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Inequalities for an operator on the space of polynomials

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Abstract

Let $\mathcal{P}_n$ be the class of all complex polynomials of degree at most n. Recently Rather et. al.[On the zeros of certain composite polynomials and an operator preserving inequalities, Ramanujan J., 54(2021) 605–612. https://doi.org/10.1007/s11139-020-00261-2] introduced an operator $N : \mathcal{P}_n \to \mathcal{P}_n$ defined by $N[P](z) := \sum_{j=0}^{k} \lambda_j \left(\frac{nz}{2}\right)^j \frac{P^{(j)}(z)}{j!},\ k \le n$ where $\lambda_j \in \mathbb{C}$, $j = 0, 1, 2, \dots, k$ are such that all the zeros of $\phi(z) = \sum_{j=0}^{k} \binom{n}{j} \lambda_j z^j$ lie in the half plane $|z| \le \left|z - \frac{n}{2}\right|$ and established certain sharp Bernstein-type polynomial inequalities. In this paper, we prove some more general results concerning the operator $N : \mathcal{P}_n \to \mathcal{P}_n$ preserving inequalities between polynomials. Our results not only contain several well known results as special cases but also yield certain new interesting results as special cases.

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1. Introduction

Polynomial inequalities have been investigated for quite some time and have important applications in all those mathematical models whose solutions lead to the problem of valuing how large or small the maximum modulus of the derivative of an algebraic polynomial can be in terms of the maximum modulus and degree of that polynomial. Polynomial inequalities are also fundamental for the proofs of many inverse theorems in polynomial approximation theory, which is concerned with approximating unknown or complicated functions by polynomials. In the first place, this concept of

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best approximation was introduced in Mathematics mainly by the work of the famous mathematician Chebyshev(1821-1894), who studied some properties of polynomial with least deviation from given continuous function. He introduced the polynomials known today as Chebyshev polynomials of first kind, which appear prominently in various extremal problems concerned with polynomial. Historically, the questions relating to approximations by polynomials have given rise to some of the interesting problems in Mathematics and engendered extensive research over the past millennium.

2. Preliminaries

Before we state some of the fundamental results in this circle of ideas, we first have a look at the symbols and notations which will be used throughout this paper.

We shall use $\mathbb{P}_n$ to denote the vector space of all polynomials of degree at most n over the field $\mathbb{C}$ of complex numbers. By $\mathcal{P}_n$, we shall be referring to the class of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree exactly n. For any $c \in \mathbb{C}$, the linear function taking $z \mapsto cz$, will be denoted by σ_c. We shall also be denoting by ψ the function which maps $z \mapsto z^n$. Further we shall use $\circ$ to denote the usual composition of functions. Thus, in this notation, for any complex function $F : \mathbb{C} \longmapsto \mathbb{C}$, the function $F \circ \sigma_R$ is defined as $F \circ \sigma_R(z) := F(\sigma_R(z)) = F(Rz)$, for $z \in \mathbb{C}$.

For $P \in \mathcal{P}_n$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{2.1}$$

and

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)| \tag{2.2}$$

Inequality (2.1) is due to S.Bernstein [5] (see also[12, 8]), whereas inequality (2.2) is a simple consequence of the maximum modulus principle [9, p.346]. Aziz and Rather [4] generalized inequalities (2.1) and (2.2) by proving *if $P \in \mathcal{P}_n$, then for every β with $|\beta| \leq 1, R > 1$ and $|z| \geq 1$,*

$$|P(Rz) - \beta P(z)| \leq |R^n - \beta||z|^n \max_{|z|=1} |P(z)|, \tag{2.3}$$

which clearly includes inequalities (2.1) and (2.2) as special cases. For $P \in \mathcal{P}_n$ *having all its zeros in* $|z| \leq 1$, Aziz and Dawood [3] proved that

