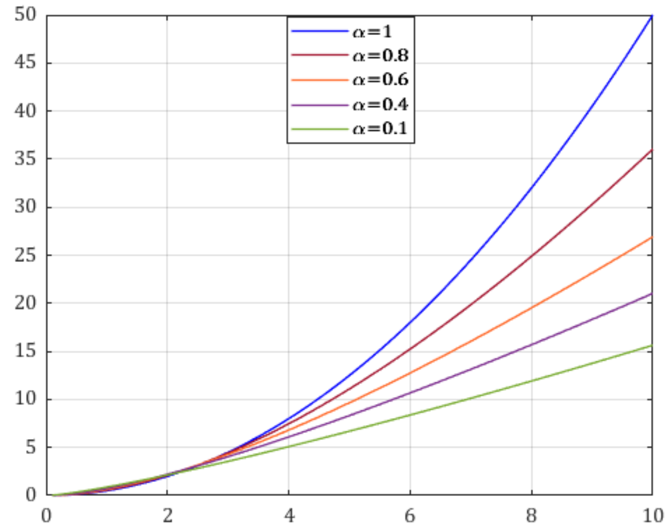

Figure 1: Graph of (3.4) for $\hat{a}=0, \beta=0.8$, $p=1$ and deference values of α.

Property 3.8. If $x(t) \in \mathrm{C}_{\delta}[\hat{a},\ a]$, then for $\alpha,\omega \in (0,1), 0 \leq \delta < 1$,

$$\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left(\mathcal{K}_{\hat{a}}^{\omega,\beta}x\right)(t)=\mathcal{K}_{\hat{a}}^{\omega,\beta}\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t).$$

That is, the fractional integral operators $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ are commutative.

Proof . We have

$$\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left(\mathcal{K}_{\hat{a}}^{\omega,\beta}x\right)(t)=\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left[\frac{1}{\omega}\sum_{n=0}^{\infty}(-\mathcal{C}_{\omega})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta+1}x\right)\right](t)$$

$$=\frac{1}{\alpha}\sum_{k=0}^{\infty}(-\mathcal{C}_{\alpha})^{k}\left({}^{RL}I_{\hat{a}}^{k\beta+1}\left[\frac{1}{\omega}\sum_{n=0}^{\infty}(-\mathcal{C}_{\omega})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta+1}x\right)\right]\right)(t)$$

$$=\frac{1}{\omega}\frac{1}{\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{k}(-\mathcal{C}_{\omega})^{n}\,{}^{RL}I_{\hat{a}}^{k\beta+1}\left({}^{RL}I_{\hat{a}}^{n\beta+1}x\right)(t)$$

$$=\frac{1}{\omega}\frac{1}{\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{k}(-\mathcal{C}_{\omega})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta+k\beta+2}x\right)(t).$$

Which completes the proof. □

4. Fractional Derivative Associated to the Fractional Integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$

After introducing the fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ of order $\alpha \in (0,1)$, it became necessary to introduce a fractional derivative of order $\alpha \in (0,1)$. In this section, we establish a fractional derivative associated with the fractional integral.

Consider the following integral equation

$$\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x\right)(t)=y(t), \quad \alpha \in (0,1), \beta > 0. \tag{4.1}$$

Using Laplace transform and applying **Lemma 3.7** leads to

$$\frac{\lambda^{\beta-1}}{\alpha\lambda^{\beta}+(1-\alpha)}\mathcal{L}\left\{x\left(t\right)\right\}\left(\lambda\right)=\mathcal{L}\left\{y\left(t\right)\right\}\left(\lambda\right),$$

or equivalently,

$$\mathcal{L}\left\{x\left(t\right)\right\}\left(\lambda\right)=\alpha\lambda\mathcal{L}\left\{y\left(t\right)\right\}\left(\lambda\right)+\frac{1-\alpha}{\lambda^{\beta-1}}\mathcal{L}\left\{y\left(t\right)\right\}\left(\lambda\right).$$

According to Lemma 3.6 and by using the inverse Laplace we get

$$x\left(t\right)=\alpha Dy\left(t\right)+\left(1-\alpha\right){}^{RL}I_{\hat{a}}^{\beta-1}y\left(t\right) \tag{4.2}$$

Based on the preceding, we will present the definition of the fractional derivative that we will be denoted by ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$. Also, we establish some properties related to this derivative.

Definition 4.1. *Let* $0<\alpha<1$, $\beta>0$ and $x\in C_{\delta}^{1}\left[\hat{a},\ a\right]$. *We define the corresponding fractional derivative to the fractional integral* $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ *as follows*

$$\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\ x\right)\left(t\right)=\alpha Dx\left(t\right)+\left(1-\alpha\right){}^{RL}I_{\hat{a}}^{\beta-1}x\left(t\right). \tag{4.3}$$

When $\alpha=1,\beta>0$, *we obtain the first-order derivative. Otherwise, if* $\alpha=0,\beta=1$ *,then* $\left({}^{\mathcal{K}}D_{\hat{a}}^{0,1}\ x\right)\left(t\right)=x\left(t\right).$

From Definition 4.1, we notice that of the fractional derivative ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ is well defined if the Riemann fractional integral. Depending on the linearity of the Riemann fractional integral, the fractional derivative ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ is also a linear operator. It is easily seen that the fractional differential operator ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ belongs to space $C_{\delta}\left[\hat{a},\ a\right]$ follows from the definition of $C_{\delta}^{1}\left[\hat{a},\ a\right]$ and Lemma 2.8.

In the following result, we derive the formula of the Laplace transforms of fractional derivative ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$. Where we have used Laplace transforms of the first derivative and Reimann fractional integral.

Lemma 4.2. *The Laplace transform of the fractional differential operator* ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ *is given by*

$$\mathcal{L}\left\{\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x\right)\left(t\right)\right\}\left(\lambda\right)=\alpha\lambda\mathcal{L}\left\{x\left(t\right)\right\}\left(\lambda\right)+\alpha\ x\left(0\right)+\left(1-\alpha\right)\frac{1}{\lambda^{\beta-1}}\ \mathcal{L}\left\{x\left(t\right)\right\}\left(\lambda\right).$$

Lemma 4.3. *Let* $x\in C_{\delta}^{1}\left[\hat{a},\ a\right]$. *Then*

$$\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x\right)\left(\hat{a}\right)\ :=\lim_{t\to\hat{a}}\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x\right)\left(t\right)=0\ ,$$

Proof . Since $x\in C_{\delta}^{1}\left[\hat{a},\ a\right]$, then $x\in C\left[\hat{a},\ a\right]$ and $Dx\left(t\right)\in C_{\delta}\left[\hat{a},\ a\right]$. From Remark 2.2 and Lemma 2.8, we have the following

$$\left\|\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x\right)\left(t\right)\right\|_{C_{\delta}}\leq\alpha\|Dx\left(t\right)\|_{C_{\delta}}+\left(1-\alpha\right)\left\|\left({}^{RL}I_{\hat{a}}^{\beta-1}x\right)\left(t\right)\right\|_{C_{\delta}},$$

$$\leq\alpha\left\|\left(t-\hat{a}\right)^{\delta}Dx\left(t\right)\right\|_{C}+\left(1-\alpha\right)\left\|\left(t-\hat{a}\right)^{\delta}\left({}^{RL}I_{\hat{a}}^{\beta-1}x\right)\left(t\right)\right\|_{C}.$$

Consequently, $\left\|\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x\right)\left(t\right)\right\|_{C_{\delta}}$ approach to zero as $t\to\hat{a}$. □

The following result shows the relationship between the fractional operators ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ and $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$. In fact, the derivative operator ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ is a left inverse of integral operator $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ and their composition is not commutative.

