Int. J. Nonlinear Anal. Appl. 13 (2022) No. 1, 685-691
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5557

Fixed point theorem for asymptotically nonexpansive mappings under a new iteration sequence in CAT(0) space

Ahmed Abdulkareem Hadi[a,*], Sabah Hassan Malih[b]

[a] Mathematics Science, Ministry of Education, Directorate of Education Baghdad Al-Rusafa 1st, Iraq
[b] Department of mathematics, College of Education for pure science (Ibn-ALHaitham), university of Baghdad, Baghdad, Iraq

(Communicated by Madjid Eshaghi Gordji)

Abstract

This paper is to define a new iterative scheme under a special sequence of asymptotically nonexpansive mapping with a special sequence. We prove some convergence, existence in CAT(0) space.

Keywords: CAT(0) space, new iteration sequence, Δ-convergent subsequence
2010 MSC: Primary 47H1099, 47H40

1. Introduction and preliminaries

Let (C, d) be a metric space.In [19] A geodesic path joining $\chi \in C$ to $\varpi \in C$ is a map c from $[0, k] \subset R$ to C if $c(0) = \chi,\ c(k) = \varpi$, and $\mathrm{d}\left(c(x), c\left(x'\right)\right) = \left|x - x'\right|$ for each $x - x' \in [0, k]$.In a geodesic metric space (C, d), the geodesic triangle $\Delta(\chi_1, \chi_2, \chi_3)$ consists of threepoints ofχ_1, χ_2, χ_3 in C between the vertices of Δ and a geodesic segment between the vertices of each pair (edge of Δ). In the Euclidean plane E^2, a comparison triangle for the geodesic triangle $\Delta(\chi_1, \chi_2, \chi_3)$ is a triangle $\overline{\Delta}\ (\chi_1, \chi_2, \chi_3) := \Delta(\overline{\chi_1}, \overline{\chi_2}, \overline{\chi_3})$ such that $d_{E^2}(\overline{\chi}_i, \overline{\chi}_j) = d(\overline{\chi}_i, \overline{\chi}_j)$ for $i, j \in \{1, 2, 3\}$.
If all geodesic triangles satisfy the axiom of comparison,is called to be a *CAT(0)* space.
If $\chi, \varpi_1, \varpi_2 \in CAT(0)$, if$\varpi_0$ is middle point of segment$[\varpi_1, \varpi_2]$, where *CAT(0)* inequality is

$$d(\chi, \varpi_0)^2 \leq \frac{1}{2} d(\chi, \varpi_1)^2 + \frac{1}{2} d(\chi, \varpi_2)^2 - \frac{1}{4}(\varpi_1, \varpi_2)^2 \,(AN)$$

*Corresponding author
Email addresses: ahmed08hadi@gmail.com (Ahmed Abdulkareem Hadi), sabahhassanmalih@gmail.com (Sabah Hassan Malih)

Received: August 2021 *Accepted:* September 2021

This is Bruhat and Tits(AN) inequality [2]. In [1] a geodesic space is a *CAT(0)* space if and only if it satisfies (AN). In [9] special sequence $\{\Psi(n)\}$ is Fibonacci sequence if $\Psi(n+1) = \Psi(n) + \Psi(n-1)$, where $\Psi(0) = \Psi(1) = 1$, $\forall\ n \geq 1$.

Definition 1.1. *A mapping of nonempty subset A ($B : A \to C$) of a $CAT(0)$ space .B is an asymptotically nonexpansive if*

$$\lim_{n\to\infty} \xi_{\Psi(n)} = 1,\ \ d\left(B^{\Psi(n)}(\chi), B^{\Psi(n)}(\varpi)\right) \leq \xi_{\Psi(n)} d(\chi, \varpi)\, \text{for both} \quad n \geq 1 \ \text{ and } \chi, \varpi \in A$$

If $\chi = B\chi, \chi \in A$ is considered a point $\chi \in A$ is called a fixed point of B. Denote the set of fixed points of B with $F(B)$.In this work we assume that A be a nonempty bounded closed and convex subset of $CAT(0)$ and a complete space