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)| \tag{2.4}$$

and

$$\min_{|z|=R>1} |P(z)| \geq R^n \min_{|z|=1} |P(z)|. \tag{2.5}$$

Equality in (2.1), (2.2),(2.3), (2.4) and (2.5) holds for polynomial $P(z) = \gamma z^n, \gamma \neq 0$. If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then the inequalities (2.1) and (2.2) can be significantly improved and their right hand side can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{2.6}$$

and

$$\max_{|z|=R>1} |P(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |P(z)| \tag{2.7}$$

Inequality (2.6) was conjectured by Erdös and verified by Lax [6], while inequality (2.7) is due to Ankeny and Rivlin [1]. Aziz and Rather [4] used (2.3) to prove that *if* $P \in \mathcal{P}_n$ *has no zero in* $|z| < 1$, *then for every* β *with* $|\beta| \leq 1$ *and* $R > 1$,

$$|P(Rz) - \beta P(z)| \leq \frac{|R^n - \beta||z|^n + |1-\beta|}{2} \max_{|z|=1} |P(z)|, \quad \text{for} \quad |z| \geq 1. \tag{2.8}$$

Inequality (2.8) is a compact generalization of inequalities (2.6) and (2.7).

Recently Rather et. al., [11], studied the comparative position of the zeros of a polynomial which is derived by the 'composition' of two polynomials and obtained the following result:

Theorem 2.1. *If all the zeros of polynomial* $f(z)$ *of degree* n *lie in* $|z| \leq r$ *and if all the zeros of the polynomial*

$$g(z) = \lambda_0 + \binom{n}{1}\lambda_1 z + \cdots + \binom{n}{k}\lambda_k z^k, \qquad k \leq n$$

lie in $|z| \leq s|z - \sigma|$, $s > 0$ *and* $\sigma \in \mathbb{C} \setminus \{0\}$, *then the polynomial*

$$h(z) = \lambda_0 f(z) + \lambda_1 f'(z)\frac{(\sigma z)}{1!} + \cdots + \lambda_k f^{(k)}(z)\frac{(\sigma z)^k}{k!}$$

has all its zeros in $|z| \leq r\max(1, s)$.

As an application of above result, they [11] introduced a linear operator $N : \mathcal{P}_n \to \mathcal{P}_n$, defined by

$$N[P](z) := \sum_{j=0}^{k} \lambda_j \left(\frac{nz}{2}\right)^j \frac{P^{(j)}(z)}{j!}, \tag{2.9}$$

where $\lambda_j \in \mathbb{C}$, $j = 0, 1, 2, \ldots, k$ are such that all the zeros of

$$\phi(z) = \sum_{j=0}^{k} \binom{n}{j}\lambda_j z^j, \quad k \leq n$$

lie in the half plane $|z| \leq \left|z - \frac{n}{2}\right|$ and established various new Bernstein type polynomial inequalities.

3. Main Results

In this paper, we prove some more general results concerning the operator $N : \mathcal{P}_n \to \mathcal{P}_n$ preserving inequalities between polynomials. Our results not only contain several well known results as special cases but also yield certain new interesting results as special cases. We begin by proving the following result:

Theorem 3.1. *If $f \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$ and $P \in \mathbb{P}_n$ is such that*

$$|P(z)| \leq |f(z)| \qquad \text{for} \quad |z| = 1,$$

then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$,

$$|N[P \circ \sigma_R](z) - \alpha N[P](z)| \leq |N[f \circ \sigma_R](z) - \alpha N[f](z)|. \tag{3.1}$$

The result is sharp and equality in (3.1) holds for $P(z) = e^{i\beta} f(z)$, $\beta \in \mathbb{R}$.

Many sharp results can be derived from Theorem 3.1. Here we mention few of these.
Setting $f(z) = Mz^n$ where $M = \max_{|z|=1} |P(z)|$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *If $P \in \mathcal{P}_n$, then for $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$,*

$$|N[P \circ \sigma_R](z) - \alpha N[P](z)| \leq |R^n - \alpha||N[\psi](z)| \max_{|z|=1} |P(z)|, \tag{3.2}$$

The result is sharp and equality in (3.2) holds for $P(z) = \lambda z^n, \lambda \neq 0, \lambda \in \mathbb{R}$.

