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Homoclinic orbits and localized solutions in discrete nonlinear Schrodinger equation with long-range interaction

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Abstract

In this paper, we use the homoclinic orbit approach without using small perturbations to prove the existence of soliton solutions of the discrete nonlinear Schrödinger equations with long-range interaction by employing the properties of the symmetries of reversible planar maps. Moreover, the long-range interaction by a potential is proportional to $1/l^{1+\alpha}$ with fractional $\alpha < 1$ and l as natural number.

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1. Introduction

Recently much attention has been paid to the analysis of discrete equations with long-range interaction driven by fractional powers of the discrete Laplacian [13, 8, 4, 18, 5]. For example in [5] the authors present the problem of a quite complete study of discrete diffusion equations with long-range interaction involving the fractional powers. In [16] unidimensional chain of linear and nonlinear oscillators with a long-range interaction power wise defined by a proportional term of $1/(n-m)^{1+\alpha}$

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($n \neq m$) has been considered. In [18], a variational cadre for a fractional difference equation on $\mathbb{Z}$ driven by the fractional discrete Laplacian has been introduced. In particular, in [13], a fractional version of the discrete nonlinear Schrödinger equation, in which the regular discrete Laplacian is changed to a discrete fractional Laplacian, has been studied.

We focus that The existence of bright solitons for various cases was then discussed by the Melnikov method assuming that the perturbation is small and for the anti-integrability method [12, 7], some localized solutions persist for small coupling cases. In [14], the variational approach can also be used, but the frequency allowed region cannot be determined by the variational method explicitly.

In this paper, we look at the the existence of time periodic and spatially localized solutions in which a complex amplitude algebraic equation is obtained.

We restrict our attention to real amplitudes and the case where corresponds to bright and dark solitons, for example [2, 9, 10].

The homoclinic orbit approach for the existence of soliton solutions of DNLS equations utilised in our paper is precisely a generalization of the work of [15] (local interaction).

The rest of of this paper is organized as follows: The second section we give some preliminaries about the reversible planar map and the homocline (heterocline) points. In addition we present the fondamental theorem for the existing of the the orbit homocline (heterocline) for a class of planar map in n dimension. In section 3 we give the conditions to proove the existing the bright and dark dark soliton for local solutions in the of discret schrodiger equations driven by fraction powers of discret laplacian.

2. Homoclinic orbits of reversible planar maps

We will present a mathematical description of the time reversal symmetry in the context of dynamical systems. In most applications of interest $\Omega = \mathbb{R}^n$. We will only consider diffeomorphisms of $\mathbb{R}^{2n}$. Let R be a smoothed diffeomorphism satisfying :

- $R \circ R =$ identity.
- The dimension of the fixed point set of R, $Fix(R)$, is n.

R is called inverse involution. A diffeomorphism T of $\mathbb{R}^{2n}$ is called R-reversible if $R \circ T = T^{-1} \circ R$. Some periodic points are easy to find; these are the symmetric periodic points described by the following proposition.

Proposition 2.1. *[11] Let $p \in Fix(R)$ and suppose $T^k(p) \in Fix(R)$ as a consequence $T^{2k}(p) = p$, then we have:*

$$T^k(p) = RT^k(p) = T^{-k}R(p) = T^{-k}(p), \quad therefore: \quad T^{2k}(p) = p.$$

Hence, symmetric periodic points may be found geometrically; we search for self-intersections of the set of fixed points of R under iteration of T. We also can find some homoclinic points of geometrically reversible diffeomorphisms of R-geometrically reversible diffeomorphisms, as is shown by the following examples:

Proposition 2.2. *[6] Let $p \in Fix(R)$ be a symmetric fixed point of T and let $W^s(p)$ and $W^u(p)$ denote the stable and unstable manifolds of p, respectively. Then $R(W^u(p)) = W^s(p)$ and $R(W^s(p)) = W^u(p)$.*
In particular, if $q \in W^u(p) \cap Fix(R)$, then q is a homoclinic point.