In Kirk [6] the existence of fixed points in $CAT(0)$ spaces then

Theorem 1.2. *[4] Let A be subset of a $CAT(0)$ space C and B satisfy Definition 1.1 Then B has a fixed point,*

A bounded sequence $\{\chi_n\}$in a metric space C. For $\chi \in C$, we get

$$r(\chi, \{\chi_n\}) = \lim_{n\to\infty} \sup d(\chi, \chi_n)$$

An asymptotic radius $r(\{\chi_n\})$ of$\{\chi_n\}$ is defiend by

$$r(\{\chi_n\}) = \inf\{r(\chi, \{\chi_n\}) : \chi \in C\},$$

and the asymptotic center $F(\{\chi_n\})$ of$\{\chi_n\}$ is defiend by

$$F(\{\chi_n\}) = \{\chi \in C : r(\chi, \{\chi_n\}) = r(\{\chi_n\})\}$$

Definition 1.3. *[7, 8] If χ is the unique asymptotic center of $\{v_n\}$ for all $\{v_n\} \subseteq \{\chi_n\}$, then a sequence $\{\chi_n\}$ in a metric space C is said to $\Delta-$ converge to $\chi \in C$,we write $\Delta - \lim_n \chi_n = \chi$ and call χ the $\Delta - limi\chi$ of $\{\chi_n\}$.*

The metric space C and $K \subseteq C$ is called $\Delta-$ compact [8]if each sequence in K has an $\Delta-$ convergence subsequence. Mapping $B : C \to K$ is called completely continuous.
In this work, B is asymptotically nonexpansive mapping.

Lemma 1.4. *[7] In a $CAT(0)$ space. Each bounded sequence has a $\Delta-$ convergent subsequence.*

Lemma 1.5. *[3] The asymptotic center of $\{\chi_n\}$ is in A of a $CAT(0)$ space and if$\{\chi_n\}$ is a bounded sequence in A.*

Lemma 1.6. *[18] Let A be of a $CAT(0)$ space$C$ and B mapping, $\{\chi_n\}$ be a bounded sequence in A,$\lim_n d(\chi_n, B\chi_n) = 0$ and $\Delta - \lim_n \chi_n = \chi$. Then $\chi = B\chi$*

Lemma 1.7. *[5] Let (C, d) be a $CAT(0)$ space.*

i. *$\chi, \varpi \in C$ and $x \in [0, 1]$, there exists a unique point $\psi \in [\chi, \varpi]$ where*

$$d(\chi, \psi) = xd(\chi, \varpi) \quad \text{and} \quad d(\varpi, \psi) = (1 - x)d(\chi, \varpi) \tag{1.1}$$

We use $(1 - x)\chi \oplus x\varpi$ for a unique point ψ hold (1).

ii. $\chi, \varpi, \psi \in C$ *and* $x \in [0,1]$, *we have*

$$d\left((1-x)\chi \oplus x\varpi, \psi\right) \leq (1-x)\, d(\chi, \psi) + x d(\varpi, \psi)$$

iii. *For* $\chi, \varpi, \psi \in C$ *and* $x \in [0,1]$, *we have*

$$d\left((1-x)\chi \oplus x\varpi, \psi\right)^2 \leq (1-x)\, d(\chi, \psi)^2 + x d(\varpi, \psi)^2 - x(1-x)\, d(\chi, \varpi)^2$$

Lemma 1.8. *[20] Let Let* $\{p_n\}$,$\{q_n\} \in R^+$*satisfying inequality*

$$p_{n+1} \leq (1+q_n)\, p_n, \quad n \geq 1. \;\; If \;\; \sum_{n=1}^{\infty} q_n < \infty,$$

then $\lim_{n\to\infty} p_n$ *exists.*

2. Main Rsults

In this section we have new iteration sequence and some theorems for fixed point in $CAT(0)$ space.