Theorem 4.4. *Let* $\alpha\in(0,1)\,, 0\leq\delta<1$.

i. *If* $x(t)\in \mathrm{C}_{\delta}[\hat{a},\ a]$, *then*
$$ {}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x(t)\right)=x(t)\,. \tag{4.4}$$

ii. *If* $x(t)\in \mathrm{C}_{\delta}^{1}[\hat{a},\ a]$, *then*
$$ {}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x(t)\right)=x(t)-x(\hat{a})E_{\beta}\left[-\mathcal{C}_{\alpha}(t-\hat{a})^{\beta}\right]. \tag{4.5}$$

Proof . Our proof is based on Theorem 3.2, Definition 4.1 and Lemma 2.10, for (i) we have

$$\begin{aligned}
{}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x(t)\right) &= {}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\left[\frac{1}{\alpha}\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta+1}x\right)(t)\,.\right]\\
&=\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n}D\left({}^{RL}I_{\hat{a}}^{n\beta+1}x\right)(t)+\frac{(1-\alpha)}{\alpha}\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n\,RL}I_{\hat{a}}^{\beta-1}\left({}^{RL}I_{\hat{a}}^{n\beta+1}x\right)(t)\\
&=\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta}x\right)(t)-\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n+1}\left({}^{RL}I_{\hat{a}}^{(n+1)\beta}x\right)(t)\\
&=x(t)\,;
\end{aligned}$$

concerning (ii),

$$\begin{aligned}
\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x(t)\right) &= \mathcal{K}_{\hat{a}}^{\alpha,\beta}\left[\alpha Dx(t)+(1-\alpha)\,{}^{RL}I_{\hat{a}}^{\beta-1}x(t)\,.\right]\\
&=\alpha\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left(Dx(t)\right)+(1-\alpha)\,\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left({}^{RL}I_{\hat{a}}^{\beta-1}x(t)\right)\\
&=\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta+1}Dx\right)(t)+\frac{(1-\alpha)}{\alpha}\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n\,RL}I_{\hat{a}}^{n\beta+1}\left({}^{RL}I_{\hat{a}}^{\beta-1}x\right)(t)\\
&=\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n}\left({}^{RL}I_{\hat{a}}^{n\beta}\left[x(t)-x(\hat{a})\right]\right)+\frac{(1-\alpha)}{\alpha}\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n\,RL}I_{\hat{a}}^{n\beta+1}\left({}^{RL}I_{\hat{a}}^{\beta-1}x\right)(t)\\
&=\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n\,RL}I_{\hat{a}}^{n\beta}\left[x(t)-x(\hat{a})\right]-\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n+1\,RL}I_{\hat{a}}^{(n+1)\beta}(x)(t)\\
&=\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n\,RL}I_{\hat{a}}^{n\beta}x(t)-\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n+1\,RL}I_{\hat{a}}^{(n+1)\beta}x(t)-\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n\,RL}I_{\hat{a}}^{n\beta}x(\hat{a})
\end{aligned}$$

Lemma 2.6 (i) now leads to

$$=x(t)-x(\hat{a})\sum_{n=0}^{\infty}(-\mathcal{C}_{\alpha})^{n}\frac{(t-\hat{a})^{n\beta}}{\Gamma(n\beta+1)}$$

Therefore,

$$\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x(t)\right)=x(t)-x(\hat{a})\,E_{\beta}\left[-\mathcal{C}_{\alpha}(t-\hat{a})^{\beta}\right].$$

□

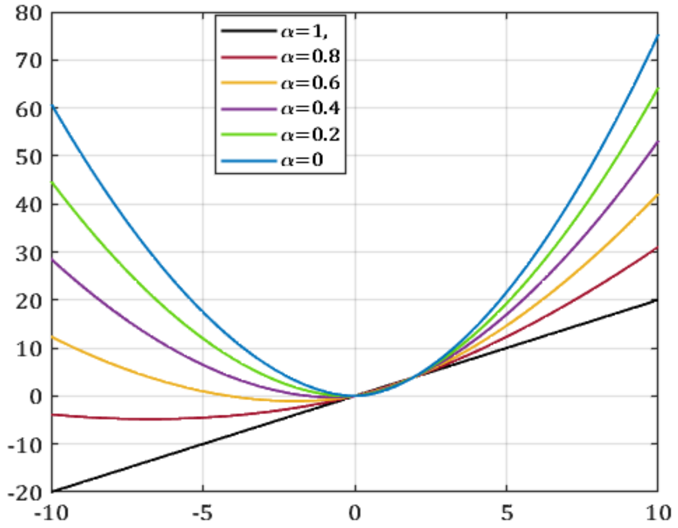

Figure 2: Graph of (4.6) for $\hat{a}=0, \beta=0.8$, $p=2$ and deference values of α

Let us consider the fractional ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$of a function $x(t)=(t-\hat{a})^p \quad ,p>-1$

$$\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}(\tau-\hat{a})^p\right)(t)=\alpha p(t-\hat{a})^{p-1}+(1-\alpha)\frac{\Gamma(p+1)}{\Gamma(\beta+p)}(t-\hat{a})^{\beta+p-1}. \tag{4.6}$$

In particular, if $p=0$, then

$$\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\ 1\right)(t)=(1-\alpha)\frac{(t-\hat{a})^{\beta-1}}{\Gamma(\beta)}$$

This means the fractional derivatives of a constant are, in general, not equal to zero.
The following is a direct result of Theorem 4.4.

Corollary 4.5. *If* $x(t)\in C_\delta^1[\hat{a},\ a]$, *and* $x(\hat{a})=0$,*then, for* $\alpha\in(0,1)\,,0\leq\delta<1$,

$$\mathcal{K}_{\hat{a}}^{\alpha,\beta}\left({}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}x(t)\right)={}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}\left(\mathcal{K}_{\hat{a}}^{\alpha,\beta}x(t)\right)=x(t).$$