Next taking $\lambda_i = 0\ \forall\ i < k$ and $\lambda_k \neq 0$ in Corollary 3.2, it follows that if $P \in \mathcal{P}_n$, then for $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$,

$$|R^k P^{(k)}(Rz) - \alpha P^{(k)}(z)| \leq \frac{n!}{(n-k)!} |R^n - \alpha||z|^{n-k} \max_{|z|=1} |P(z)|. \tag{3.3}$$

Remark 3.3. *For $k = 0$, inequality (3.3) reduces to inequality (2.3).*

On the other hand, if we choose $P(z) = mz^n$ in Theorem 3.1 where $m = \min_{|z|=1} |f(z)|$, we obtain the following result.

Corollary 3.4. *If $f(z)$ is polynomial of degree n having all its zeros in $|z| \leq 1$, then for $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$, we have*

$$|N[f \circ \sigma_R](z) - \alpha N[f](z)| \geq |R^n - \alpha||N[\psi](z)|m, \tag{3.4}$$

The result is sharp as shown by polynomial $f(z) = \lambda z^n, \lambda \neq 0, \lambda \in \mathbb{R}$.
Setting $\lambda_i = 0\ \forall\ i < k$ and $\lambda_k \neq 0$, inequality (3.4) yields

$$|R^k f^{(k)}(Rz) - \alpha f^{(k)}(z)| \geq \frac{n!}{(n-k)!} |R^n - \alpha||z|^{n-k} m, \tag{3.5}$$

which for $k = 0$, leads to

$$|f(Rz) - \alpha f(z)| \geq |R^n - \alpha||z|^n \min_{|z|=1} |f(z)|. \tag{3.6}$$

Remark 3.5. *On taking $\alpha = 0$, inequality (3.6) reduces to inequality (2.5).*

Next we present the following result for the class of polynomials having no zero inside the unit circle.

Theorem 3.6. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$*

$$|N[P \circ \sigma_R](z) - \alpha N[P](z)| \leq \frac{1}{2}\Big(|R^n - \alpha||N[\psi](z)| + |1-\alpha||\lambda_0|\Big)M, \tag{3.7}$$

where $M = \max_{|z|=1}|P(z)|$. The result is best possible and equality in (3.7) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

Remark 3.7. *For $k = 0$ inequality (3.7) reduces to inequality (2.8) which includes inequalities (2.6) and (2.7) as special cases.*

Further, if we set $\lambda_i = 0\ \forall\ i < k$ and $\lambda_k \neq 0$ in Theorem 3.6, we get the following result.

Corollary 3.8. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$*

$$|R^k P^{(k)}(Rz) - \alpha P^{(k)}(z)| \leq \frac{n!}{2(n-k)!}\Big(|R^n - \alpha||z|^{n-k}\Big)M.$$

Taking in particular $k = 1$, we get

$$|RP'(Rz) - \alpha P'(z)| \leq \frac{n}{2}\Big(|R^n - \alpha||z|^{n-1}\Big)M.$$

A polynomial $P \in \mathcal{P}_n$ is said be self-inversive if $P(z) = P^\star(z)$ where $P^\star(z) := z^n\overline{P(1/\overline{z})}$. It is known [10] that the inequality (2.6) also holds if $P \in \mathcal{P}_n$ is self-inversive polynomial. In this direction we prove the following result:

Theorem 3.9. *If $P \in \mathcal{P}_n$ is a self-inversive polynomial, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$,*

$$|N[P \circ \sigma_R](z) - \alpha N[P](z)| \leq \frac{1}{2}\Big(|R^n - \alpha||N[\psi](z)| + |1-\alpha||\lambda_0|\Big)M, \tag{3.8}$$

where $M = \max_{|z|=1}|P(z)|$. Equality in (3.8) holds for $P(z) = z^n + 1$.

Remark 3.10. By taking $\lambda_i = 0\ \forall\ i < k$ and $\lambda_k \neq 0$ we get the following inequality which contains a result due to O'hara and Rodriguez [10] as a special case.

$$|R^k P^{(k)}(Rz) - \alpha P^{(k)}(z)| \leq \frac{n!}{2(n-k)!}\Big(|R^n - \alpha||z|^{n-k}\Big)M.$$

4. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma follows from Theorem 2.1.