Theorem 2.1. *Let* A *be a subset of a* $CAT(0)$ *space* C *and let* B *be a mapping,* $\left\{\xi_{\Psi(n)}\right\}$ *satisfying* $\xi_{\Psi(n)} \geq 1$ *and* $\sum_{n=1}^{\infty}\left(\xi_{\Psi(n)} - 1\right) < \infty$. *Let*$\{\eta_n\}, \{\mu_n\}, \{\theta_n\}$ *be real sequences in* $[0,1]$. *For a given* $\chi_1 \in A$, *concider the sequence* $\{\chi_n\}, \{\varpi_n\}$ *and* $\{\psi_n\}$ *defined new Iteration Sequence by*

$$\begin{aligned} \psi_n &= \theta_n B^{\Psi(n)} \chi_n \oplus (1-\theta_n)\, \chi_n \\ \varpi_n &= \mu_n B^{\Psi(n)} \psi_n \oplus (1-\mu_n)\, \chi_n \qquad n \geq 1 \\ \chi_{n+1} &= \eta_n B^{\Psi(n)} \varpi_n \oplus (1-\eta_n)\, \chi_n \end{aligned}$$

If $F(B)$ *is nonempty set of fixed point, and* $\forall \zeta \in F(B)$ *then* $\lim_n d(\chi_n, \zeta)$ *exists for all*$\zeta \in F(B)$

Proof . $F(B) \neq \varnothing$ *by Theorem 1.2 for all* $\zeta \in F(B)$, *then*

$$\begin{aligned} d(\psi_n, \zeta) &= d\left(\theta_n B^{\Psi(n)} \chi_n \oplus (1-\theta_n)\, \chi_n, \zeta\right) \leq \theta_n d\left(B^{\Psi(n)} \chi_n, \zeta\right) + (1-\theta_n)\, d(\chi_n, \zeta) \\ &\leq \theta_n \xi_{\Psi(n)} d(\chi_n, \zeta) + (1-\theta_n)\, d(\chi_n, \zeta) = \left(1 - \theta_n \xi_{\Psi(n)} - \theta_n\right) d(\chi_n, \zeta) \end{aligned} \tag{2.1}$$

Also

$$d(\varpi_n, \zeta) = d\left(\mu_n B^{\Psi(n)} \psi_n \oplus (1-\mu_n)\, \chi_n, \zeta\right) \leq \mu_n \xi_{\Psi(n)} d(\psi_n, \zeta) + (1-\mu_n)\, d(\chi_n, \zeta) \tag{2.2}$$

By (2.1) *and* (2.2), *we have*

$$\begin{aligned} d(\chi_{n+1}, \zeta) &= d\left(\eta_n B^{\Psi(n)} \varpi_n \oplus (1-\eta_n)\, \chi_n, \zeta\right) \leq \eta_n \xi_{\Psi(n)} d(\varpi_n, \zeta) + (1-\eta_n)\, d(\chi_n, \zeta) \\ &\leq \eta_n \xi_{\Psi(n)} \left[\mu_n \xi_{\Psi(n)} d(\psi_n, \zeta) + (1-\mu_n)\, d(\chi_n, \zeta)\right] + (1-\eta_n)\, d(\chi_n, \zeta) \\ &\leq \eta_n \xi_{\Psi(n)} \left[\mu_n \xi_{\Psi(n)} \left(1 - \theta_n \xi_{\Psi(n)} - \theta_n\right) d(\chi_n, \zeta) + (1-\mu_n)\, d(\chi_n, \zeta)\right] + (1-\eta_n)\, d(\chi_n, \zeta) \\ &= \left(\eta_n \mu_n \theta_n \xi_{\Psi(n)}^2 + \eta_n \mu_n \xi_{\Psi(n)} + \eta_n\right) \left(\xi_{\Psi(n)} - 1\right) d(\chi_n, \zeta) + d(\chi_n, \zeta) \\ &\leq \left(\xi_{\Psi(n)}^2 + \xi_{\Psi(n)} + 1\right) \left(\xi_{\Psi(n)} - 1\right) d(\chi_n, \zeta) + d(\chi_n, \zeta) \\ &= \left[1 + \left(\xi_{\Psi(n)}^2 + \xi_{\Psi(n)} + 1\right) \left(\xi_{\Psi(n)} - 1\right)\right] d(\chi_n, \zeta) \end{aligned}$$