It should be noted that the fractional derivatives operators ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ are, in general, not commutative operators. To illustrate this, we set that $x(t)=e^t$. Hence,

$$\begin{aligned}
{}^{\mathcal{K}}D_0^{\frac{1}{2},\frac{3}{2}}\left({}^{\mathcal{K}}D_0^{\frac{1}{3},\frac{3}{2}}\left(e^t\right)\right)&={}^{\mathcal{K}}D_0^{\frac{1}{2},\frac{3}{2}}\left[\frac{1}{3}D\left(e^t\right)+\frac{2}{3}I_0^{\frac{1}{2}}\left(e^t\right)\right]\\
&={}^{\mathcal{K}}D_0^{\frac{1}{2},\frac{3}{2}}\left[\frac{1}{3}e^t+\frac{2}{3}t^{\frac{1}{2}}E_{1,\frac{3}{2}}(t)\right]\\
&=\frac{1}{2}D\left[\frac{1}{3}e^t+\frac{2}{3}t^{\frac{1}{2}}E_{1,\frac{3}{2}}(t)\right]+\frac{1}{2}I_0^{\frac{1}{2}}\left[\frac{1}{3}e^t+\frac{2}{3}t^{\frac{1}{2}}E_{1,\frac{3}{2}}(t)\right]\\
&=\left[\frac{1}{6}e^t+\frac{1}{6}t^{-\frac{1}{2}}E_{1,\frac{1}{2}}(t)\right]+\left[\frac{1}{6}t^{\frac{1}{2}}E_{1,\frac{3}{2}}(t)+\frac{1}{3}tE_{1,2}(t)\right].
\end{aligned}$$

similarly,

$${}^{\mathcal{K}}D_0^{\frac{1}{3},\frac{3}{2}}\left({}^{\mathcal{K}}D_0^{\frac{1}{2},\frac{3}{2}}\left(e^t\right)\right)=\left[\frac{1}{6}e^t+\frac{1}{12}t^{-\frac{1}{2}}E_{1,\frac{1}{2}}(t)\right]+\left[\frac{1}{3}t^{\frac{1}{2}}E_{1,\frac{3}{2}}(t)+\frac{1}{3}tE_{1,2}(t)\right].$$

5. A New Fractional Derivative

Our goal in this section is to introduce a new fractional derivative and establish its properties.

Definition 5.1. *Let* $\alpha\in(0,1)$, $\beta>0$ *and let* $x(t)\in C_\delta[\hat{a},\ a]$ *such that* $\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)(t)\in C_\delta^1[\hat{a},\ a]$. *The fractional derivative for the function* $x(t)$ *of order* α,β *and type* $0\le\mu\le1$, *is defined as follows:*

$$\left(D_{\hat{a},\mu}^{\alpha,\beta}x\right)(t)=\left({}^{C}D_{\hat{a}}^{1-\mu}\mathcal{K}_{\hat{a}}^{1-\alpha,\beta RL}D_{\hat{a}}^{\mu}x\right)(t), \tag{5.1}$$

where ${}^{RL}D,{}^{c}D$ *are the Riemann-Liouville and Caputo fractional derivatives, respectively.*

It is not hard to show that the expression (5.1) is well defined. Indeed our condition $\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)(t)\in C_\delta^1[\hat{a},\ a]$ leads to ${}^{RL}D_{\hat{a}}^{\mu}x=D\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\in C_\delta[\hat{a},\ a]$. According to Lemma 3.5, we have $\mathcal{K}_{\hat{a}}^{1-\alpha,\beta}\left({}^{RL}D_{\hat{a}}^{\mu}x\right)\in C_\delta^1[\hat{a},\ a]$, this means $D\left(\mathcal{K}_{\hat{a}}^{1-\alpha,\beta RL}D_{\hat{a}}^{\mu}x\right)\in C_\delta[\hat{a},\ a]$. Now we already know from Lemma 2.8 that ${}^{RL}I_{\hat{a}}^{\mu}D\left(\mathcal{K}_{\hat{a}}^{1-\alpha,\beta RL}D_{\hat{a}}^{\mu}x\right)\in C_\delta[\hat{a},\ a]$, it follows that $\left({}^{C}D_{\hat{a}}^{1-\mu}\mathcal{K}_{\hat{a}}^{1-\alpha,\beta RL}D_{\hat{a}}^{\mu}x\right)(t)\in C_\delta[\hat{a},\ a]$.

It is worth noting that the above definition contains special cases existing previously, as follows:

1. When $\mu=1$, we get the Hattaf fractional derivative of Caputo sense ${}^{C}D_{\hat{a}}^{\alpha,\beta}$ [19] given by

$$\left(D_{\hat{a},1}^{\alpha,\beta}x\right)(t)=\left(\mathcal{K}_{\hat{a}}^{1-\alpha,\beta}(Dx)\right)(t)=\frac{1}{1-\alpha}\int_{\hat{a}}^{t}E_\beta\left[-\lambda_\alpha(t-s)^\beta\right](Dx)(s)\ ds.\left(\lambda_\alpha=\frac{1}{\mathcal{C}_\alpha}\right)$$

2. When $\mu=0$, we get the Hattaf fractional derivative of Riemann-Liouville derivative sense ${}^{R}D_{\hat{a}}^{\alpha,\beta}$ [19] given by

$$\left(D_{\hat{a},0}^{\alpha,\beta}x\right)(t)=\left(D\ \mathcal{K}_{\hat{a}}^{1-\alpha,\beta}x\right)(t)=\frac{1}{1-\alpha}\frac{d}{dt}\int_{\hat{a}}^{t}E_\beta\left[-\lambda_\alpha(t-s)^\beta\right]x(s)\ ds.$$

3. When $\mu=1,\beta=\alpha$ we obtain the Atangana-Baleanu (ABC)fractional derivative of Caputo sense${}^{ABC}D_{\hat{a}}^{\alpha}$ [17] given by

$$\left(D_{\hat{a},1}^{\alpha,\alpha}x\right)(t)=\left(\ \mathcal{K}_{\hat{a}}^{1-\alpha,\alpha}Dx\right)(t)=\frac{1}{1-\alpha}\int_{\hat{a}}^{t}E_\alpha\left[-\lambda_\alpha(t-s)^\alpha\right](Dx)(s)\ ds.$$

4. When $\mu=0,\beta=\alpha$ we obtain the Atangana-Baleanu(ABR) fractional derivative of Riemann-Liouville sense${}^{ABR}D_{\hat{a}}^{\alpha,\beta}$ [17]] given by

$$\left(D_{\hat{a},0}^{\alpha,\alpha}x\right)(t)=\left(\ D\mathcal{K}_{\hat{a}}^{1-\alpha,\alpha}x\right)(t)=\frac{1}{1-\alpha}\frac{d}{dt}\int_{\hat{a}}^{t}E_\alpha\left[-\lambda_\alpha(t-s)^\alpha\right]x(s)\ ds.$$

5. When $\mu=1,\beta=1$ we have the Caputo-Fabrizio fractional derivative [15] given by

$$\left(D_{\hat{a},1}^{\alpha,1}x\right)(t)=\left(\mathcal{K}_{\hat{a}}^{1-\alpha,1}(Dx)\right)(t)=\frac{1}{1-\alpha}\int_{\hat{a}}^{t}exp\left[-\lambda_\alpha(t-s)\right](Dx)(s)\ ds.$$

By the series formula (3.2), for any α,β,μ, x satisfying the conditions from Definition 5.1, obviously to see that the fractional derivative $D_{\hat{a},\mu}^{\alpha,\beta}$ can be expressed as

$$\left(D_{\hat{a},\mu}^{\alpha,\beta}x\right)(t)=\frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left({}^{C}D_{\hat{a}}^{1-\mu RL}I_{\hat{a}}^{n\beta+1RL}D_{\hat{a}}^{\mu}x\right)(t), \tag{5.2}$$

The above series is helpful for us to show some properties of the fractional derivative $D_{\hat{a},\mu}^{\alpha,\beta}$, for instance, the linearity of the fractional derivative $D_{\hat{a},\mu}^{\alpha,\beta}$ which we get from the linearity of the operators ${}^{RL}D,{}^{c}D,{}^{RL}I$, and others will be presented later.