Let $x \in W^u(p)$ such that $\lim_{n\longrightarrow\infty} T^{-n}(x) = p$, and so we have:

$$p = R \lim_{n\longrightarrow\infty}(T^{-n}(x)) = \lim_{n\longrightarrow\infty} T^n(R(x)),$$

in such a way that $R(x) \in W^s(p)$, where $RW^u(p) \subset W^s(p)$. We also have $RW^s(p) \subset W^u(p)$, such that $RW^u(p) = W^s(p)$. If $q \in W^u(p) \cap Fix(R)$, so $q = R(q) \in W^s(p) \cap Fix(R)$ also, so that q is a homoclinic point [6]. Therefore, to produce homoclinic points for reversible diffeomorphisms, it is sufficient to find the intersections of $w^u(p)$ with $Fix(R)$. We note that both of the above propositions are true in much more general terms. Homoclinic points that are also in $Fix(R)$ are classified as symmetric homoclinic points. Such a point is called a regular homoclinic point if the unstable variety (and thus also the stable variety) meets $Fix(R)$ transversely to the homoclinic point.

Proposition 2.3. *[11] Let p be a symmetric fixed point and let q be a symmetric homoclinic point in $W^u(p)$. Let N be any neighborhood of p in $Fix(R)$. Then there exists an infinite number of periodic symmetric points in N.*

Proposition 2.4. *[6] Let p be a non-symmetric periodic point. Suppose $q \in w^u(p) \cap Fix(R)$. Then $q \in w^u(p) \cap w^s(R(p))$. Thus some heteroclinic points can be found geometrically as symmetric homoclinic points. Regular symmetric heteroclinic points are defined as regular homoclinic points.*

Proposition 2.5. *[6] Let T be an R-reversible diffeomorphism of the plane and assume that p is a non-symmetric saddle point for T. Suppose that a branch of $w^u(p)$ and a branch of $w^s(p)$ intersect t. Suppose that a branch of $w^s(p)$ meets $Fix(R)$ transversely. There are then an infinite number of many symmetric periodic orbits entering any neighborhood of p and $R(p)$.*

A class of classical reversible planar maps is derived from symmetric difference equations of the form [11, 15]:

$$x_{n+1} + x_{n-1} = g(x_n), \tag{2.1}$$

In this article we treat the most general case:

$$\sum_{l=1}^{\frac{N}{2}-1} \frac{x_{n+l} + x_{n-l}}{l^{1+\alpha}} = g(x_n), \tag{2.2}$$

often appear in the discussion of stationary states of long-range interactions, l is the distance between oscillators and α a fractional value. The system (2.1) can be expressed as a planar application, denoted T.

We calculate the application T for the stationary states of the oscillators. We derive the application T for the order M such that $M = \frac{N}{2} - 1$.

$$\begin{cases} x_{1,n+1} = x_{2,n} \\ x_{2,n+1} = x_{3,n} \\ \dots\dots\dots\dots \\ x_{M,n+1} = x_{M+1,n} \\ x_{M+1,n+1} = x_{M+2,n} \\ \dots\dots\dots\dots \\ x_{2M-1,n+1} = x_{2M,n} \\ x_{2M,n+1} = \phi_{n+M} \end{cases} \tag{2.3}$$

$$x_{2M,n+1} = -x_{1,n} - (M)^{1+\alpha} (\sum_{l=1}^{M-1} \frac{x_{(M+1)-l,n} + x_{(M+1)+l,n}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(x_{(M+1),n}).$$

$$T(x_1, x_2, ..., x_{2M}) = (x_2, x_3, ..., x_{2M}, -x_1 - (M)^{1+\alpha} (\sum_{l=1}^{M-1} \frac{x_{(M+1)-l} + x_{(M+1)+l}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(x_{(M+1)})).$$

Calculate T^{-1}:
We set:

$$H(t_1, t_2, ..., t_{2M-1}) = -(M)^{1+\alpha} (\sum_{l=1}^{M-1} \frac{t_{M-l} + t_{M+l}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(t_M).$$

