

Comparison of Harder stability and Rus stability of Mann iteration procedure and their equivalence

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Abstract

In this paper, we study the stability of Mann iteration procedure in two directions, namely one due to Harder and the other one due to Rus with respect to a map $T : K \rightarrow K$ where K is a nonempty closed convex subset of a normed linear space X and that satisfies the property: there exist $\delta \in (0, 1)$ and $L \geq 0$ such that $\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\|$ for $x, y \in K$. Also, we show that the stability of the Mann iteration procedure in the sense of Rus may not imply that of Harder for weak contraction maps. Further, we compare and study the equivalence of these two stabilities under certain hypotheses and provide examples to illustrate the phenomena.

Keywords: Fixed point, Mann iteration procedure, stability in the sense of Harder, limit shadowing property, stability in the sense of Rus.

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1. Introduction

Throughout this paper, we assume that $(X, \|\cdot\|)$ is a normed linear space, K is a nonempty closed convex subset of X and $T : K \rightarrow K$ is a selfmap of K . A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote the set of all fixed points of T by $F(T)$.

In 1953, Mann [6] introduced an iteration procedure, namely Mann iteration procedure as follows: For $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n \quad \text{for } n = 0, 1, 2, \dots \quad (1.1)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$.

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In 1988, Harder and Hicks [4] established a systematic study of the stability of a general fixed point iteration procedure by defining the stability of general iteration procedure.

Definition 1.1. [4] Let (X, d) be a metric space, $T : X \rightarrow X$ be a map, $x_0 \in X$ and assume that the iteration procedure is defined by

$$x_{n+1} = f(T, x_n) \tag{1.2}$$

for $n = 0, 1, \dots$. Suppose that the sequence $\{x_n\}_{n=0}^\infty$ converges to a fixed point p of T . We say that the fixed point iteration procedure (1.2) is T -stable if for an arbitrary sequence $\{y_n\}_{n=0}^\infty$ in X , $\lim_{n \rightarrow \infty} \epsilon_n = 0$ if and only if $\lim_{n \rightarrow \infty} y_n = p$ where $\epsilon_n = d(y_{n+1}, f(T, y_n))$ for $n = 0, 1, 2, \dots$. In this case, we also say that the iteration procedure (1.2) is stable in the sense of Harder.

If we consider the Mann iteration procedure (1.1) in place of (1.2) then we say that the Mann iteration procedure is T -stable or stable in the sense of Harder.

Rhoades [10, 11] and Osilike [7, 8] studied the stability of Mann iteration procedure for the class of maps $T : K \rightarrow K$ that satisfies the condition

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\| \tag{1.3}$$

for some $\delta \in (0, 1)$, $L \geq 0$ and for all $x, y \in K$. Here we note that the map that satisfies condition (1.3) may not have a fixed point in complete normed linear spaces.

Berinde [2] introduced the concept of weak contraction map, i.e., there exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Ty\| \tag{1.4}$$

for all $x, y \in K$ and proved that the weak contraction map has a fixed point in complete metric spaces but does not ensure the uniqueness of fixed point in complete normed linear spaces.

In order to get the existence and uniqueness of fixed point of T , Babu, Sandhya and Kameswari [1] introduced the following condition : there exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L \min\{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\} \tag{1.5}$$

for all $x, y \in K$. A map T that satisfies (1.5) is said to be ‘B-weak contraction’. Every ‘B-weak contraction’ has a unique fixed point in complete metric space [1].

The concept of limit shadowing property was first introduced by Eirola, Nevanlinna and Pilyugin [3]. Rus[12] introduced GM-algorithm in terms of admissible perturbations and studied its stability by using limit shadowing property in the metric space setting. According to Rus [12], the Mann iteration procedure (1.1) is a special case of GM-algorithm [12] on a normed linear space. Rus [12] defined limit shadowing property and stability of Mann iteration procedure (1.1) as follows.

Definition 1.2. [12] Let X be a normed linear space, $K \subseteq X$ and K be convex. A map $T : K \rightarrow K$ is said to have limit shadowing property with respect to the Mann iteration procedure (1.1) if for any arbitrary sequence $\{y_n\}_{n=0}^\infty$ in K , $\lim_{n \rightarrow \infty} \epsilon_n = 0$ implies that there exists $x_0 \in K$ such that $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$, where $x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n$ and $\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n\|$ for $n = 0, 1, \dots$.

Definition 1.3. [12] The Mann iteration procedure is said to be stable in the sense of Rus with respect to an operator T if it is convergent with respect to T and the operator T has the limit shadowing property.

Ioana Timis [13] applied Rus definition for the stability of Picard iteration procedure and compared the two stabilities (one due to Rus and the other one due to Harder) and proved that Rus stability is more general than that of Harder.

We apply the following lemma in our subsequent discussion.

Lemma 1.4. [5] *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers. Assume that there exists a constant $0 \leq h < 1$ such that $a_{n+1} \leq ha_n + b_n$ for all n , and $\lim_{n \rightarrow \infty} b_n = 0$. Then $\lim_{n \rightarrow \infty} a_n = 0$.*

In this paper, we study the stability of the Mann iteration procedure (1.1) in the sense of Rus.

In Section 2, we prove that the Mann iteration procedure (1.1) is stable in the sense of Harder implies that it is stable in the sense of Rus and provide an example to show that its converse is not true. In Section 3, we study the stability of Mann iteration procedure (1.1) in the sense of Rus for a map that satisfies condition (1.3) and obtain the stability in the sense of Harder by proving the equivalence of the two stabilities in normed linear spaces. In Section 4, we show that the Mann iteration procedure (1.1) is stable in the sense of Rus may not imply the stability in the sense of Harder for weak contraction maps.