Remark 5.2. *If* α= 0, *then*
$\left(D^{0,\beta}_{\widehat{a},\mu}x\right)(t) = \left({}^{C}D^{1-\mu}_{\hat{a}}\ I\ {}^{RL}D^{\mu}_{\hat{a}}x\right)(t)$, *particularly*

$$\left(D^{0,\beta}_{\widehat{a},\mu}x\right)(t) = \begin{cases} x(t) - x(\hat{a}), & \mu = 1, \\ x(t), & \mu = 0. \end{cases}$$

Theorem 5.3. *Let* α,β,μ and x *be satisfying the conditions from Definition 5.1. Then*

$$\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(t) = \left(D^{\alpha,\beta}_{\widehat{a},0}x\right)(t) - \frac{\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})}{1-\alpha}(t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_{\alpha}(t-s)^{\beta}\right] \tag{5.3}$$

Proof. Using the formula (5.2) and semigroup property of Riemann-Liouville fractional integral, we have

$$\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(t) = \frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_{\alpha})^{n}\left({}^{RL}I^{n\beta+\mu RL}_{\hat{a}}D^{\mu}_{\hat{a}}x\right)(t)$$

Applying Lemma 2.12, we obtain

$$\begin{aligned}
\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(t) &= \frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_{\alpha})^{nRL}I^{n\beta}_{\hat{a}}\left(x(t) - \frac{\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})}{\Gamma(\mu)}(t-\hat{a})^{\mu-1}\right) \\
&= \frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_{\alpha})^{nRL}I^{n\beta}_{\hat{a}}x(t) - \frac{1}{1-\alpha}\frac{\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})}{\Gamma(\mu)}\sum_{n=0}^{\infty}(-\lambda_{\alpha})^{nRL}I^{n\beta}_{\hat{a}}(t-\hat{a})^{\mu-1} \\
&= \frac{1}{1-\alpha}\frac{d}{dt}\sum_{n=0}^{\infty}(-\lambda_{\alpha})^{nRL}I^{n\beta+1}_{\hat{a}}x(t) - \frac{\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_{\alpha})^{n}\frac{(t-\hat{a})^{n\beta+\mu-1}}{\Gamma(n\beta+\mu)} \\
&= \left(D\ \mathcal{K}^{1-\alpha,\beta}_{\hat{a}}x\right)(t) - \frac{\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})}{1-\alpha}(t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_{\alpha}(t-s)^{\beta}\right].
\end{aligned}$$

This completes the proof. □

In particular, when $\mu = 1$, we have

$$\left(D^{\alpha,\beta}_{\widehat{a},1}x\right)(t) = \left(D^{\alpha,\beta}_{\widehat{a},0}x\right)(t) - \frac{x(\hat{a})}{1-\alpha}E_{\beta}\left[-\lambda_{\alpha}(t-s)^{\beta}\right]. \tag{5.4}$$

Theorem 5.4. *The fractional derivative* $D^{\alpha,\beta}_{0,\mu}$ *has the following Laplace transformation*

$$\mathcal{L}\left\{\left(D^{\alpha,\beta}_{0,\mu}x\right)(t)\right\}(\lambda) = \left[\frac{\lambda^{\beta}\mathcal{L}\{x(t)\}(\lambda) - \lambda^{\beta-\mu}\left({}^{RL}I^{1-\mu}_{0}x\right)(0)}{(1-\alpha)\lambda^{\beta}+\alpha}\right]$$

Proof. Using Remark 2.14 (third relation), we obtain

$$\mathcal{L}\left\{\left(D^{\alpha,\beta}_{0,\mu}x\right)(t)\right\}(\lambda) = \lambda^{1-\mu}\mathcal{L}\left\{\left(\mathcal{K}^{1-\alpha,\beta RL}_{0}D^{\mu}_{0}x\right)(t)\right\} - \lambda^{-\mu}\left(\mathcal{K}^{1-\alpha,\beta RL}_{0}D^{\mu}_{0}x\right)(\hat{a}),$$

follows from Lemma 3.6

$$\mathcal{L}\left\{\left(D^{\alpha,\beta}_{0,\mu}x\right)(t)\right\}(\lambda) = \lambda^{1-\mu}\mathcal{L}\left\{\left(\mathcal{K}^{1-\alpha,\beta RL}_{0}D^{\mu}_{0}x\right)(t)\right\}$$

Lemma 3.7 now yields

$$\mathcal{L}\left\{\left(D^{\alpha,\beta}_{0,\mu}x\right)(t)\right\}(\lambda)=\lambda^{1-\mu}\left[\frac{\lambda^{\beta-1}}{(1-\alpha)\,\lambda^{\beta}+\alpha}\mathcal{L}\left\{\left({}^{RL}D^{\mu}_{0}x\right)(t)\right\}(\lambda)\right]$$

Remark 2.14 (second relation) shows that

$$\mathcal{L}\left\{\left(D^{\alpha,\beta}_{0,\mu}x\right)(t)\right\}(\lambda)=\left[\frac{\lambda^{\beta}\mathcal{L}\left\{x(t)\right\}(\lambda)-\lambda^{\beta-\mu}\left({}^{RL}I^{1-\mu}_{0}x\right)(0)}{(1-\alpha)\,\lambda^{\beta}+\alpha}\right].\qquad\square$$

Theorem 5.5. *Let* α,β,μ and x *be satisfying the conditions from Definition 5.1. Then the fractional differential operator* $D^{\alpha,\beta}_{\widehat{a},\mu}$ *is bounded:*

$$\left\|\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(t)\right\|_{C_\delta}\le K\left\|{}^{RL}D^{\mu}_{\hat{a}}x\right\|_{C_\delta},$$

where

$$K=\frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\left[1+\ (a-\hat{a})\,E_{\beta}\left[\lambda_{\alpha}(a-\hat{a})^{\beta}\right]\right].$$

Proof. By Lemma 2.8, we have

$$\begin{aligned}
\left\|\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(t)\right\|_{C_\delta}\le\ &(a-\hat{a})^{\mu}\ \frac{\Gamma(1-\delta)}{\Gamma(\mu+1-\delta)}\left\|D\left(\mathcal{K}^{1-\alpha,\beta RL}_{\hat{a}}D^{\mu}_{\hat{a}}x\right)(s)\right\|_{C_\delta}\\
&=(a-\hat{a})^{\mu}\ \frac{\Gamma(1-\delta)}{\Gamma(\mu+1-\delta)}\left\|\frac{1}{1-\alpha}\frac{d}{dt}\int_{\hat{a}}^{t}E_{\beta}\left[-\lambda_{\alpha}(t-s)^{\beta}\right]\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(s)\ ds\right\|_{C_\delta}\\
&=\ \frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\left\|\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(t)+\int_{\hat{a}}^{t}\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(s)\left(\frac{\partial}{\partial t}E_{\beta}\left[-\lambda_{\alpha}(t-s)^{\beta}\right]\right)ds\right\|_{C_\delta}
\end{aligned}$$