Since$\left\{\xi_{\Psi(n)}\right\}$ *is bounded, there exists* $G > 0$,

$$d(\chi_{n+1}, \zeta) \leq \left(1 + G\left(\xi_{\Psi(n)} - 1\right)\right) d(\chi_n, \zeta)$$

By Lemma 1.8 and the fact that $\sum_{n=1}^{\infty}\left(\xi_{\Psi(n)} - 1\right) < \infty$, *we get* $\lim_{n\to\infty} d(\chi_n, \zeta)$ *exists.* □

Theorem 2.2. *Let* $A,\ C,\ B, \left\{\xi_{\Psi(n)}\right\}, \{\eta_n\}, \{\mu_n\}, \{\theta_n\}, \{\chi_n\}, \{\varpi_n\}, \{\psi_n\}$ *are as in Theorem 2.1*

I. *If* $0 < \lim_{n\to\infty} \inf \eta_n \le \lim_{n\to\infty} \sup \eta_n < 1$, *then* $\lim_{n\to\infty} d\left(B^{\Psi(n)}\varpi_n, \chi_n\right) = 0$

II. *If* $0 < \lim_{n\to\infty} \inf \mu_n \le \lim_{n\to\infty} \sup \mu_n < 1$ *and* $\lim_{n\to\infty} \inf \eta_n > 0$, *then* $\lim_{n\to\infty} d\left(B^{\Psi(n)}\psi_n, \chi_n\right) = 0$

Proof . *B has a fixed point ζ in A. Choose an arbitrary number $2 > 0$ and a number $r > 0$ such that $A \subseteq B$, and $A - A \subseteq B_r$. It follows from Lemma 1.7 that*

$$\begin{aligned} d(\psi_n - \zeta)^2 &= d\left(\theta_n\left(B^{\Psi(n)}\chi_n - \zeta\right) \oplus (1-\theta_n)(\chi_n - \zeta)\right)^2 \\ &\le \theta_n d\left(B^{\Psi(n)}\chi_n - \zeta\right)^2 \oplus (1-\theta_n) d(\chi_n - \zeta)^2 - (\theta_n) d\left(B^{\Psi(n)}\chi_n - \chi_n\right) \\ &\le \theta_n \xi^2_{\Psi(n)} d(\chi_n - \zeta)^2 \oplus (1-\theta_n) d(\chi_n - \zeta)^2 \\ &\le \left(1 + \theta_n \xi^2_{\Psi(n)} - \theta_n\right) d(\chi_n - \zeta)^2 \end{aligned}$$

Also

$$\begin{aligned} d(\varpi_n - \zeta)^2 &\le d\left(\mu_n\left(B^{\Psi(n)}\psi_n - \zeta\right) \oplus (1-\mu_n)(\chi_n - \zeta)\right)^2 \\ &\le \mu_n d\left(B^{\Psi(n)}\psi_n - \zeta\right)^2 \oplus (1-\mu_n) d(\chi_n - \zeta)^2 - \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \zeta\right) \\ &\le \theta_n \xi^2_{\Psi(n)} d(\psi_n - \zeta)^2 \oplus (1-\theta_n) d(\chi_n - \zeta)^2 - \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \chi_n\right) \end{aligned}$$