2. Comparison of Harder stability and Rus stability of Mann iteration procedure

Proposition 2.1. Let K be a nonempty closed convex subset of a normed linear space X , $T : K \rightarrow K$ be a selfmap of K with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^\infty$ be an arbitrary sequence in $(0, 1]$. Let $x_0 \in K$. We assume that the Mann iteration procedure (1.1) converges to a fixed point p of T and it is T -stable then T has a unique fixed point.

Proof . Let q be a fixed point of T in K with $q \neq p$. We consider the sequence $\{y_n\}_{n=0}^\infty$ where $y_n = q$ for $n = 0, 1, 2, \dots$. Then $\lim_{n \rightarrow \infty} \epsilon_n = \lim_{n \rightarrow \infty} \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T y_n\| = 0$ but $\lim_{n \rightarrow \infty} y_n = q \neq p$, a contradiction. Hence T has a unique fixed point. \square

Theorem 2.2. Under the hypotheses of Proposition 2.1, if the Mann iteration procedure (1.1) is stable in the sense of Harder then it is stable in the sense of Rus.

Proof . Let $\{y_n\}_{n=0}^\infty$ be a sequence in K and $\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T y_n\|$ for $n = 0, 1, 2, \dots$. We assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Since the Mann iteration procedure is T -stable, we have $\lim_{n \rightarrow \infty} y_n = p$.

By Proposition 2.1, T has a unique fixed point p and hence for any $x_0 \in K$, the sequence $\{x_n\}$ defined by (1.1) converges to p .

Since $\|y_n - x_n\| \leq \|y_n - p\| + \|x_n - p\|$ for all n , we have $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Therefore T has the limit shadowing property with respect to the Mann iteration procedure (1.1) and hence the Mann iteration procedure (1.1) is stable in the sense of Rus. \square

The following example suggests that the converse of Theorem 2.2 is not true. One more example (Example 4.1) is given in this direction in Section 4.

Example 2.3. Let $X = \mathbb{R}$ be equipped with the usual norm on \mathbb{R} and $K = [0, 1]$. We define $T : K \rightarrow K$ by

$$Tx = \begin{cases} \frac{2}{3} & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1, \end{cases}$$

so that $F(T) = \{\frac{2}{3}, 1\}$. Let $\alpha_0 = 1$ and $\alpha_n = \frac{n}{n+1}$ for $n = 1, 2, \dots$. It is easy to see that for any $x_0 \in [0, 1]$ the sequence $\{x_n\}_{n=0}^\infty$ generated by the Mann iteration procedure (1.1) converges to

a fixed point of T . For example, if $x_0 \in [0, 1)$ then $x_n = \frac{2}{3}$ for $n = 1, 2, \dots$ so that $\lim_{n \rightarrow \infty} x_n = \frac{2}{3}$ and if $x_0 = 1$ then $x_n = 1$ for all n so that $\lim_{n \rightarrow \infty} x_n = 1$. Let $\{y_n\}_{n=0}^\infty$ be a sequence in $[0, 1]$, $\epsilon_n = |y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T y_n|$ and $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Case (i): Suppose there exist a positive integer N such that $y_n \in [0, 1)$ for $n \geq N$. In this case, $\epsilon_n = |y_{n+1} - \frac{y_n}{n+1} - \frac{2n}{3(n+1)}|$ for all $n \geq N$. Therefore

$$\begin{aligned} |y_{n+1} - \frac{2}{3}| &\leq |y_{n+1} - \frac{y_n}{n+1} - \frac{2n}{3(n+1)}| + |\frac{y_n}{n+1} + \frac{2n}{3(n+1)} - \frac{2}{3}| \\ &= \epsilon_n + \frac{1}{n+1} |y_n - \frac{2}{3}| \\ &\leq \epsilon_n + \frac{1}{2} |y_n - \frac{2}{3}|, \end{aligned}$$

for all $n \geq N$. By Lemma 1.4, we have $\lim_{n \rightarrow \infty} y_n = \frac{2}{3}$. Here, we choose an arbitrary point $x_0 \in [0, 1)$ so that $\lim_{n \rightarrow \infty} x_n = \frac{2}{3}$ and hence $\lim_{n \rightarrow \infty} |y_n - x_n| = 0$.

Case(ii): Suppose that there is a positive integer N such that $y_n = 1$ for $n \geq N$. In this case, $\lim_{n \rightarrow \infty} y_n = 1$. If $x_0 = 1$ then $\lim_{n \rightarrow \infty} x_n = 1$ so that $\lim_{n \rightarrow \infty} |y_n - x_n| = 0$.

Case(iii): Suppose that $y_n \in [0, 1)$ for infinite values of n and $y_n = 1$ for infinitely many values of n . Let n_1 be the smallest positive integer such that $y_{n_1} \in [0, 1)$ and $y_{n_1+1} = 1$. By choosing $n_1 < n_2 < \dots < n_{k-1}$, let n_k be the smallest positive integer such that $y_{n_k} \in [0, 1)$ and $y_{n_k+1} = 1$. Thus we have constructed a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \in [0, 1)$ and $y_{n_k+1} = 1$ for $k = 1, 2, \dots$. Therefore

$$\begin{aligned} \epsilon_{n_k} &= |y_{n_k+1} - (1 - \alpha_{n_k})y_{n_k} - \alpha_{n_k} T y_{n_k}| \\ &= |1 - \frac{y_{n_k}}{n_k + 1} - \frac{2n_k}{3(n_k + 1)}| \\ &= |\frac{n_k + 3}{3(n_k + 1)} - \frac{y_{n_k}}{n_k + 1}|, \end{aligned}$$