Analysis similar as in the proof of Lemma 3.5 shows that

$$\begin{aligned}
\left\|\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(t)\right\|_{C_\delta}\le&\frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\left\|\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(t)\right\|_{C_\delta}\\
&+\frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\sum_{n=0}^{\infty}(\lambda_{\alpha})^{n\,RL}I^{n\beta}_{\hat{a}}\left\|\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(t)\right\|_{C_\delta}\\
\le&\frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\left\|\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(t)\right\|_{C_\delta}\\
&+\frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\left\|\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(t)\right\|_{C_\delta}\sum_{n=0}^{\infty}(\lambda_{\alpha})^{n}\frac{1}{\Gamma(n\beta+1)}(t-\hat{a})^{n\beta+1}\\
\le&\frac{(a-\hat{a})^{\mu}\Gamma(1-\delta)}{(1-\alpha)\Gamma(\mu+1-\delta)}\left[1+\ (a-\hat{a})\,E_{\beta}\left[\lambda_{\alpha}(a-\hat{a})^{\beta}\right]\right]\left\|\left({}^{RL}D^{\mu}_{\hat{a}}x\right)(t)\right\|_{C_\delta}.
\end{aligned}$$

This completes the proof. $\square$

From the above theorem, the following identicality can be obtained immediately

$$\left(D^{\alpha,\beta}_{\widehat{a},\mu}x\right)(\hat{a})\ :=\lim_{t\to\hat{a}}\left(onD^{\alpha,\beta}_{\hat{a},\mu}x\right)(t)=0\ .\qquad(5.5)$$

Now, let us consider, the fractional derivative $D^{\alpha,\beta}_{\widehat{a},\mu}$, of a particular function $x(t) = (t-\hat{a})^p$, $p > -1$.

$$\begin{aligned}\left(D^{\alpha,\beta}_{\widehat{a},\mu}(\tau-\hat{a})^p\right)(t) &= \frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left({}^{C}D^{1-\mu}_{\hat{a}}{}^{RL}I^{n\beta+1}_{\hat{a}}{}^{RL}D^{\mu}_{\hat{a}}(\tau-\hat{a})^p\right)(t)\\ &= \frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left({}^{C}D^{1-\mu}_{\hat{a}}{}^{RL}I^{n\beta+1}_{\hat{a}}\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}(\tau-\hat{a})^{p-\mu}\right)(t)\\ &= \frac{1}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left({}^{C}D^{1-\mu}_{\hat{a}}\frac{\Gamma(p+1)}{\Gamma(n\beta+2+p-\mu)}(\tau-\hat{a})^{n\beta+1+p-\mu}\right)(t)\\ &= \frac{\Gamma(p+1)}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left(\frac{(t-\hat{a})^{n\beta+p}}{\Gamma(n\beta+1+p)}\right)\end{aligned}$$

Or, equivalent

$$\left(D^{\alpha,\beta}_{\widehat{a},\mu}(\tau-\hat{a})^p\right)(t) = \frac{\Gamma(p+1)}{1-\alpha}(t-\hat{a})^p E_{\beta,\mathrm{p}+1}\left[-\lambda_\alpha(t-s)^\beta\right]. \tag{5.6}$$

In particular, if $p=0$, then

$$\left(D^{\alpha,\beta}_{\widehat{a},\mu}1\right)(t) = \frac{1}{1-\alpha}(t-\hat{a})\,E_\beta\left[-\lambda_\alpha(t-s)^\beta\right]$$

This means that the fractional derivatives $D^{\alpha,\beta}_{\widehat{a},\mu}$ of a constant are, in general, not equal to zero.

On the other hand, for $0<\mu\le 1$,

$$\left(D^{\alpha,\beta}_{\widehat{a},\mu}(\tau-\hat{a})^{\mu-1}\right)(t) = 0, \tag{5.7}$$

Indeed,

$$\begin{aligned}\left(D^{\alpha,\beta}_{\widehat{a},\mu}(\tau-\hat{a})^{\mu-1}\right)(t) &= \left({}^{C}D^{1-\mu}_{\hat{a}}\mathcal{K}^{1-\alpha,\beta}_{\hat{a}}D\left[{}^{RL}I^{1-\mu}_{\hat{a}}(\tau-\hat{a})^{\mu-1}\right]\right)(t)\\ &= \left({}^{C}D^{1-\mu}_{\hat{a}}\mathcal{K}^{1-\alpha,\beta}_{\hat{a}}D\left[\Gamma(\mu)\right]\right) = 0\end{aligned}$$

In the following, we will establish a fractional integral associated with the fractional derivative. For this, let us consider the following

$$\left(D^{\alpha,\beta}_{0,\mu}x\right)(t) = u(t) \tag{5.8}$$

Using the Laplace transform, we get

$$\frac{\lambda^\beta\mathcal{L}\{x(t)\}(\lambda) - \lambda^{\beta-\mu}\left({}^{RL}I^{1-\mu}_{0}x\right)(0)}{(1-\alpha)\lambda^\beta+\alpha} = \mathcal{L}\{u(t)\}(\lambda)$$

Therefore

$$\mathcal{L}\{x(t)\}(\lambda) = \lambda^{-\mu}\left({}^{RL}I^{1-\mu}_{0}x\right)(0) + (1-\alpha)\mathcal{L}\{u(t)\}(\lambda) + \frac{\alpha}{\lambda^\beta}\mathcal{L}\{u(t)\}(\lambda),$$

applying the inverse Laplace leads to

$$x(t) = \frac{\left({}^{RL}I^{1-\mu}_{0}x\right)(0)}{\Gamma(\mu)}t^{\mu-1} + (1-\alpha)u(t) + \alpha\left({}^{RL}I^{\beta}_{0}u\right)(t).$$

Relying on the above work, we present the following definition.

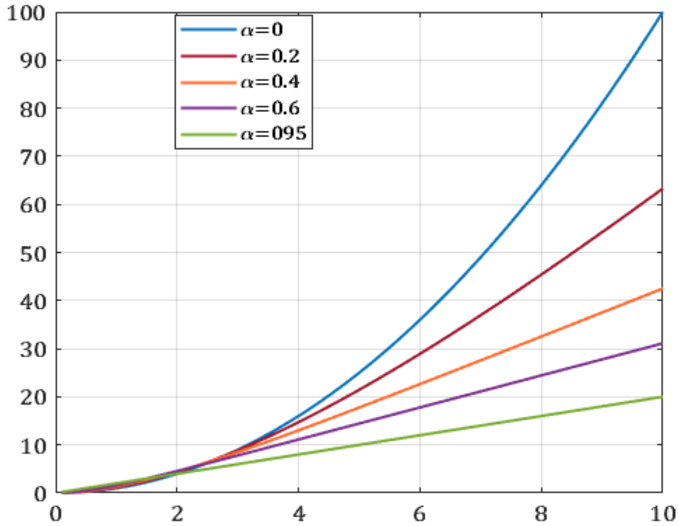

Figure 3: Graph of (5.6) for $\hat{a}=0, \beta=1$, $p=2$ and deference values of α.