Using the change of variable:

$$\begin{aligned} t_1 &= x_2, \\ t_2 &= x_3, \\ &... \\ t_{2M-1} &= x_{2M}, \end{aligned}$$

then:

$$T(x_1, x_2, ..., x_{2M}) = (t_1, t_2, ..., t_{2M-1}, -x_1 + H(t_1, t_2, ..., t_{2M-1})),$$

and:

$$\begin{aligned} T^{-1} \circ T(x_1, x_2, ..., x_{2M}) =& T^{-1}(t_1, t_2, ..., t_{2M-1}, -x_1 + H(t_1, t_2, ..., t_{2M-1})) \\ =& (x_1 + H(t_1, t_2, ..., t_{2M-1}) - H(t_1, t_2, ..., t_{2M-1}), t_1, t_2, ..., t_{2M-1}) \\ =& (x_1, x_2, ..., x_{2M}), \end{aligned}$$

then

$$T^{-1}(t_1, t_2, ..., t_{2M-1}, y) = (-y + H(t_1, t_2, ..., t_{2M-1}), t_1, t_2, ..., t_{2M-1}),$$

and

$$\begin{aligned} T^{-1}(x_1, x_2, ..., x_{2M}) =& (-x_{2M} + H(t_1, t_2, ..., t_{2M-1}), x_1, x_2, ..., x_{2M-1}) \\ =& (-x_{2M} - (M)^{1+\alpha} (\sum_{l=1}^{M-1} \frac{x_{M-l} + x_{M+l}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(x_M), x_1, x_2, ..., x_{2M-1}), \end{aligned}$$

Furthermore, T is a diffeomorphism of class $\mathcal{C}^1$ if $(M)^{1+\alpha} g = H$ is $\mathcal{C}^1$.

We suppose that H is ever of class $\mathcal{C}^1$ is an odd function. We verify that T is R-reversible with the respect of the involution:

$$R_1(x_1, x_2, ..., x_M, x_{M+1}, ..., x_{2M}) = (x_{2M}, x_{2M-1}, ..., x_1),$$

and R_2-reversible with respect to involution:

$$R_2(x_1, x_2, ..., x_M, x_{M+1}, ..., x_{2M}) = (-x_{2M}, -x_{2M-1}, ..., -x_1),$$

as though H is an odd function. Note that the set of fixed points $Fix(R_1)$ and $Fix(R_2)$ are determined by the lines S_1 and S_2, respectively. Let:

$$f(x_{M+1}) = g(x_{M+1}) - 2x_{M+1}\sum_{l=1}^{M}\frac{1}{l^{1+\alpha}}, \tag{2.4}$$

Theorem 2.6. *Assume that:*
(i) $f(z)$ *is a* C^1 *and odd function, and has only three real zeros,* $-z_0, 0,$ *and* $z_0 (z_0 > 0)$ *with* $f(0) > 0$*;*
(ii) $\sup_{z\geq z'}(f(z) + 2z\sum_{l=1}^{M-1}\frac{1}{l^{1+\alpha}}) < 0$ *for some* $z' > z_0$*.*
Then the planar map T *has a homoclinic orbit.*

Proof . As f is an odd function and has three distinct real zeros, we can suppose that its real zeros are $-z_0, 0$ and z_0 with $z_0 > 0$. The function is therefore a function with three distinct real zeros.

Meanwhile, the planar map T has three fixed points $P(-z_0, ..., -z_0), O(0, ..., 0)$ and $Q(z_0, ..., z_0)$, which are all symmetric with respect to the involution R_1. The origin O is hyperbolic if $f'(0) > 0$. Moreover, the unstable manifolds The unstable manifold $W^u(O)$ and the stable manifold $W^s(O)$ are tangent to the stable and unstable eigenspaces $E_s(0)$ and $E_u(0)$ of the Jacobian matrix of T at the origin of the Jacobian matrix of T.

First we show that the intersection of $W^u(O)$ with the interior of the segment EQ is nonempty, where $E(0, ..., 0, z_0, ..., z_0)$. One may easily check that one branch of $W^u(O)$ initially enters the interior of the triangle $\triangle OEQ$, denoted by *int* $(\triangle OEQ)$.