for all k . Since $\lim_{k \rightarrow \infty} \epsilon_{n_k} = 0$, we have $\lim_{k \rightarrow \infty} (\frac{n_k+3}{3n_k+3} - \frac{y_{n_k}}{n_k+1}) = 0$. As the sequence $\{y_{n_k}\}$ is bounded, $\lim_{k \rightarrow \infty} \frac{y_{n_k}}{n_k+1} = 0$. Hence $\lim_{k \rightarrow \infty} \frac{n_k+3}{3n_k+3} = 0$. But $\lim_{k \rightarrow \infty} \frac{n_k+3}{3n_k+3} = \frac{1}{3}$, a contradiction. Hence, we do not need discuss Case (iii). Therefore by the above cases it follows that T has the limit shadowing property with respect to the Mann iteration procedure and hence it is stable in the sense of Rus. Here we observe that the Mann iteration procedure given in this example is not stable in the sense of Harder by Proposition 2.1.

In the following example, we show that the Mann iteration procedure (1.1) is neither stable in the sense of Harder nor in the sense of Rus and hence this example motivates us to consider a map that satisfies condition (1.3) in the study of stability analysis of Mann iteration procedure (1.1) in Section 3.

Example 2.4. Let $[-1, 1]$ be a closed convex subset of \mathbb{R} , where \mathbb{R} is the set of all real numbers with usual norm on \mathbb{R} . We define $T : [-1, 1] \rightarrow [-1, 1]$ by

$$Tx = \begin{cases} \frac{2}{3} + x & \text{if } x \in [-1, -\frac{2}{3}) \\ 1 + 2x & \text{if } x \in [-\frac{2}{3}, 0]. \\ 1 - 2x & \text{if } x \in [0, \frac{2}{3}] \\ \frac{2}{3} - x & \text{if } x \in (\frac{2}{3}, 1] \end{cases}$$

Then $F(T) = \{\frac{1}{3}\}$. Let $\alpha_n = \frac{1}{n+1}$ for $n = 0, 1, 2, \dots$. We now show that for any x_0 in $[-1, 1]$, $\{x_n\}$ defined by (1.1) converges to $\frac{1}{3}$.

Case(i): Let $x_0 \in [-1, -\frac{2}{3})$. In this case, $x_1 = \frac{2}{3} + x_0 \in [-\frac{1}{3}, 0)$ so that $Tx_1 = \frac{7}{3} + 2x_0$ and

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{3(1 + x_0)}{2} \in [0, \frac{1}{2}).$$

Therefore $x_3 = \frac{2x_2 + Tx_2}{3} = \frac{1}{3}$. On continuing this process, we get $x_n = \frac{1}{3}$, for $n = 3, 4, \dots$

Case(ii): Let $x_0 \in [-\frac{2}{3}, 0]$.

Sub-Case(i): Let $x_0 \in [-\frac{2}{3}, -\frac{1}{2}]$. Here $x_1 = Tx_0 = 1 + 2x_0 \in [-\frac{1}{3}, 0]$ and

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{3x_1 + 1}{2} = 2 + 3x_0 \in [0, \frac{1}{2}].$$

Therefore $x_3 = \frac{2x_2 + Tx_2}{3} = \frac{2x_2 + 1 - 2x_2}{3} = \frac{1}{3}$. Thus $x_n = \frac{1}{3}$ for $n = 3, 4, \dots$

Sub-Case(ii): Let $x_0 \in [-\frac{1}{2}, -\frac{1}{6}]$ so that $x_1 = Tx_0 = 1 + 2x_0 \in [0, \frac{2}{3}]$,

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{x_1 + 1 - 2x_1}{2} = -x_0 \in [\frac{1}{6}, \frac{1}{2}], \quad \text{and} \quad x_3 = \frac{2x_2 + Tx_2}{3} = \frac{2x_2 + 1 - 2x_2}{3} = \frac{1}{3}.$$

Thus $x_n = \frac{1}{3}$ for $n = 3, 4, \dots$

Sub-Case(iii): Let $x_0 \in (-\frac{1}{6}, 0]$ so that $x_1 = Tx_0 = 1 + 2x_0 \in (\frac{2}{3}, 1]$,

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{x_1 + \frac{2}{3} - x_1}{2} = \frac{1}{3}.$$

Therefore $x_n = \frac{1}{3}$ for $n = 2, 3, \dots$

Case(iii): Let $x_0 \in [0, \frac{2}{3}]$.

Sub-Case(i): Let $x_0 \in [0, \frac{1}{6})$ so that $x_1 = 1 - 2x_0 \in (\frac{2}{3}, 1]$,

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{x_1 + \frac{2}{3} - x_1}{2} = \frac{1}{3}.$$

Therefore $x_n = \frac{1}{3}$ for $n = 2, 3, \dots$

Sub-Case(ii): Let $x_0 \in [\frac{1}{6}, \frac{1}{2}]$ so that $x_1 = 1 - 2x_0 \in [0, \frac{2}{3}]$,

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{x_1 + 1 - 2x_1}{2} = \frac{1 - (1 - 2x_0)}{2} = x_0 \quad \text{and} \quad x_3 = \frac{2x_2 + Tx_2}{3} = \frac{2x_0 + 1 - 2x_0}{3} = \frac{1}{3}.$$

Therefore $x_n = \frac{1}{3}$ for $n = 3, 4, \dots$

Sub-Case(iii): Let $x_0 \in [\frac{1}{2}, \frac{2}{3}]$ so that $x_1 = Tx_0 = 1 - 2x_0 \in [-\frac{1}{3}, 0]$,

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{x_1 + 1 - 2x_1}{2} = x_0.$$