Definition 5.6. *The fractional integral associated with the fractional derivative* $\left(D_{\widehat{a},\mu}^{\alpha,\beta}x\right)$ *, is defined by*

$$\left(I_{\hat{a}}^{\alpha,\beta}x\right)(t)=(1-\alpha)\,x\,(t)+\alpha\left({}^{RL}I_{\hat{a}}^{\beta}x\right)(t)\,. \tag{5.9}$$

We obtain the original function when $\alpha=0$. Otherwise, we have

$$\left(I_{\hat{a}}^{1,1}x\right)(t)=\int_{\hat{a}}^{t}x\,(s)\,ds.$$

The fractional integral $I_{\hat{a}}^{\alpha,\beta}$ defined in [19] as the fractional integral associated with the generalized fractional Riemann-Liouville derivative sense.

The Laplace transform of the fractional integral $I_{0}^{\alpha,\beta}$ is given by

$$\mathcal{L}\left\{\left(I_{0}^{\alpha,\beta}x\right)(t)\right\}(\lambda)=\frac{(1-\alpha)\,\lambda^{\beta}+\alpha}{\lambda^{\beta}}\mathcal{L}\left\{x\,(t)\right\}(\lambda)\,. \tag{5.10}$$

From Lemma 2.8, it can be directly verified that the fractional integral $I_{\hat{a}}^{\alpha,\beta}$ is bounded.

Theorem 5.7. *The fractional integration operator* $I_{\hat{a}}^{\alpha,\beta}$ *with* $0<\alpha<1, \beta>0$ *is bounded in* $\mathrm{C}_{\delta}\left[\hat{a},\ a\right]$*:*

$$\left\|\left(I_{\hat{a}}^{\alpha,\beta}x\right)(t)\right\|_{\mathrm{C}_{\delta}}\leq C\|x\|_{\mathrm{C}_{\delta}},$$

where

$$C=(1-\alpha)+\frac{\alpha(a-\hat{a})^{\beta}\Gamma(1-\delta)}{\Gamma(\beta+1-\delta)}\ .$$

In the following theorem, we derive the composition relations between the fractional integration operator $I_{\hat{a}}^{\alpha,\beta}$ *with the fractional differentiation operator* $D_{\widehat{a},\mu}^{\alpha,\beta}$.

Theorem 5.8. *The equalities*

$$\left(I_{\hat{a}}^{\alpha,\beta}D_{\widehat{a},\mu}^{\alpha,\beta}x\right)(t)=x\left(t\right)-\frac{\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)}{\Gamma\left(\mu\right)}(t-\hat{a})^{\mu-1}, \tag{5.11}$$

and

$$\left(D_{\widehat{a},\mu}^{\alpha,\beta}I_{\hat{a}}^{\alpha,\beta}x\right)(t)=x\left(t\right)-\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)(t-\hat{a})^{\mu-1}E_{\beta}\left[-\lambda_{\alpha}(t-\hat{a})^{\beta}\right], \tag{5.12}$$

are valid whenever α,β,μ and x *satisfy the conditions from Definition 5.1.*

Proof. According to Theorem 5.3, we obtain

$$\left(I_{\hat{a}}^{\alpha,\beta}D_{\widehat{a},\mu}^{\alpha,\beta}x\right)(t)=I_{\hat{a}}^{\alpha,\beta}\left(D_{\widehat{a},0}^{\alpha,\beta}x\right)(t)-\frac{\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)}{1-\alpha}\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n}}{\Gamma\left(\beta n+\mu\right)}I_{\hat{a}}^{\alpha,\beta}\left(t-\hat{a}\right)^{\beta n+\mu-1}$$

$$=(1-\alpha)\left(D_{\widehat{a},0}^{\alpha,\beta}x\right)(t)+\alpha^{RL}I_{\hat{a}}^{\beta}\left(D_{\widehat{a},0}^{\alpha,\beta}x\right)(t)+\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n}}{\Gamma\left(\beta n+\mu\right)}\left(t-\hat{a}\right)^{\beta n+\mu-1}$$

$$+\frac{\alpha}{1-\alpha}\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n}}{\Gamma\left(\beta n+\mu\right)}{}^{RL}I_{\hat{a}}^{\beta}\left(t-\hat{a}\right)^{\beta n+\mu-1}$$

$$=\sum_{n=0}^{\infty}\left(-\lambda_{\alpha}\right)^{n}\left({}^{RL}I_{\hat{a}}^{n\beta}x\right)(t)+\frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}\left(-\lambda_{\alpha}\right)^{n}{}^{RL}I_{\hat{a}}^{\beta}\left({}^{RL}I_{\hat{a}}^{n\beta}x\right)(t)$$

$$+\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n}}{\Gamma\left(\beta n+\mu\right)}\left(t-\hat{a}\right)^{\beta n+\mu-1}$$

$$+\frac{\alpha\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)}{1-\alpha}\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n}}{\Gamma\left(\beta\left(n+1\right)+\mu\right)}\left(t-\hat{a}\right)^{\beta(n+1)+\mu-1}$$

$$=\sum_{n=0}^{\infty}\left(-\lambda_{\alpha}\right)^{n}\left({}^{RL}I_{\hat{a}}^{n\beta}x\right)(t)-\sum_{n=0}^{\infty}\left(-\lambda_{\alpha}\right)^{n+1}\left({}^{RL}I_{\hat{a}}^{(n+1)\beta}x\right)(t)$$

$$+\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n}}{\Gamma\left(\beta n+\mu\right)}\left(t-\hat{a}\right)^{\beta n+\mu-1}$$

$$-\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)\sum_{n=0}^{\infty}\frac{\left(-\lambda_{\alpha}\right)^{n+1}}{\Gamma\left(\beta\left(n+1\right)+\mu\right)}\left(t-\hat{a}\right)^{\beta(n+1)+\mu-1}$$

$$=x(t)-\frac{\left({}^{RL}I_{\hat{a}}^{1-\mu}x\right)\left(\hat{a}\right)}{\Gamma(\mu)}(t-\hat{a})^{\mu-1}.$$

For identity (5.11), we have

$$\begin{aligned}\left(D^{\alpha,\beta}_{\widehat{a},\mu}I^{\alpha,\beta}_{\hat{a}}x\right)(t)=&\left(D^{\alpha,\beta}_{\widehat{a},0}I^{\alpha,\beta}_{\hat{a}}x\right)(t)-\frac{\left({}^{RL}I^{1-\mu}_{\hat{a}}I^{\alpha,\beta}_{\hat{a}}x\right)(\hat{a})}{1-\alpha}(t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_\alpha(t-s)^\beta\right]\\
=&(1-\alpha)\left(D^{\alpha,\beta}_{\widehat{a},0}x\right)(t)+\alpha\left({}^{RL}I^{\beta}_{\hat{a}}D^{\alpha,\beta}_{\widehat{a},0}x\right)(t)\\
&-\frac{1}{1-\alpha}\left[(1-\alpha)\,{}^{RL}I^{1-\mu}_{\hat{a}}x(t)+\alpha^{RL}I^{1-\mu RL}_{\hat{a}}I^{\beta}_{\hat{a}}x(t)\right](\hat{a})(t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_\alpha(t-s)^\beta\right]\\
=&\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left({}^{RL}I^{n\beta}_{\hat{a}}x\right)(t)+\frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}(-\lambda_\alpha)^{n\,RL}I^{\beta}_{\hat{a}}\left({}^{RL}I^{n\beta}_{\hat{a}}x\right)(t)\\
&-\left[\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})+\frac{\alpha}{1-\alpha}\left({}^{RL}I^{\beta+1-\mu}_{\hat{a}}x\right)(\hat{a})\right](t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_\alpha(t-s)^\beta\right]\\
=&\sum_{n=0}^{\infty}(-\lambda_\alpha)^n\left({}^{RL}I^{n\beta}_{\hat{a}}x\right)(t)-\sum_{n=0}^{\infty}(-\lambda_\alpha)^{n+1}\left({}^{RL}I^{(n+1)\beta}_{\hat{a}}x\right)(t)\\
&-\left[\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})\right](t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_\alpha(t-s)^\beta\right]\\
=&x(t)-\left[\left({}^{RL}I^{1-\mu}_{\hat{a}}x\right)(\hat{a})\right](t-\hat{a})^{\mu-1}E_{\beta,\mu}\left[-\lambda_\alpha(t-s)^\beta\right]\end{aligned}$$