Lemma 4.1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leq r$, then all the zeros of $N[P](z)$ also lie in $|z| \leq r$.*

Lemma 4.2. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then for $R > 1$ and $|z| = 1$,*

$$|P(Rz)| > |P(z)|.$$

The proof of above Lemma is simple, we omit the details.

Lemma 4.3. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > 1$,*

$$|N[P \circ \sigma_R](z) - \alpha N[P]|(z) \leq |N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)|, \tag{4.1}$$

where $P^*(z) = z^n\overline{P(1/\overline{z})}$. The result is best possible and equality in (4.1) holds for $P(z) = z^n + 1$.

Proof . Since $P \in \mathcal{P}_n$ has all its zeros in $|z| \geq 1$, therefore in view of [12, Lemma 11.5.2], it follows that for every $\gamma \in \mathbb{C}$ with $|\gamma| > 1$, the polynomial $F(z) = P(z) - \gamma P^*(z)$ has all its zeros in $|z| \leq 1$. Applying Lemma 4.2 to the polynomial $F(z)$, we conclude that for $R > 1$,

$$|F(z)| < |F(Rz)| \qquad \text{for} \quad |z| = 1.$$

Using Rouche's Theorem it follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ that all the zeros of polynomial $g(z) = F(Rz) - \alpha F(z)$ lie in $|z| < 1$. Invoking Lemma 4.1 with $s = 1, t = n/2$ and noting that the operator N is linear, it follows that for $|\alpha| \leq 1,\ |\gamma| > 1$ and $R > 1$, all the zeros of the polynomial

$$\begin{aligned} G(z) &= N[g](z) \\ &= (N[P \circ \sigma_R](z) - \alpha N[P](z)) - \gamma(N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)) \end{aligned} \tag{4.2}$$

lie in $|z| < 1$. This implies

$$|N[P \circ \sigma_R](z) - \alpha N[P](z)| \leq |N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)|, \text{ for } |z| \geq 1. \tag{4.3}$$

If inequality (4.3) is not true, then for some point $z = z_0$ with $|z_0| \geq 1$ we have

$$|N[P \circ \sigma_R](z_0) - \alpha N[P](z_0)| > |N[P^* \circ \sigma_R](z_0) - \alpha N[P^*](z_0)|.$$

Since all zeros of $P^*(z)$ lie in $|z| \leq 1$, therefore it follows (as in the case of $F(z)$) that all the zeros of $P^*(Rz) - \alpha P^*(z)$ lie in $|z| < 1$. Applying Lemma 4.1, it follows that all the zeros of $N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)$ lie in $|z| < 1$, so that $N[P^* \circ \sigma_R](z_0) - \alpha N[P^*](z_0) \neq 0$. Taking

$$\gamma = \frac{N[P \circ \sigma_R](z_0) - \alpha N[P](z_0)}{N[P^* \circ \sigma_R](z_0) - \alpha N[P^*](z_0)}$$

and noting that γ is well defined complex number with $|\gamma| > 1$, with this choice of γ in equation (4.2), it follows that $G(z_0) = 0, |z_0| \geq 1$, which is contradiction to the fact that all the zeros of $G(z)$ lie in $|z| < 1$. This completes the proof of Lemma (4.3). □

Finally we need the following lemma.

Lemma 4.4. *If $P \in \mathcal{P}_n$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1,\ R > 1$ and $|z| \geq 1$,*

$$\begin{aligned} |N[P \circ \sigma_R](z) - \alpha N[P](z)| &+ |N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)| \\ &\leq \big(|R^n - \alpha||N[\psi](z)| + |1 - \alpha||\lambda_0|\big)M, \end{aligned} \tag{4.4}$$

where $M = \max_{|z|=1} |P(z)|, P^*(z) = z^n\overline{P(1/\overline{z})}$.