As $x_3 = \frac{2x_2 + Tx_2}{3} = \frac{2x_0 + 1 - 2x_0}{3} = \frac{1}{3}$, we have $x_n = \frac{1}{3}$ for $n = 3, 4, \dots$

Case(iv): Let $x_0 \in (\frac{2}{3}, 1]$. Therefore $x_1 = Tx_0 = \frac{2}{3} - x_0 \in [-\frac{1}{3}, 0)$,

$$x_2 = \frac{x_1 + Tx_1}{2} = \frac{x_1 + 1 + 2x_1}{2} = \frac{3x_1 + 1}{2} \in [0, \frac{1}{2}) \quad \text{and} \quad x_3 = \frac{2x_2 + Tx_2}{3} = \frac{2x_2 + 1 - 2x_2}{3} = \frac{1}{3}.$$

Hence $x_n = \frac{1}{3}$ for $n = 3, 4, \dots$

Thus in all the above cases we have shown that for any $x_0 \in [-1, 1]$, the sequence $\{x_n\}$ converges to the fixed point $\frac{1}{3}$. We consider the sequence $\{y_n\}_{n=0}^\infty \subseteq [-1, 1]$, where $y_n = \frac{(-1)^n}{n+1}$ for $n = 0, 1, 2, \dots$. Since $y_n \in [-\frac{1}{2}, \frac{1}{3}]$, for $n = 1, 2, \dots$, we have $Ty_n = \frac{n-1}{n+1}$ so that

$$\begin{aligned} \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n| \\ &= \left| \frac{(-1)^{n+1}}{n+2} + \frac{n(-1)^{n+1}}{(n+1)^2} - \frac{(n-1)}{(n+1)^2} \right| \\ &= \left| \frac{(-1)^{n+1}(2n^2 + 4n + 1) - (n^2 + n - 2)}{(n+2)(n+1)^2} \right| \end{aligned}$$

for $n = 0, 1, 2, \dots$. Therefore

$$\epsilon_n = \begin{cases} \left| \frac{-3n^2 - 5n + 1}{n^3 + 4n^2 + 5n + 2} \right| & \text{if } n \text{ is even} \\ \frac{n^2 + 3n + 3}{n^3 + 4n^2 + 5n + 2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\lim_{n \rightarrow \infty} \epsilon_n = 0$ but $\lim_{n \rightarrow \infty} y_n = 0 \neq \frac{1}{3}$ so that the Mann iteration procedure is not stable in the sense of Harder. Moreover, for any $x_0 \in [-1, 1]$, $\lim_{n \rightarrow \infty} |y_n - x_n| = \frac{1}{3} \neq 0$ so that T does not satisfy the limit shadowing property. Therefore it is not stable in the sense of Rus.

Hence the following question is natural.

Question: Under what hypothesis, the converse of Theorem 2.2 holds ?

Its answer is given in Theorem 3.2 of Section 3.

3. Equivalence of Harder stability and Rus stability of Mann iteration procedure

Theorem 3.1. Let K be a nonempty closed convex subset of a normed linear space X , $T : K \rightarrow K$ be a selfmap that satisfies the condition (1.3). We assume that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $(0, 1]$ such that $\alpha \leq \alpha_n$ for some $\alpha > 0$ and $n = 0, 1, 2, \dots$. Then the Mann iteration procedure (1.1) is stable in the sense of Rus.

Proof . Since $F(T) \neq \emptyset$ and T has at most one fixed point, $F(T)$ is singleton, say $\{p\}$. Let $x_0 \in K$ be arbitrary and $\{x_n\}_{n=0}^\infty$ be the sequence generated by the Mann iteration procedure (1.1). We consider

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\| \\ &= (1 - \alpha_n(1 - \delta))\|x_n - p\| \\ &\leq (1 - \alpha(1 - \delta))\|x_n - p\| \text{ for } n = 0, 1, 2, \dots \\ &\leq (1 - \alpha(1 - \delta))^2\|x_{n-1} - p\| \\ &\vdots \\ &\leq (1 - \alpha(1 - \delta))^{n+1}\|x_0 - p\|. \end{aligned}$$

Since $0 < 1 - \alpha(1 - \delta) < 1$, we have $\lim_{n \rightarrow \infty} (1 - \alpha(1 - \delta))^{n+1} = 0$. Therefore $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$ so that the sequence $\{x_n\}$ converges to p .

Let $\{y_n\}_{n=0}^\infty$ be an arbitrary sequence in K and $\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n\|$. We assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. We consider

$$\begin{aligned}
 \|y_{n+1} - x_{n+1}\| &\leq \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n\| + \|(1 - \alpha_n)y_n + \alpha_nTy_n - x_{n+1}\| \\
 &\leq \epsilon_n + (1 - \alpha_n)\|y_n - x_n\| + \alpha_n\|Ty_n - Tx_n\| \\
 &\leq \epsilon_n + (1 - \alpha_n)\|y_n - x_n\| + \alpha_n[\delta\|y_n - x_n\| + L\|x_n - Tx_n\|] \\
 &\leq \epsilon_n + (1 - \alpha_n(1 - \delta))\|y_n - x_n\| + L\alpha_n\|x_n - Tx_n\| \\
 &\leq \epsilon_n + (1 - \alpha(1 - \delta))\|y_n - x_n\| + L\alpha_n\|x_n - Tx_n\| \\
 &= \epsilon_n + (1 - \alpha(1 - \delta))\|y_n - x_n\| + L\|x_{n+1} - x_n\|.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $0 < 1 - \alpha(1 - \delta) < 1$, by applying Lemma 1.4, we have $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$. Thus T has the limit shadowing property with respect to the Mann iteration procedure and hence it is stable in the sense of Rus. \square

The following theorem provides criteria for the equivalence of T -stabilities due to Harder and due to Rus.