For each point $A(x_1, x_2, ..., x_{2M}) \in int(\triangle OEQ)$, one has that

$$\begin{aligned}
0 &< x_1 < x^{N'+1} < z_0,\\
0 &< x_2 < x_{M+2} < z_0,\\
&....\\
0 &< x_M < x_{2M} < z_0
\end{aligned}$$

and the coordinates of the image point $T(A)$ are

$$T(x_1, x_2, ..., x_{2M}) = (x_2, x_3, ..., x_{2M}, -x_1 - (M)^{1+\alpha}(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l} + x_{(M+1)+l}}{l^{1+\alpha}}) + (M)^{1+\alpha}g(x_{(M+1)})).$$

Moreover, since $f(x_{(M+1),n})$ is positive for $x_{(M+1),n} \in (0, z_0)$, the distance from point $T(A)$ to the line S_1 is greater than the distance from A to S_1. Consequently, the unstable manifold $W^u(O)$ in the interior of $\triangle OEQ$ does not intersect the segments OE and OQ.

In the following, we prove by contradiction that $W^u(O)$ meets the segment EQ.

Assume that the branch of $W^u(O)$ in the first quadrant always lies in the interior of $\triangle OEQ$. Take a point $B \in W^u(O) \cap int(\triangle OEQ)$. Then all the image points $T^n(B) \in int(\triangle OEQ)$ for $n = 1, 2, \cdots$. Moreover, the sequences of x_1-coordinates, x_2-coordinates,..., x_{2M}-coordinates of $T^n(B)$ are both strictly increasing and bounded above, hence convergent to $(x_1)^*, (x_2)^*, ..., (x_{2M})^*$, respectively. As a consequence, the sequence of points $\{T^n(B)\}$ is convergent to $N((x_1)^*, (x_2)^*, ..., (x_{2M})^*)$, which is a fixed point of T.

From the facts $(x_1)^* > 0, (x_2)^* > 0, ..., (x_{2M})^* > 0$, it follows that $N = Q$. On the other hand, the sequence of the distance between $T^n(B)$ and S_1 is also strictly increasing, implying that $N \neq Q$, a contradiction. Therefore, the unstable manifold $W^u(O)$ pierces the segment EQ.

Secondly, we show that $W^u(O)$ in the first quadrant meets the line S_1 at some point. Denote by

$$H_0(x_{1,0}, x_{2,0}, ..., x_{M,0}, z_0, ..., z_0)$$

the intersection point of $W^u(O)$ with the segment EQ.

Let $H_{n+1} = T(H_n), n = 0, 1, \cdots$. The coordinates of H_n are $(x_{1,n}, x_{2,n}, ..., x_{M,n}, x_{M+1,n}, ..., x_{2M,n})$ .It then follows that $x_{M+1,n} > z_0$. Since

$$f(x_{M+1,n}) + 2x_{M+1,n} \sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < 0 \ for \ x_{M+1,n} > x_{M+1,0}$$

we derive from assumption (ii) that:

$$\sup_{x_{M+1} \geq x_{M+1,M}} (f(x_{M+1,n}) + 2x_{M+1,n} \sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}}) < 0.$$

We denote

$$\sup_{x_{M+1} \geq x_{M+1,M}} (f(x_{M+1,n}) + 2x_{M+1,n} \sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}}) = -a \qquad (a > 0).$$

Suppose that $W^u(O)$ in the first quadrant does not intersect the line S_1. Then $W^u(O)$ lies between the z-axis and the line S_1, and thus the points H_n lie above the line S_1.

$$f(x_{M+1,n}) + 2x_{M+1,n} \sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < -a \quad for \quad n = 1, 2, \cdots .$$

Let d_n denotes the distance of H_n to the line S_1. Then

$$d_n = \frac{\sqrt{2}}{2} \sum_{j=1}^{M} (x_{M+j,n} - x_{j,n}), \quad n = 0, 1, ... \tag{2.5}$$