This completes the proof. □

As particular cases of the previous theorem, if $\mu=0$, then the fractional integration operator $I^{\alpha,\beta}_{\hat{a}}$ is a left and right inverse to the fractional differentiation operator $D^{\alpha,\beta}_{\widehat{a},0}$, i.e.

$$\left(I^{\alpha,\beta}_{\hat{a}}D^{\alpha,\beta}_{\widehat{a},0}x\right)(t)=\left(D^{\alpha,\beta}_{\widehat{a},0}I^{\alpha,\beta}_{\hat{a}}x\right)(t)=x(t). \tag{5.13}$$

On the other hand, if $\mu=1$, then they satisfy the following Newton-Leibniz formula

$$\left(I^{\alpha,\beta}_{\hat{a}}D^{\alpha,\beta}_{\widehat{a},1}x\right)(t)=x(t)-x(\hat{a}), \tag{5.14}$$

also, we obtain

$$\left(D^{\alpha,\beta}_{\widehat{a},1}I^{\alpha,\beta}_{\hat{a}}x\right)(t)=x(t)-x(\hat{a})\,E_\beta\left[-\lambda_\alpha(t-s)^\beta\right]. \tag{5.15}$$

For the function $x(t)=(t-\hat{a})^p$, $p>-1$. We have

$$\left(I^{\alpha,\beta}_{\hat{a}}(\tau-\hat{a})^p\right)(t)=(1-\alpha)(t-\hat{a})^p+\alpha\frac{\Gamma(p+1)}{\Gamma(\beta+p+1)}(t-\hat{a})^{\beta+p}. \tag{5.16}$$

6. An application

Since the outbreak epidemic of the Corona (COVID-19) in the Chinese city of Wuhan in 2019, researchers have rushed to provide mathematical modeling of it. For example, in [23], the authors used the Caputo fractional derivative to present a mathematical model for the transmission of COVID-19. Baleanu et al. [24]presented a fractional-order model for the Coronavirus (COVID-19) transmission with Caputo–Fabrizio derivative. In [19], the author considered the following model:

$$\begin{cases}{}^{C}D^{\alpha,\beta}_{\hat{a}}\mathcal{I}(t)=k\mathcal{S}(t)\mathcal{I}(t)-(s+m)\mathcal{I}(t),\\ {}^{C}D^{\alpha,\beta}_{\hat{a}}\mathcal{S}(t)=\mathcal{A}-m\mathcal{S}(t)-k\mathcal{S}(t)\mathcal{I}(t),\\ {}^{C}D^{\alpha,\beta}_{\hat{a}}\mathcal{R}(t)=s\mathcal{I}(t)-m\mathcal{R}(t).\end{cases} \tag{6.1}$$

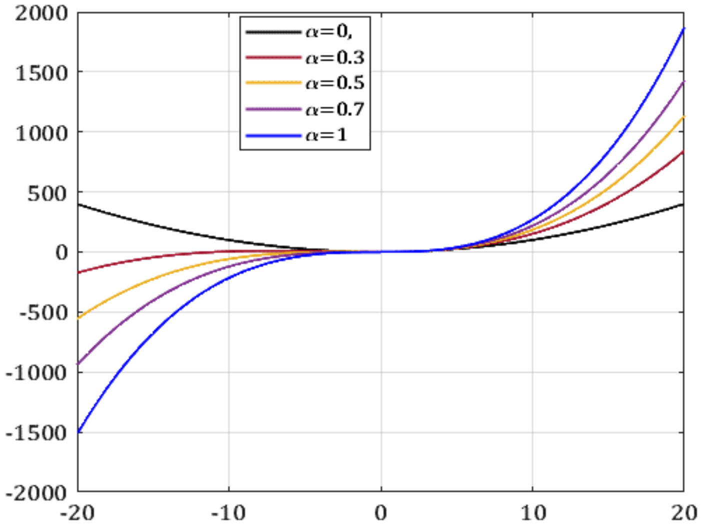

Figure 4: Graph of (5.16) for $\hat{a}=0, \beta=0.8$, $p=2$ and deference values of α.

Where ${}^{C}D_{\hat{a}}^{\alpha,\beta}$ is the generalized Caputo-Fabrizio fractional derivative with $\alpha \in (0,1), \beta > 0. \mathcal{R}(t), \mathcal{I}(t)$ and $\mathcal{S}(t)$ are the numbers of removed individuals, infected and susceptible at time t, respectively. The parameters s, k, m and $\mathcal{A}$ constitute the removal rate, the infection rate, the natural death rate and the recruitment rate, respectively.

Let us denote the total population by $\mathcal{H}(t)$. Then

$$ {}^{C}D_{\hat{a}}^{\alpha,\beta}\mathcal{H}(t) = \mathcal{A} - m\mathcal{H}(t). \tag{6.2}$$

The fractional differential equation (6.2) has an important role in the science of viruses and epidemics. In particular, $\mathcal{H}(t)$ can describe the intensity of healthy CD4+ T cells for H.I.V infection, where $\mathcal{A}$ and m represent the rate of production of CD4+ T cells and the rate of dying it, respectively. In this section, we consider the following model

$$\left(D_{0,\mu}^{\alpha,\beta}\mathcal{H}\right)(t) = \mathcal{A} - m\mathcal{H}(t). \tag{6.3}$$

Where $\alpha \in (0,1), \beta > 0$ and $0 \leq \mu \leq 1$. Applying Laplace transform to both sides (6.3), we get that

$$\mathcal{L}\left\{\left(D_{0,\mu}^{\alpha,\beta}\mathcal{H}\right)(t)\right\}(\lambda) = \frac{\mathcal{A}}{\lambda} - m\mathcal{L}\{\mathcal{H}(t)\}(\lambda).$$