Theorem 3.2. Let K be a nonempty closed convex subset of a normed linear space X , $T : K \rightarrow K$ be a selfmap that satisfies the condition (1.3). We assume that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $(0, 1]$ such that $\sum_{n=0}^\infty \alpha_n = \infty$. Then the Mann iteration procedure (1.1) is stable in the sense of Harder if and only if it is stable in the sense of Rus.

Proof . Since $F(T) \neq \emptyset$, let p be the fixed point of T in K . Let $x_0 \in K$ and $\{x_n\}_{n=0}^\infty$ be the sequence generated by the Mann iteration procedure (1.1). We consider

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - p\| \\
 &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - Tp\| \\
 &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n[\delta\|x_n - p\| + L\|p - Tp\|] \\
 &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\| \\
 &= [1 - \alpha_n(1 - \delta)]\|x_n - p\| \\
 &\leq [1 - \alpha_n(1 - \delta)][1 - \alpha_{n-1}(1 - \delta)]\|x_{n-1} - p\| \\
 &\vdots \\
 &\leq \prod_{k=0}^n [1 - \alpha_k(1 - \delta)]\|x_0 - p\| \quad \text{for } n = 0, 1, 2, \dots \tag{3.1}
 \end{aligned}$$

By the mean value theorem, we have $1 - x \leq e^{-x}$ for all $x > 0$ so that $0 < [1 - \alpha_k(1 - \delta)] \leq e^{-\alpha_k(1 - \delta)}$ for all k and hence $0 \leq \prod_{k=0}^n (1 - \alpha_k(1 - \delta)) \leq e^{-\sum_{k=0}^n \alpha_k(1 - \delta)}$ for $n = 0, 1, 2, \dots$.

Since $\sum_{n=0}^\infty \alpha_n = \infty$, we have $\lim_{n \rightarrow \infty} \prod_{k=0}^n (1 - \alpha_k(1 - \delta)) = 0$. Hence from (3.1) it follows that $\lim_{n \rightarrow \infty} x_n = p$. Thus we have proved that for any $x_0 \in K$, the sequence $\{x_n\}$ converges to p .

If the Mann iteration procedure (1.1) is T -stable then by Theorem 2.2 it is stable in the sense of Rus. We now assume that the Mann iteration procedure is stable in the sense of Rus. Let $\{y_n\}_{n=0}^\infty$ be an arbitrary sequence in K and set $\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n\|$. We assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Since T has the limit shadowing property with respect to the Mann iteration procedure, there exists $x_0 \in K$ such that $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ where $\{x_n\}$ is defined by (1.1). Since $\|y_n - p\| \leq \|y_n - x_n\| + \|x_n - p\|$, we have $\lim_{n \rightarrow \infty} y_n = p$.

Conversely we assume that $\lim_{n \rightarrow \infty} y_n = p$. We consider

$$\begin{aligned}
 \epsilon_n &= \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n\| \\
 &\leq \|y_{n+1} - p\| + \|(1 - \alpha_n)y_n + \alpha_nTy_n - p\| \\
 &\leq \|y_{n+1} - p\| + (1 - \alpha_n)\|y_n - p\| + \alpha_n\|Ty_n - p\|
 \end{aligned}$$

$$\begin{aligned} &\leq \|y_{n+1} - p\| + (1 - \alpha_n)\|y_n - p\| + \alpha_n\delta\|y_n - p\| \\ &= \|y_{n+1} - p\| + (1 - \alpha_n(1 - \delta))\|y_n - p\|. \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Hence the Mann iteration procedure is stable in the sense of Harder. \square

Corollary 3.3. [[7, Theorem 5]] Under the hypotheses of Theorem 3.1, the Mann iteration procedure (1.1) is stable in the sense of Harder.

Proof . Since $\alpha_n \geq \alpha$ for $n = 0, 1, 2, \dots$, we have $\sum_{n=0}^{\infty} \alpha_n = \infty$ and hence the conclusion follows from Theorem 3.1 and Theorem 3.2. \square

The following example is in support of Theorem 3.1 and Corollary 3.3 and also it answer the following question: Can we replace the condition “there exists $\alpha \in (0, 1)$ such that $\alpha_n \geq \alpha$ for $n = 0, 1, 2, \dots$ ” in either Theorem 3.1 or Corollary 3.3 by the condition $\sum_{n=0}^{\infty} \alpha_n = \infty$? Case(ii) of the following suggest that its answer is ‘no’.

Example 3.4. Let $X = \mathbb{R}$ be a normed linear space with the usual norm and $K = [0, 1]$. We define $T : [0, 1] \rightarrow [0, 1]$ by

$$Tx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{x}{2} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Here $F(T) = \{0\}$ and Păcurar [9] showed that T satisfies the condition (1.3) with $\delta = \frac{1}{2}$ and $L = 1$.

Case(i): Let $\alpha_0 = 1, \alpha_n = \frac{n}{n+1}$ for $n = 1, 2, \dots$ so that $\alpha_n \geq \frac{1}{2}$ and hence $\sum_{n=0}^{\infty} \alpha_n = \infty$. First we prove that for any $x_0 \in [0, 1]$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.1), converges to the fixed point 0.