Consequently, one has that for $n = 0, 1, ...,$

$$\begin{aligned}
d_{n+1} &= \frac{\sqrt{2}}{2}\sum_{j=1}^{M}(x_{M+j,n+1} - x_{j,n+1}) \\
&= \frac{\sqrt{2}}{2}[(x_{M+2,n} - x_{2,n}) + (x_{M+3,n} - x_{3,n}) + ... + (x_{2M,n} - x_{M,n}) + \\
&\quad (-x_{1,n} - (M)^{1+\alpha}(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,n} + x_{(M+1)+l,n}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(x_{(M+1),n}) - x_{M+1,n})] \\
&= \frac{\sqrt{2}}{2}[(x_{M+1,n} - x_{1,n}) + (x_{M+2,n} - x_{2,n}) + (x_{M+3,n} - x_{3,n}) + ... + (x_{2M,n} - x_{M,n}) - \\
&\quad (M)^{1+\alpha}(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,n} + x_{(M+1)+l,n}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(x_{(M+1),n}) - 2x_{M+1,n}] \\
&= d_n + \frac{\sqrt{2}}{2}(-(M)^{1+\alpha}(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,n} + x_{(M+1)+l,n}}{l^{1+\alpha}}) + (M)^{1+\alpha} g(x_{(M+1),n}) - 2x_{M+1,n}) \\
&= d_n + \frac{\sqrt{2}}{2}(-(M)^{1+\alpha}(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,n} + x_{(M+1)+l,n}}{l^{1+\alpha}}) + (M)^{1+\alpha}(f(x_{(M+1),n}) + \\
&\quad 2x_{M+1,n}\sum_{l=1}^{M}\frac{1}{l^{1+\alpha}}) - 2x_{M+1,n}) \\
&= d_n + (M)^{1+\alpha}\frac{\sqrt{2}}{2}(-(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,n} + x_{(M+1)+l,n}}{l^{1+\alpha}}) + f(x_{(M+1),n}) + 2x_{M+1,n}\sum_{l=1}^{M-1}\frac{1}{l^{1+\alpha}})
\end{aligned}$$

It follows that

$$\begin{cases}
d_1 = d_0 + (M)^{1+\alpha}\frac{\sqrt{2}}{2}(-(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,0}+x_{(M+1)+l,0}}{l^{1+\alpha}}) + f(x_{(M+1),0}) + 2x_{M+1,0}\sum_{l=1}^{M-1}\frac{1}{l^{1+\alpha}}) \\
d_2 = d_1 + (M)^{1+\alpha}\frac{\sqrt{2}}{2}(-(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,1}+x_{(M+1)+l,1}}{l^{1+\alpha}}) + f(x_{(M+1),1}) + 2x_{M+1,1}\sum_{l=1}^{M-1}\frac{1}{l^{1+\alpha}}) \\
.... \\
d_{n+1} = d_n + (M)^{1+\alpha}\frac{\sqrt{2}}{2}(-(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,n}+x_{(M+1)+l,n}}{l^{1+\alpha}}) + f(x_{(M+1),n}) + 2x_{M+1,n}\sum_{l=1}^{M-1}\frac{1}{l^{1+\alpha}})
\end{cases} \quad (2.6)$$

and hence

$$\begin{aligned}
0 &\leq \sqrt{2}d_{n+1} \\
&= \sqrt{2}d_0 + \sum_{i=0}^{n}\sqrt{2}\,(M)^{1+\alpha}(-(\sum_{l=1}^{M-1}\frac{x_{(M+1)-l,i} + x_{(M+1)+l,i}}{l^{1+\alpha}}) + f(x_{(M+1),i}) + 2x_{M+1,i}\sum_{l=1}^{M-1}\frac{1}{l^{1+\alpha}}) \\
&\leq \sqrt{2}d_0 - nb - na, \quad b > 0 \quad and \quad a > 0.
\end{aligned}$$

Letting $n \to \infty$, one obtains a contradiction. Therefore, the intersection of $W^u(O)$ with the line S_1 is nonempty.