Thus

$$\begin{aligned} d(\chi_{n+1} - \zeta)^2 &= d\left(\eta_n\left(B^{\Psi(n)}\varpi_n - \zeta\right) \oplus (1-\eta_n)(\chi_n - \zeta)\right)^2 \\ &\le \eta_n d\left(B^{\Psi(n)}\varpi_n - \zeta\right)^2 \oplus (1-\eta_n) d(\chi_n - \zeta)^2 - \eta_n(1-\eta_n)\left(B^{\Psi(n)}\varpi_n - \chi_n\right) \\ &\le \eta_n \xi^2_{\Psi(n)} d(\varpi_n - \zeta) \oplus (1-\eta_n) d(\chi_n - \zeta)^2 - \eta_n(1-\eta_n) d\left(B^{\Psi(n)}\varpi_n - \chi_n\right) \\ &\le \eta_n \xi^2_{\Psi(n)}\left(\mu_n \xi^2_{\Psi(n)} d(\psi_n - \zeta)^2 \oplus (1-\mu_n) d(\chi_n - \zeta)^2 - \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \chi_n\right)\right. \\ &\quad \left.\oplus (1-\eta_n) d(\chi_n - \zeta)^2 - \eta_n(1-\eta_n) d\left(B^{\Psi(n)}\varpi_n - \chi_n\right)\right) \\ &\le \eta_n \xi^2_{\Psi(n)} \cdot \mu_n \xi^2_{\Psi(n)}\left(1 \oplus \theta_n \xi^2_{\Psi(n)} - \theta_n\right) d(\chi_n - \zeta)^2 \oplus \eta_n \xi^2_{\Psi(n)}(1-\mu_n) d(\chi_n - \zeta)^2 \\ &\quad - \eta_n \xi^2_{\Psi(n)} \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \chi_n\right) \oplus (1-\eta_n) d(\chi_n - \zeta)^2 \\ &\quad - \eta_n(1-\eta_n) d\left(B^{\Psi(n)}\varpi_n - \chi_n\right) \\ &= d(\chi_n - \zeta)^2 \oplus \left(\eta_n \mu_n \theta_n\left(\xi^2_{\Psi(n)}\right)^2 \oplus \eta_n \mu_n \xi^2_{\Psi(n)} \oplus \eta_n\right)\left(\xi^2_{\Psi(n)} - 1\right) d(\chi_n - \zeta)^2 \\ &\quad - \eta_n \xi^2_{\Psi(n)} \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \chi_n\right) - \eta_n(1-\eta_n) d\left(B^{\Psi(n)}\varpi_n - \chi_n\right) \\ &\le d(\chi_n - \zeta)^2 \oplus \left(\left(\xi^2_{\Psi(n)}\right)^2 + \xi^2_{\Psi(n)} + 1\right)\left(\xi^2_{\Psi(n)} - 1\right) d(\chi_n - \zeta)^2 \\ &\quad - \eta_n \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \chi_n\right) - \eta_n(1-\eta_n) d\left(B^{\Psi(n)}\varpi_n - \chi_n\right) \end{aligned} \tag{2.3}$$

The convergence of $\left\{\xi_{\Psi(n)}\right\}$ and the bounded property of G imply that there exists a constant $G > 0$ where $\left(\left(\xi^2_{\Psi(n)}\right)^2 \oplus \xi^2_{\Psi(n)} \oplus 1\right) d(\chi_n - \zeta)^2 \le G$. Then from (2.3) we obtain

$$\eta_n(1-\eta_n) d\left(B^{\Psi(n)}\varpi_n - \chi_n\right) \le d(\chi_n - \zeta)^2 - d(\chi_{n+1} - \zeta)^2 \oplus G\left(\xi^2_{\Psi(n)} - 1\right) \tag{2.4}$$

And

$$\eta_n \mu_n(1-\mu_n) d\left(B^{\Psi(n)}\psi_n - \chi_n\right) \le d(\chi_n - \zeta)^2 - d(\chi_{n+1} - \zeta)^2 \oplus G\left(\xi^2_{\Psi(n)} - 1\right) \tag{2.5}$$

I. *If* $0<\liminf_{n\to\infty}\eta_n\leq\limsup_{n\to\infty}\eta_n<1$, *there exists some real number* $\rho>0$ *and a natural number* N_0, *such that*

$$\eta_n(1-\eta_n)=\eta_n(1-\eta_n)^2\oplus\eta_n^2(1-\eta_n)\geq\rho>0,\forall n>N_0$$