By induction on n , it is easy to see that for any $x_0 \in [0, 1]$, $x_n \leq \frac{x_1}{n!}$ for $n = 1, 2, \dots$ so that $\lim_{n \rightarrow \infty} x_n = 0$.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $[0, 1]$ and $\epsilon_n = |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n|$ for $n = 0, 1, \dots$. We assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. We consider

$$\begin{aligned} |y_{n+1}| &\leq |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n| + |(1 - \alpha_n)y_n + \alpha_nTy_n| \\ &\leq \epsilon_n + \frac{|y_n|}{n+1} + \frac{n}{n+1}|Ty_n| \\ &\leq \epsilon_n + \frac{y_n}{n+1} + \frac{n}{n+1} \frac{y_n}{2} \\ &= \epsilon_n + \frac{(n+2)y_n}{2(n+1)}. \end{aligned}$$

Therefore $y_{n+1} \leq \epsilon_n + \frac{3}{4}y_n$ for $n = 1, 2, \dots$, and hence by Lemma 1.4, we have $\lim_{n \rightarrow \infty} y_n = 0$. Moreover, for any $x_0 \in [0, 1]$, we have $\lim_{n \rightarrow \infty} x_n = 0$ so that $\lim_{n \rightarrow \infty} |y_n - x_n| = 0$. Therefore T has the limit shadowing property with respect to the Mann iteration procedure and hence the Mann iteration procedure is stable in the sense of Rus.

Now we assume that $\lim_{n \rightarrow \infty} y_n = 0$. Therefore by using the continuity of T at 0 it follows that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Hence the Mann iteration procedure (1.1) is stable in the sense of Harder and in the sense of Rus.

Case(ii): Let $\alpha_n = \frac{1}{n+1}$ $n = 0, 1, 2, \dots$ so that $\sum_{n=0}^{\infty} \alpha_n = \infty$. By induction on n , it is easy to see that for any $x_0 \in [0, 1]$, $x_n \leq \frac{x_1}{n}$ for $n = 1, 2, \dots$ so that $\lim_{n \rightarrow \infty} x_n = 0$, i.e., the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the Mann iteration procedure (1.1) converges to the fixed point 0 of T . We consider

the sequence $\{y_n\}$ where $y_n = \frac{n}{2(n+2)}$ for $n = 0, 1, 2, \dots$.

$$\begin{aligned} \text{Since } y_n \in (0, \frac{1}{2}], Ty_n = 0 \text{ and } \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n| \\ &= |\frac{n+1}{2n+6} - (1 - \frac{1}{n+1})\frac{n}{2n+4}| \\ &= |\frac{n+1}{2n+6} - \frac{n^2}{(n+1)(2n+4)}| \\ &= \frac{n^2+5n+2}{2(n+1)(n+2)(n+3)}. \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \epsilon_n = 0$. But $\lim_{n \rightarrow \infty} y_n = \frac{1}{2} \neq 0$. Hence the Mann iteration procedure is not stable in the sense of Harder. Since for any $x_0 \in [0, 1]$, $\lim_{n \rightarrow \infty} |y_n - x_n| = \frac{1}{2} \neq 0$ so that T does not satisfy the limit shadowing property with respect to the Mann iteration procedure (1.1) and hence it is not stable in the sense of Rus. Thus neither of the stabilities hold for Mann iteration procedure.

Note: Example 3.4 illustrates the importance of the hypotheses in proving Theorem 3.1 and Theorem 3.2.

Corollary 3.5. Let K be a nonempty closed convex subset of a Banach space X , and $T : K \rightarrow K$ be a B-weak contraction map, that is, T satisfies (1.5). Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $(0, 1]$ such that $\sum_{n=0}^\infty \alpha_n = \infty$. Then the Mann iteration procedure (1.1) is stable in the sense of Harder if and only if it is stable in the sense of Rus.

Proof . By Theorem 2.3 of [1], T has a unique fixed point p in K . Since T satisfies the condition (1.3), the conclusion follows from Theorem 3.2. \square

Corollary 3.6. Under the hypotheses of Corollary 3.5, if $\alpha_n \geq \alpha$ for some $\alpha > 0$ then the Mann iteration procedure (1.1) is stable in the sense of Rus as well as it is stable in the sense of Harder.

Proof . Follows from Theorem 3.1 and Corollary 3.5. \square

The following is in support of Corollary 3.5.

Example 3.7. Let $[0, \frac{1}{2}]$ be a closed convex subset of the Banach Space \mathbb{R} equipped with the usual norm and $T : [0, \frac{1}{2}] \rightarrow [0, \frac{1}{2}]$ be a selfmap defined by

$$Tx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{4}] \\ \frac{x^2}{2} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}]. \end{cases}$$

Then $F(T) = \{0\}$ and T is a B-weak contraction map, i.e., T satisfies condition (1.5) with $\delta = \frac{1}{2}$ and $L = 1$.

Case(i): We choose $\alpha_0 = 1, \alpha_n = \frac{n}{n+1}$ so that $\sum_{n=0}^\infty \alpha_n = \infty$.

We show that for any $x_0 \in [0, \frac{1}{2}]$, the sequence $\{x_n\}_{n=0}^\infty$ generated by the Mann iteration procedure (1.1) converges to 0. If $x_0 \in [0, \frac{1}{2}]$ then by induction on n , it is easy to see that $x_n \leq \frac{x_1}{n!}$ for $n = 1, 2, \dots$ so that $\lim_{n \rightarrow \infty} x_n = 0$.