Finally, from proposition 2.2 it follows that $W^u(O)$ and $W^s(O)$ intersect at some point q on S_1, implying the existence of a homoclinic orbit. □

Theorem 2.7. *Assume that $f(z)$ is a $\mathcal{C}^1$ and odd function, and has only three real zeros, $-z_0$, 0 and z_0 ($z_0 > 0$) with $f'(z_0) > 0$. So the planar map T has a heteroclinic orbit*

Proof . The reversible map T has three fixed points, two of which, $P(-z0,-z0,...,-z_0,-z_0)$ and $Q(-z0,-z0,...,-z_0,-z_0)$, are hyperbolic if $f'(z_0)>0$.Similarly to the proof of the previous theorem one can verify that $W_u(Q)$ intersects the $x_1,x_2,...,x_M$ axis at $H(x_1,x_2,...,x_M,0,0,...,0)$ with

$$\begin{aligned} 0 &< x_1 < z_0, \\ 0 &< x_2 < z_0, \\ &.... \\ 0 &< x_M < z_0 \end{aligned}$$

Simple calculations show that $T(H)$ and H are symmetric with respect to S_2.Then the intersection of $W_u(Q)$ with S_2 is nonempty. Consequently, from proposition 2.2 it follows that the intersection of $W^u(Q)$ with $W^s(P)$ is nonempty, and hence the planar map T has a heteroclinic orbit. □

3. Localized solutions in nonlinear Schrodinger lattices

From non-local discrete equations driven by fractional powers of the discrete Laplacian. This equation is valid for all n :

$$i\frac{\partial\psi_n}{\partial t}+h(|\ \psi_n\ |)\psi_n+J\sum_{l=1}^{\frac{N}{2}-1}\frac{\psi_{n+l}-2\psi_n+\psi_{n-l}}{l^{1+\alpha}}=0 \tag{3.1}$$

where h is C^1 function.
Great attention has been paid to localized solutions of the form $\psi_n=\phi_n e^{iwt}$ where ϕ_n are time independent. Such solutions are time periodic and spatially localized.

Our goal is to proving the homoclinic orbit approach by exploiting the properties of reversible planar systems.
The equation (3.1) becomes:

$$-w\phi_n e^{iwt}+h(|\ \phi_n\ |)\phi_n e^{iwt}+Je^{iwt}\sum_{l=1}^{\frac{N}{2}-1}\frac{\phi_{n+l}-2\phi_n+\phi_{n-l}}{l^{1+\alpha}}=0$$

so

$$w\phi_n-h(|\ \phi_n\ |)\phi_n=J\sum_{l=1}^{\frac{N}{2}-1}\frac{\phi_{n+l}-2\phi_n+\phi_{n-l}}{l^{1+\alpha}}$$

where:

$$w\phi_n-h(|\ \phi_n\ |)\phi_n=J\sum_{l=1}^{\frac{N}{2}-1}\frac{\phi_{n+l}+\phi_{n-l}}{l^{1+\alpha}}-2J\sum_{l=1}^{\frac{N}{2}-1}\frac{\phi_n}{l^{1+\alpha}} \tag{3.2}$$

so

$$\sum_{l=1}^{\frac{N}{2}-1}\frac{\phi_{n+l}+\phi_{n-l}}{l^{1+\alpha}}=\frac{1}{J}(w\phi_n-h(|\ \phi_n\ |)\phi_n)+2\phi_n\sum_{l=1}^{\frac{N}{2}-1}\frac{1}{l^{1+\alpha}} \tag{3.3}$$

we set:

$$g(\phi_n) = f(\phi_n) + 2\phi_n \sum_{l=1}^{\frac{N}{2}-1} \frac{1}{l^{1+\alpha}} \tag{3.4}$$

We will compute the application T of order M such that $M = \dfrac{N}{2} - 1$.
By using a new variable :

$$\begin{cases} x_{1,n} = \phi_{n-M} \\ x_{2,n} = \phi_{n-(M-1)} \\ \dots\dots\dots\dots \\ x_{M,n} = \phi_{n-1} \\ x_{M+1,n} = \phi_n \\ x_{M+2,n} = \phi_{n+1} \\ \dots\dots\dots\dots \\ x_{2M,n} = \phi_{n+(M-1)} \end{cases}$$

so:

$$\begin{cases} x_{1,n+1} = \phi_{n-(M-1)} = x_{2,n} \\ x_{2,n+1} = \phi_{n-(M-2)} = x_{3,n} \\ \dots\dots\dots\dots \\ x_{M,n+1} = \phi_n = x_{M+1,n} \\ x_{M+1,n+1} = \phi_{n+1} = x_{M+2,n} \\ \dots\dots\dots\dots \\ x_{2M-1,n+1} = \phi_{n+2} = x_{2M,n} \\ x_{2M,n+1} = \phi_{n+M} \end{cases}$$