It follows from inequality (2.4) *that for any natural number* $m>N_0$

$$\begin{aligned}\sum_{n=N_0}^{m}d\left(B^{\Psi(n)}\varpi_n-\chi_n\right)&\leq\sum_{n=N_0}^{m}\eta_n(1-\eta_n)d\left(B^{\Psi(n)}\varpi_n-\chi_n\right)\\&\leq d(\chi_{N_0}-\zeta)^2-d(\chi_{m+1}-\zeta)^2\oplus G\sum_{n=N_0}^{m}\left(\xi_{\Psi(n)}^2-1\right)\\&\leq d(\chi_{N_0}-\zeta)^2\oplus G\sum_{n=N_0}^{m}\left(\xi_{\Psi(n)}^2-1\right)\end{aligned}\tag{2.6}$$

It is easy to verify that $\chi^2-1\leq 2\chi(\chi-1)$ *for* $a\geq 1$ *by the application of the Lagrange mean value theorem. This together with the assumption* $\sum_{n=1}^{\infty}\left(\xi_{\Psi(n)}-1\right)<\infty$ *implies that* $\sum_{n=1}^{\infty}\left(\xi_{\Psi(n)}^2-1\right)<\infty$. *Let* $m\to\infty$ *in inequality* (2.6); *we get* $\sum_{n=N_0}^{\infty}d\left(B^{\Psi(n)}\varpi_n-\chi_n\right)<+\infty$ *and therefore* $\lim_{n\to\infty}d\left(B^{\Psi(n)}\varpi_n-\chi_n\right)=0$. *It follows that*

$$\lim_{n\to\infty}d\left(B^{\Psi(n)}\varpi_n-\chi_n\right)=0$$

.

II. *If* $0<\liminf_{n\to\infty}\mu_n\leq\limsup_{n\to\infty}\mu_n<1$ *and* $\liminf_{n\to\infty}\eta_n>0$, *then* $\lim_{n\to\infty}d\left(B^{\Psi(n)}\psi_n,\chi_n\right)=0$, *using a similar method, together with inequality* (2.5), *it can be proved that* $\lim_{n\to\infty}d\left(B^{\Psi(n)}\psi_n-\chi_n\right)=0$.

□

Theorem 2.3. *Let* A *be a subset of a* $CAT(0)$ *space* C *and let* B *be a mapping with* $\left\{\xi_{\Psi(n)}\right\}$ *satisfying* $\left\{\xi_{\Psi(n)}\right\}\geq 1$ *and* $\sum_{n=1}^{\infty}\left(\xi_{\Psi(n)}-1\right)<\infty$. *Let* $\{\eta_n\},\ \{\mu_n\},\{\theta_n\}$ *be real sequence in* $[0,1]$ *satisfying*

I. $0<\liminf_{n\to\infty}\eta_n\leq\limsup_{n\to\infty}\eta_n<1$ *and*
II. $0<\liminf_{n\to\infty}\mu_n\leq\limsup_{n\to\infty}\mu_n<1$

For a given $\chi_1\in A$, *define*

$$\begin{aligned}\psi_n&=\theta_nB^{\Psi(n)}\chi_n\oplus(1-\theta_n)\chi_n\\\varpi_n&=\mu_nB^{\Psi(n)}\psi_n\oplus(1-\mu_n)\chi_n\qquad n\geq 1\\\chi_{n+1}&=\eta_nB^{\Psi(n)}\varpi_n\oplus(1-\eta_n)\chi_n\end{aligned}$$

If $F(B)$ *is nonempty set of fixed point, and* $\forall\zeta\in F(B)$ *then* $\lim_{n\to\infty}d(B\chi_n,\chi_n)=0$
Proof . *from Theorem 2.2, we have*

$$\lim_{n\to\infty}d\left(B^{\Psi(n)}\varpi_n,\chi_n\right)=0\ \ and\ \ \lim_{n\to\infty}d\left(B^{\Psi(n)}\psi_n,\chi_n\right)=0.$$