Let $\{y_n\}_{n=0}^\infty$ be a sequence in $[0, \frac{1}{2}]$ and set $\epsilon_n = |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n|$. We assume that

$\lim_{n \rightarrow \infty} \epsilon_n = 0$. Therefore

$$\begin{aligned} |y_{n+1}| &\leq |y_{n+1} - (1 - \alpha_n)y_n - Ty_n| + |(1 - \alpha_n)y_n + \alpha_nTy_n| \\ &= \epsilon_n + \frac{y_n}{n + 1} + \frac{nTy_n}{n + 1} \\ &\leq \epsilon_n + \frac{y_n}{n + 1} + \frac{ny_n^2}{2(n + 1)} \\ &\leq \epsilon_n + \frac{(n + 2)y_n}{2(n + 1)} \\ &\leq \epsilon_n + \frac{3}{4}y_n \end{aligned}$$

for $n = 0, 1, 2, 3, \dots$. By Lemma 1.4, we have $\lim_{n \rightarrow \infty} y_n = 0$. Moreover, for any $x_0 \in [0, \frac{1}{2}]$, $\lim_{n \rightarrow \infty} x_n = 0$ so that $\lim_{n \rightarrow \infty} |y_n - x_n| = 0$.

Thus T has the limit shadowing property with respect to the Mann iteration procedure and hence the Mann iteration procedure is stable in the sense of Rus. Now we assume that $\lim_{n \rightarrow \infty} y_n = 0$. Therefore

$$\begin{aligned} \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n| \\ &\leq y_{n+1} + \frac{y_n}{n + 1} + \frac{n}{n + 1} \frac{y_n^2}{2} \end{aligned}$$

so that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Therefore the Mann iteration procedure (1.1) is stable in the sense of Harder as well as in the sense of Rus.

Case(ii): We choose $\alpha_n = \frac{1}{n+1}$ for $n = 0, 1, 2, \dots$ so that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Therefore T satisfies the hypotheses of Corollary 3.5. We show that for any $x_0 \in [0, \frac{1}{2}]$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the Mann iteration procedure (1.1) converges to 0. If $x_0 \in [0, \frac{1}{2}]$ then by induction on n , it is easy to see that $x_n \leq \frac{x_1}{n}$ for $n = 1, 2, \dots$ so that $\lim_{n \rightarrow \infty} x_n = 0$. Let $y_n = \frac{n}{4(n+4)}$ for $n = 0, 1, 2, \dots$ be a sequence in $[0, \frac{1}{2}]$. Therefore $Ty_n = 0$ for all n and

$$\begin{aligned} \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n| \\ &= \left| \frac{n + 1}{4(n + 5)} - \frac{n^2}{4(n + 1)(n + 4)} \right| \\ &= \frac{n^2 + 9n + 4}{4(n + 1)(n + 4)(n + 5)}, \end{aligned}$$

so that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. But $\lim_{n \rightarrow \infty} y_n = \frac{1}{4} \neq 0$. Therefore, the Mann iteration procedure is not stable in the sense of Harder. Moreover, for any $x_0 \in [0, \frac{1}{2}]$, $\lim_{n \rightarrow \infty} |y_n - x_n| = \frac{1}{4} \neq 0$ showing that T does not satisfy the limit shadowing property with respect to the Mann iteration procedure. Hence neither of the stabilities hold for this Mann iteration procedure.

4. Stability of Mann iteration procedure with respect to weak contraction

In this section, we answer the following question.

Question: “Does the conclusion of Theorem 3.1 and Theorem 3.2 hold if T is a weak contraction map ?”

The following example suggests that its answer is 'No'.

Example 4.1. Let $X = \mathbb{R}$ where \mathbb{R} is the set of all real numbers with usual norm on \mathbb{R} . We define $T : [0, 1] \rightarrow [0, 1]$ by

$$Tx = \begin{cases} \frac{2}{3}x & \text{if } x \in [0, \frac{1}{2}], \\ \frac{2}{3}x + \frac{1}{3} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Păcurar [9] showed that T is a weak contraction map, i.e., T satisfies (1.4) with $\delta = \frac{2}{3}$ and $L = 6$. Here $F(T) = \{0, 1\}$. We choose $\alpha_0 = 1, \alpha_n = \frac{n}{n+1}$ for $n = 1, 2, \dots$ so that $\frac{1}{2} \leq \alpha_n$ for $n = 0, 1, \dots$.

Case(i): First we show that for any $x_0 \in [0, 1]$, the sequence $\{x_n\}_{n=0}^\infty$ generated by the Mann iteration procedure (1.1) converges to a fixed point of T .

Sub-Case (i): Let $x_0 \in [0, \frac{1}{2}]$ so that $x_1 = Tx_0 = \frac{2x_0}{3} \in [0, \frac{1}{3}]$. By induction on n , we show that $x_n \in [0, \frac{1}{3}]$ and $x_n = \frac{(2n+1)!}{6^n(n!)^2}x_1$ for $n = 1, 2, 3, \dots$. We assume that $x_n \in [0, \frac{1}{3}]$ and $x_n = \frac{(2n+1)!}{6^n(n!)^2}x_1$ for some positive integer n . Then $x_{n+1} = \frac{2n+3}{3n+3}x_n$ so that $x_{n+1} \in [0, \frac{1}{3}]$ and $x_{n+1} = \frac{2n+3}{3n+3} \frac{(2n+1)!}{6^n(n!)^2}x_1 = \frac{(2n+3)!}{6^{n+1}((n+1)!)^2}x_1$. Therefore $x_n = \frac{(2n+1)!}{6^n(n!)^2}x_1$ for $n = 1, 2, 3, \dots$.

Since $\lim_{n \rightarrow \infty} \frac{(2n+1)!}{6^n(n!)^2} = 0$, the sequence $\{x_n\}$ converges to 0.