We have:

$$\sum_{l=1}^{M} \frac{\phi_{n+l} + \phi_{n-l}}{l^{1+\alpha}} = g(\phi_n) \tag{3.5}$$

$g(\phi_n)$ is written as:

$$g(\phi_n) = (\phi_{n-1} + \phi_{n+1}) + \frac{1}{2^{1+\alpha}}(\phi_{n-2} + \phi_{n+2}) + \frac{1}{3^{1+\alpha}}(\phi_{n-3} + \phi_{n+3}) + \dots\dots + \frac{1}{(N')^{1+\alpha}}(\phi_{n-M} + \phi_{n+M})$$

$$\phi_{n+M} = -(M)^{1+\alpha}\left(\sum_{l=1}^{M-1} \frac{\phi_{n-l} + \phi_{n+l}}{l^{1+\alpha}}\right) - \phi_{n-M} + (M)^{1+\alpha} g(\phi_n)$$

from which one takes

$$f(x_{M+1}) = \left(\frac{1}{J}(wx_{M+1} - h(|x_{M+1}|)x_{M+1}\right)$$

where $f(z) = (\frac{1}{J}(wz - h(|z|)z)$. Define $h_\infty = \lim_{r\to+\infty} h(r)$ if the limits exists, other wise $h_\infty = \infty$.

Theorem 3.1. *1-Assume that h is strictly increasing in $[0, +\infty[$. Then there exist a bright Solitons of the form $\phi_n e^{i\omega t}$ with $h(0) + 2\sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < \omega < h_\infty$ for the system (3.1) with $J > 0$.*
2-Assume that h is strictly decreasing in $[0, +\infty[$. Then there exist a bright Solitons of the form $\phi_n e^{i\omega t}$ with $h_\infty + 2\sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < \omega < h(0)$ for the system (3.1) with $J < 0$.

Proof . Assume that h is strictly increasing and $J > 0$. Then it follows that $f(z)$ has only three zeros if $h(0) + 2\sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < \omega < h_\infty$ and $f'(0) = (\omega - h(0))/J < 0$ for $J > 0$. Therefore system (3.1) admits bright solitons solutions by the theorem 2.6. Similarly the other cases can proved by theorem 2.6. □

Theorem 3.2. *Assume that $h'(r) > 0$ (< 0) for $r \in [0, +\infty[$. Then there exist a dark Solitons of the form $\phi_n e^{i\omega t}$ with $h(0) + 2\sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < \omega < h_\infty$ $(h_\infty + 2\sum_{l=1}^{M-1} \frac{1}{l^{1+\alpha}} < \omega < h(0))$ for the system* (3.1) *with $J < 0$ (> 0).*

Proof . The proof is abvious by the theorem 2.7. □

4. Conclusion

The existence of bright soliton solutions has been investigated by this method for a discrete Schrodinger equation with short term interaction in [15] and also by the variational method in [17]. The frequency ω associated with the sequence ϕ_n in which is a minimizer for some variational problem. Thus, one must first solve a variational problem to obtain a minimizer, and then derive the corresponding frequency. One cannot explicitly determine the allowed region of the ω frequency by the variational method. However, our approach gives the frequency ω and the corresponding sequence ϕ_n simultaneously, and thus one can obtain the existence interval of the frequency ω. Another efficient tool to find soliton solutions is the anti-integrability method [1, 7]. On the other hand, if we apply the anti-integrability method, we need the coupling strength to be small in order to apply the implicit function theorem. On the other hand, in our result, the existence of bright solitons implies that the coupling strength could be large.

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