Thus

$$d\left(B^{\Psi(n)}\chi_n,\chi_n\right)\le d\left(B^{\Psi(n)}\chi_n,B^{\Psi(n)}\varpi_n\right)+d\left(B^{\Psi(n)}\varpi_n,\chi_n\right)\le\xi_{\Psi(n)}d\left(\chi_n,\varpi_n\right)+d\left(B^{\Psi(n)}\varpi_n,\chi_n\right)$$
$$\le\xi_{\Psi(n)}\mu_n d\left(B^{\Psi(n)}\psi_n,\chi_n\right)+d\left(B^{\Psi(n)}\varpi_n,\chi_n\right)\to 0\ As\ \ n\to\infty \tag{2.7}$$

So that

$$\begin{aligned}d\left(\chi_{n+1},B^{\Psi(n)}\chi_{n+1}\right)&\le d\left(\chi_{n+1},\chi_n\right)+d\left(B^{\Psi(n)}\chi_{n+1},B^{\Psi(n)}\chi_n\right)+d\left(B^{\Psi(n)}\chi_n,\chi_n\right)\\&\le d\left(\chi_{n+1},\chi_n\right)+\xi_{\Psi(n)}d\left(\chi_{n+1},\chi_n\right)+d\left(B^{\Psi(n)}\chi_n,\chi_n\right)\\&=\left(1+\xi_{\Psi(n)}\right)d\left(\eta_nB^{\Psi(n)}\varpi_n\oplus\left(1-\eta_n\right)\chi_n,\chi_n\right)+d\left(B^{\Psi(n)}\chi_n,\chi_n\right)\\&\le\left(1+\xi_{\Psi(n)}\right)\eta_nd\left(B^{\Psi(n)}\varpi_n,\chi_n\right)+d\left(B^{\Psi(n)}\chi_n,\chi_n\right)\to 0\ \ As\ n\to\infty\end{aligned}\tag{2.8}$$

By (2.7) and (2.8), we have

$$\begin{aligned}d\left(\chi_{n+1},B\chi_{n+1}\right)&\le d\left(\chi_{n+1},B^{\Psi(n)+1}\chi_{n+1}\right)+d\left(B^{\Psi(n)+1}\chi_{n+1},B\chi_{n+1}\right)\\&\le d\left(\chi_{n+1},B^{\Psi(n)+1}\chi_{n+1}\right)+\xi_{\Psi(1)}d\left(B^{\Psi(n)}\chi_{n+1},\chi_{n+1}\right)\to 0.\ \ As\ n\to\infty\end{aligned}$$

Which implies$\lim_{n\to\infty}d\left(B\chi_n,\chi_n\right)=0$ *as desired* □

Theorem 2.4. *Let A be a subset of a $CAT(0)$ space C and let B be a mapping with $\left\{\xi_{\Psi(n)}\right\}$ satisfying $\left\{\xi_{\Psi(n)}\right\}\ge 1$ and $\sum_{n=1}^{\infty}\left(\xi_{\Psi(n)}-1\right)<\infty$. Let $\{\eta_n\},\ \{\mu_n\},\{\theta_n\}\in[0,1]$ satisfying*

I. $0<\lim_{n\to\infty}\inf\eta_n\le\lim_{n\to\infty}\sup\eta_n<1$ *and*
II. $0<\lim_{n\to\infty}\inf\mu_n\le\lim_{n\to\infty}\sup\mu_n<1$

Such that $\chi_1\in A$, define

$$\begin{aligned}\psi_n&=\theta_nB^{\Psi(n)}\chi_n\oplus\left(1-\theta_n\right)\chi_n\\\varpi_n&=\mu_nB^{\Psi(n)}\psi_n\oplus\left(1-\mu_n\right)\chi_n,\qquad n\ge 1\\\chi_{n+1}&=\eta_nB^{\Psi(n)}\varpi_n\oplus\left(1-\eta_n\right)\chi_n\end{aligned}$$