Sub-Case(ii): Let $x_0 \in (\frac{1}{2}, 1]$ so that $x_1 = Tx_0 = \frac{2}{3}x_0 + \frac{1}{3} \in (\frac{2}{3}, 1] \subset (\frac{1}{2}, 1]$. By induction on n , we show that $x_n \in (\frac{1}{2}, 1]$ and $|x_n - 1| = \frac{(2n+1)!}{6^n(n!)^2}|x_1 - 1|$ for $n = 1, 2, \dots$. We assume that $x_n \in (\frac{1}{2}, 1]$ and $|x_n - 1| = \frac{(2n+1)!}{6^n(n!)^2}|x_1 - 1|$ for some $n \geq 1$. Then $x_{n+1} = \frac{(2n+3)x_n + n}{3n+3}$ so that $x_{n+1} \in (\frac{1}{2}, 1]$ and

$$|x_{n+1} - 1| = \frac{2n+3}{3n+3}|x_n - 1| = \frac{(2n+3)!}{6^{n+1}((n+1)!)^2}|x_1 - 1|$$

for $n = 1, 2, 3, \dots$. Therefore $|x_n - 1| = \frac{(2n+1)!}{6^n(n!)^2}|x_1 - 1|$ for $n = 1, 2, 3, \dots$ and hence $\lim_{n \rightarrow \infty} x_n = 1$.

Case(ii): Now we show that T has the limit shadowing property with respect to the Mann iteration procedure (1.1). Let $\{y_n\}_{n=0}^\infty$ be an arbitrary sequence in $[0, 1]$ and set $\epsilon_n = |y_{n+1} - (1 - \alpha_n)y_n - \alpha_nTy_n|$ for $n = 0, 1, 2, \dots$. We assume that $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Sub-Case (i): Suppose there is a positive integer n_0 such that $y_n \in [0, \frac{1}{2}]$ for $n \geq n_0$. Then $Ty_n = \frac{2y_n}{3}$ and $\epsilon_n = |y_{n+1} - \frac{(2n+3)y_n}{3n+3}|$ for $n \geq n_0$. Therefore

$$|y_{n+1}| \leq \left| y_{n+1} - \frac{2n+3}{3n+3}y_n \right| + \frac{2n+3}{3n+3}|y_n| \leq \epsilon_n + \frac{5}{6}|y_n|$$

for $n \geq n_0$. By Lemma 1.4, $\lim_{n \rightarrow \infty} y_n = 0$. We choose an arbitrary point $x_0 \in [0, \frac{1}{2}]$ so that $\lim_{n \rightarrow \infty} x_n = 0$ and hence $\lim_{n \rightarrow \infty} |y_n - x_n| = 0$.

Sub-Case(ii): Suppose there is an integer n_0 such that $y_n \in (\frac{1}{2}, 1]$ for $n \geq n_0$. Then $Ty_n = \frac{2}{3}y_n + \frac{1}{3}$ and

$$\epsilon_n = \left| y_{n+1} - \frac{y_n}{n+1} - \frac{n}{n+1} \left(\frac{2}{3}y_n + \frac{1}{3} \right) \right| = \left| y_{n+1} - \frac{(2n+3)y_n}{3n+3} - \frac{n}{3n+3} \right|,$$

for $n \geq n_0$. Therefore,

$$\begin{aligned} |y_{n+1} - 1| &\leq \left| y_{n+1} - \frac{(2n+3)y_n}{3n+3} - \frac{n}{3n+3} \right| + \left| \frac{(2n+3)y_n}{3n+3} + \frac{n}{3n+3} - 1 \right| \\ &\leq \epsilon_n + \frac{2n+3}{3n+3}|y_n - 1| \\ &\leq \epsilon_n + \frac{5}{6}|y_n - 1| \end{aligned}$$

for all $n \geq n_0$. By Lemma 1.4, $\lim_{n \rightarrow \infty} |y_n - 1| = 0$ so that the sequence $\{y_n\}$ converges to 1. We choose an arbitrary point $x_0 \in (\frac{1}{2}, 1]$ so that $\lim_{n \rightarrow \infty} x_n = 1$ and hence $\lim_{n \rightarrow \infty} |y_n - x_n| = 0$.

Sub-Case(iii): Suppose $y_n \in [0, \frac{1}{2}]$ for infinitely many values of n and $y_n \in (\frac{1}{2}, 1]$ for infinitely many values of n . Let n_1 be the smallest positive integer such that $y_{n_1} \in [0, \frac{1}{2}]$ and $y_{n_1+1} \in (\frac{1}{2}, 1]$. By choosing $n_1 < n_2 < \dots < n_{k-1}$, let n_k be the smallest positive integer such that $y_{n_k} \in [0, \frac{1}{2}]$ and $y_{n_k+1} \in (\frac{1}{2}, 1]$. Thus we have constructed a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ such that $y_{n_k} \in [0, \frac{1}{2}]$ and $y_{n_k+1} \in (\frac{1}{2}, 1]$ for all k . Therefore $\lim_{n \rightarrow \infty} \epsilon_n = 0$ implies that

$$\lim_{k \rightarrow \infty} \left| y_{n_k+1} - \frac{(2n_k + 3)y_{n_k}}{3n_k + 3} \right| = 0.$$

Since $\frac{2n_k+3}{3n_k+3}y_{n_k} \in [0, \frac{5}{12}]$, $|y_{n_k+1} - (\frac{2n_k+3}{3n_k+3})y_{n_k}| > \frac{1}{12}$ for all k so that $\lim_{k \rightarrow \infty} |y_{n_k+1} - \frac{(2n_k+3)y_{n_k}}{3n_k+3}| \neq 0$, a contradiction. Hence we need not discuss Sub-Case(iii) of Case(ii).

Therefore by Case(ii), T has the limit shadowing property with respect to the Mann iteration procedure (1.1) and hence it is stable in the sense of Rus. But, by the Proposition 2.1, we observe that the Mann iteration procedure (1.1) is not stable in the sense of Harder.

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