From Theorem 5.5, we obtain

$$\frac{\lambda^{\beta}\mathcal{L}\{\mathcal{H}(t)\}(\lambda) - \lambda^{\beta-\mu}\left({}^{RL}I_0^{1-\mu}\mathcal{H}\right)(0)}{(1-\alpha)\lambda^{\beta}+\alpha} = \frac{\mathcal{A}}{\lambda} - m\mathcal{L}\{\mathcal{H}(t)\}(\lambda),$$

and

$$\lambda^{\beta}\mathcal{L}\{\mathcal{H}(t)\}(\lambda) - \lambda^{\beta-\mu}\left({}^{RL}I_0^{1-\mu}\mathcal{H}\right)(0) = \frac{(1-\alpha)\lambda^{\beta}\mathcal{A}+\alpha\mathcal{A}}{\lambda} - \left[(1-\alpha)\lambda^{\beta}m+\alpha m\right]\mathcal{L}\{\mathcal{H}(t)\}(\lambda),$$

hence,

$$\lambda^{\beta}\mathcal{L}\{\mathcal{H}(t)\}(\lambda) + \left[(1-\alpha)\lambda^{\beta}m+\alpha m\right]\mathcal{L}\{\mathcal{H}(t)\}(\lambda) = \lambda^{\beta-\mu}\left({}^{RL}I_0^{1-\mu}\mathcal{H}\right)(0) + \frac{(1-\alpha)\lambda^{\beta}\mathcal{A}+\alpha\mathcal{A}}{\lambda}.$$

Therefore,

$$\left([1+(1-\alpha)\,m]\,\lambda^{\beta}+\alpha m\right)\mathcal{L}\left\{\mathcal{H}\left(t\right)\right\}(\lambda)=\lambda^{\beta-\mu}\left({}^{RL}I_{0}^{1-\mu}\mathcal{H}\right)(0)+\frac{(1-\alpha)\,\lambda^{\beta}\mathcal{A}+\alpha\mathcal{A}}{\lambda}.$$

For simplicity, let $z_\alpha = 1+(1-\alpha)\,m$. Then

$$\mathcal{L}\left\{\mathcal{H}\left(t\right)\right\}(\lambda)=\frac{\lambda^{\beta-\mu}}{\lambda^{\beta}+\frac{\alpha m}{z_\alpha}}\frac{\left({}^{RL}I_{0}^{1-\mu}\mathcal{H}\right)(0)}{z_\alpha}+\frac{\lambda^{\beta-1}}{\lambda^{\beta}+\frac{\alpha m}{z_\alpha}}\frac{(1-\alpha)\,\mathcal{A}}{z_\alpha}+\frac{\alpha\mathcal{A}}{z_\alpha\lambda^{\beta+1}+\alpha m\lambda}$$

Or equivalent,

$$\mathcal{L}\left\{\mathcal{H}\left(t\right)\right\}(\lambda)=\frac{\lambda^{\beta-\mu}}{\lambda^{\beta}+\frac{\alpha m}{z_\alpha}}\frac{\left({}^{RL}I_{0}^{1-\mu}\mathcal{H}\right)(0)}{z_\alpha}+\frac{\lambda^{\beta-1}}{\lambda^{\beta}+\frac{\alpha m}{z_\alpha}}\frac{(1-\alpha)\,\mathcal{A}}{z_\alpha}+\frac{\mathcal{A}}{m\lambda}-\frac{\mathcal{A}}{m}\,\frac{\lambda^{\beta-1}}{\lambda^{\beta}+\frac{\alpha m}{z_\alpha}}.$$

Using the inverse Laplace leads to

$$\begin{aligned}\mathcal{H}\left(t\right)&=\frac{\mathcal{A}}{m}+\frac{\left({}^{RL}I_{0}^{1-\mu}\mathcal{H}\right)(0)}{z_\alpha}t^{\mu-1}E_{\beta,\mu}\left(-\frac{\alpha m}{z_\alpha}t^{\beta}\right)+\left(\frac{(1-\alpha)\,\mathcal{A}}{z_\alpha}-\frac{\mathcal{A}}{m}\right)E_{\beta}\left(-\frac{\alpha m}{z_\alpha}t^{\beta}\right)\\&=\frac{\mathcal{A}}{m}+\frac{\left({}^{RL}I_{0}^{1-\mu}\mathcal{H}\right)(0)}{z_\alpha}t^{\mu-1}E_{\beta,\mu}\left(-\frac{\alpha m}{z_\alpha}t^{\beta}\right)-\frac{\mathcal{A}}{mz_\alpha}E_{\beta}\left(-\frac{\alpha m}{z_\alpha}t^{\beta}\right).\end{aligned}\tag{6.4}$$

In particular, if $\mu=1$, then

$$\mathcal{H}\left(t\right)=\frac{\mathcal{A}}{m}+\left(\frac{\mathcal{H}\left(0\right)}{z_\alpha}-\frac{\mathcal{A}}{mz_\alpha}\right)E_{\beta}\left(-\frac{\alpha m}{z_\alpha}t^{\beta}\right).\tag{6.5}$$

Which is a solution of the fractional differential equation (6.2).

7. Conclusion

The fractional-order derivatives presented in [15, 17] have been of interest to many researchers because they describe many real-world problems more accurately than others since they contain non-local and non-singular kernels [20, 21, 22]. In this work, we introduced a new fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ ($0<\alpha<1, \beta>0$) by generalized the fractional integral given in [16] where we replaced the kernel from the exponential to Mittag-Leffler, and we expressed it by a series of Riemann-Liouville fractional integrals, which helped to study some properties related to it, where we found the Laplace transform of $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$and showed that the fractional integration operator $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ is bounded from $\mathrm{C}_{\delta}\left[\hat{a},\ a\right]$ into $\mathrm{C}\left[\hat{a},\ a\right]$ and the fractional integrals operators $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ are commutative. We used the Laplace transforms to derived the corresponding fractional derivative ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ and we established its properties where we proved that it is a left inverse of integral operator $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ and in general, is not the right inverse; also, we showed that the fractional derivatives operators ${}^{\mathcal{K}}D_{\hat{a}}^{\alpha,\beta}$ are, in general, not commutative. Via composition, our fractional integral $\mathcal{K}_{\hat{a}}^{\alpha,\beta}$ with the classical Riemann-Liouville and Caputo derivatives, we introduced a new fractional derivative $D_{\hat{a},\mu}^{\alpha,\beta}$ $(0\leq\mu\leq1)$ which is an interpolated fractional derivative between the generalized fractional derivatives in a sense Riemann-Liouville and Caputo with non-singular kernels presented in [19]. We obtain the Hattaf fractional derivative of Riemann-Liouville derivative sense ${}^{R}D_{\hat{a}}^{\alpha,\beta}$ [19] when $\mu=0$ and the Hattaf fractional derivative of Caputo sense ${}^{C}D_{\hat{a}}^{\alpha,\beta}$ [19] if $\mu=1$. We expressed the fractional derivative $D_{\hat{a},\mu}^{\alpha,\beta}$ by a series of compositions of classical Riemann-Liouville fractional integrals with classical Riemann-Liouville and Caputo derivatives, which may be more applicable in numerical investigation than the original formula. Also, we derived the Laplace transform of derivative $D^{\alpha,\beta}_{\widehat{a},\mu}$ and showed that fractional differential operator $D^{\alpha,\beta}_{\widehat{a},\mu}$ is bounded in $C_\delta\,[\hat{a},\ a]$. We also determined the corresponding fractional integral of the fractional derivative $D^{\alpha,\beta}_{\widehat{a},\mu}$ with some properties related to it. As a numerical example, we applied all these fractional operators to the power function $(t-\hat{a})^p$, $p>-1$ accompanied by graphics. Coronavirus (COVID-19) transmission model was our application to the fractional derivative $D^{\alpha,\beta}_{\widehat{a},\mu}$.

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