Proof . Since $|P(z)| \le M$ for $|z| = 1$, hence by Rouche's Theorem it follows that for $\mu \in \mathbb{C},\ |\mu| > 1$, $F(z) = P(z) - \mu M$ has all zeros in $|z| \ge 1$. Applying Lemma (4.3) to $F(z)$, it follows that $\alpha \in \mathbb{C},\ |\alpha| \le 1,\ R > 1$ and $|z| \ge 1$,

$$|N[F \circ \sigma_R](z) - \alpha N[F](z)| \le |N[F^* \circ \sigma_R](z) - \alpha N[F^*](z)|, \tag{4.5}$$

where $F^*(z) = z^n \overline{F(1/\overline{z})} = P^*(z) - \overline{\mu} z^n M$. Hence for $\alpha, \mu \in \mathbb{C}$ with $|\mu| > 1,\ |\alpha| \le 1,\ R > 1$ and $|z| \ge 1$, we obtain

$$\begin{aligned} &|N[P \circ \sigma_R](z) - \alpha N[P](z) - \mu(1-\alpha)\lambda_0 M| \\ &\qquad \le |N[P^* \circ \sigma_R](z) - \alpha N[P^*](z) - \overline{\mu}(R^n - \alpha)N[\psi](z)M|. \end{aligned} \tag{4.6}$$

Choosing argument of μ such that

$$\begin{aligned} &|N[P^* \circ \sigma_R](z) - \alpha N[P^*](z) - \overline{\mu}(R^n - \alpha)N[\psi](z)M| \\ &\qquad = |\mu||R^n - \alpha||N[\psi](z)|M - |N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)|, \end{aligned}$$

which is possible by inequality (3.2), we obtain from (4.6) that for $|\mu| > 1,\ |\alpha| \le 1,\ R > 1$ and $|z| \ge 1$,

$$\begin{aligned} &|N[P \circ \sigma_R](z) - \alpha N[P](z)| + |N[P^* \circ \sigma_R](z) - \alpha N[P^*](z)| \\ &\qquad \le |\mu|(|R^n - \alpha||N[\psi](z)| + |1-\alpha||\lambda_0|)M. \end{aligned}$$

Letting $|\mu| \to 1$, we get inequality (4.4). This completes the proof of lemma 4.4. □

5. Proof of the Theorems

Proof .[Proof of Theorem 3.1] By hypothesis $f(z)$ is a polynomial of degree n having all zeros in $|z| \le 1$ and $P \in \mathcal{P}_n$ such that

$$|P(z)| \le |f(z)| \qquad \text{for} \quad |z| = 1. \tag{5.1}$$

If z_k is a zero of $f(z)$ of multiplicity t_k on the unit circle $|z| = 1$, then it is evident from (5.1) that z_k is also a zero of $P(z)$ of multiplicity at least t_k. Let $f_1(z) = \prod\limits_{z_k \in X} (z - z_k)^{t_k}$ where $X = \{z_k \in \mathbb{C} : f(z_k) = 0, |z_k| = 1\}$. Then from (5.1), we have

$$\left|\frac{P(z)}{f_1(z)}\right| \le \left|\frac{f(z)}{f_1(z)}\right| \quad \text{for} \quad |z| = 1.$$

By Rouche's theorem for every $\gamma \in \mathbb{C}$ with $|\gamma| > 1$, the polynomial $u(z) = \frac{P(z) - \gamma f(z)}{f_1(z)}$ has $n - \sum t_k$ zeros in $|z| < 1$. Since the polynomial $f_1(z)$ has $\sum t_k$ zeros on $|z| = 1$, the polynomial $g(z) = u(z)f_1(z) = P(z) - \gamma f(z)$ has all the n zeros in $|z| \le 1$. Applying Lemma 4.2 to the polynomial $g(z)$, it follows that for $R > 1$ and $|z| = 1$,

$$|g(z)| < |g(Rz)|.$$

Since all the zeros of $g(Rz)$ lie in $|z| \le 1/R < 1$, therefore by Rouche's Theorem all the zeros of the polynomial

$$T(z) = g(Rz) - \alpha g(z)$$

lie in $|z|<1$ for every $\alpha\in\mathbb{C}$ with $|\alpha|\le 1$. Using Lemma 4.1 with $s=1, t=n/2$ and noting that the operator N is linear, it follows that for $|\alpha|\le 1,\ |\gamma|>1$ and $R>1$, all the zeros of the polynomial

$$\begin{aligned} N[T](z) &= N[g\circ\sigma_R](z)-\alpha N[g](z)\\ &= (N[P\circ\sigma_R](z)-\alpha N[P](z))-\gamma(N[f\circ\sigma_R](z)-\alpha N[f](z)) \end{aligned} \tag{5.2}$$

lie in $|z|<1$. This implies,

$$|N[P\circ\sigma_R](z)-\alpha N[P](z)|\le|N[f\circ\sigma_R](z)-\alpha N[f](z)| \quad \text{for } |z|\ge 1. \tag{5.3}$$

For if (5.3) is not true, then there exists a point z_0 with $|z_0|\ge 1$, such that

$$|N[P\circ\sigma_R](z_0)-\alpha N[P](z_0)|>|N[f\circ\sigma_R](z_0)-\alpha N[f](z_0)|,$$

Since all zeros of $f(z)$ lie in $|z|\le 1$, therefore it follows (as in the case of $g(z)$) that all the zeros of $f(Rz)-\alpha f(z)$ lie in $|z|<1$. Applying Lemma 4.1, it follows that all the zeros of $N[f\circ\sigma_R](z)-\alpha N[f](z)$ lie in $|z|<1$, so that $N[f\circ\sigma_R](z_0)-\alpha N[f](z_0)\ne 0$. Taking

$$\gamma=\frac{N[P\circ\sigma_R](z_0)-\alpha N[P](z_0)}{N[f\circ\sigma_R](z_0)-\alpha N[f](z_0)}$$

and noting that γ is a well defined complex number with $|\gamma|>1$ and with this choice of γ, from (5.2), we get $N[T](z_0)=0,\ |z_0|\ge 1$. This contradicts the fact that all the zeros of $N[T](z)$ lie in $|z|<1$ and thus establishes (5.3). This completes the proof of Theorem 3.1. □

Proof .[Proof of Theorem 3.6] Since $P\in\mathcal{P}_n$, having all its zeros in $|z|\ge 1$, therefore from Lemma 4.3 and Lemma 4.4 it follows that for $|\alpha|\le 1,\ R>1$ and $|z|\ge 1$,

$$\begin{aligned} &2|N[P\circ\sigma_R](z)-\alpha N[P](z)|\\ &\quad\le|N[P\circ\sigma_R](z)-\alpha N[P](z)|+|N[P^*\circ\sigma_R](z)-\alpha N[P^*](z)|\\ &\quad\le\big(|R^n-\alpha||N[\psi](z)|+|1-\alpha||\lambda_0|\big)M, \end{aligned}$$

which is equivalent to inequality (3.7). This completes the proof of Theorem 3.6. □

Proof .[Proof of Theorem 3.9] Since $P\in\mathcal{P}_n$ is a self-inversive polynomial, therefore,

$$P(z)=P^*(z)\qquad\forall\ z\in\mathbb{C},$$

this gives,

$$|N[P\circ\sigma_R](z)-\alpha N[P](z)|=|N[P^*\circ\sigma_R](z)-\alpha N[P^*](z)|\quad\forall\ z\in\mathbb{C}.$$

Combining this with Lemma 4.4, we get for $|\alpha|\le 1,\ R>1$ and $|z|\ge 1$,

$$\begin{aligned} &2|N[P\circ\sigma_R](z)-\alpha N[P](z)|\\ &\quad=|N[P\circ\sigma_R](z)-\alpha N[P](z)|+|N[P^*\circ\sigma_R](z)-\alpha N[P^*](z)|\\ &\quad\le\big(|R^n-\alpha||N[\psi](z)|+|1-\alpha||\lambda_0|\big)M, \end{aligned}$$

which is equivalent to inequality (3.8) and completes the proof of Theorem 3.9. □

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