If $F(B)$ is nonempty set of fixed point, and $\forall\zeta\in F(B)$, then $\{\chi_n\}$ $\Delta-$ converges to a fixed point of$B$

Proof . *Since Theorem 2.2 $\lim_{n\to\infty}d\left(\chi_n,B\chi_n\right)=0$. Let $\omega_w\left(\chi_n\right)=\bigcup F\left(\{\upsilon_n\}\right)$ where,$\{\upsilon_n\}$ of $\{\chi_n\}$. We claim that $\omega_w\left(\chi_n\right)\subset F(B)$. Let $\upsilon\in\omega_w\left(\chi_n\right)$, $\{\upsilon_n\}\subseteq\{\chi_n\}$ where $F(\{\upsilon_n\})=\{\upsilon\}$. By Lemmas 1.4 and 1.5, then $\{y_n\}\subseteq\{\upsilon_n\}$ and $\Delta-\lim_n y_n=y\in A$. Since $\lim_n d\left(y_n,By_n\right)=0$, then $y\in F(B)$ by Lemma 1.6. Assume that $\upsilon=y$. Suppose not, since $y\in F(B)$, by Theorem 2.2 $\lim_n d\left(\chi_n,y\right)$ exists. By the uniqueness of asymptotic centers,*

$$\begin{aligned}\limsup_n\left(y_n,y\right)\le\limsup_n\left(y_n,\upsilon\right)\le\limsup_n d\left(\upsilon_n,\upsilon\right)<\limsup_n d\left(\upsilon_n,y\right)\\=\limsup_n d\left(\chi_n,y\right)=\limsup_n d\left(y_n,y\right)\end{aligned}$$

A contradiction, and hence $\upsilon=y\in F(B)$, it suffices to prove that $\omega_w\left(\chi_n\right)$ consists of exactly one point. □

3. Open problem

The study for results in papers [10, 11, 12, 13, 14, 15, 16, 17] under new iteration.

4. Conclusion

The idea in this research includes obtaining a new iteration that is subject to a sequence and describing this iterationin obtaining the fixed point theorems in $CAT(0)$ space.

References

[1] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, vol. 319. Springer Science & Business Media, 2013.

[2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Publ. Math.Institut Hautes Études Sci. 41 (1972) 5–251.

[3] S. Dhompongsa, W. A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. 8 (2007).

[4] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. Theory, Methods Appl. 65 (2006) 762–772.

[5] S. Dhompongsa and B. Panyanak, *On Δ_ϵ convergence theorems in CAT (0) spaces*, Comput. Math. Appl. 56 (2008) 2572–2579.

[6] W.A. Kirk, *Geodesic geometry and fixed point theory*, In Seminar of mathematical analysis (Malaga/Seville), 2002/2003), 64 (2003) 195–225.

[7] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. Theory, Methods Appl. 68 (2008) 3689–3696.

[8] T.C. Lim, *Remarks on some fixed point theorems,* Proc. Am. Math. Soc. 60 (1976) 179–182.

[9] S.H. Malih, *Common Fixed Point for a Pair of Asymptotically Nonexpansive and Multivalued Mapping Under Fibonacci Iteration Sequence in CAT (0) Space*, J. Phys. Conf. Ser. 1879 (2021).

[10] S.H. Malih and S.S. Abed, *Approximating random fixed points under a new iterative sequence*, J. Interdiscip. Math. 22 (2019) 1407–1414.

[11] S.H. Malih, *Fixed point theorems of modified Mann and Ishikawa iterations*, J. Interdiscip. Math. 24 (2021) 1093–1097.

[12] S.H. Malih, *Stability and convergence theorems of pointwise asymptotically nonexpansive random operator in Banach space*, Int. J. Nonlinear Anal. Appl. 12 (2021) 119–127.

[13] S.H. Malih, *Random fixed point for random Fibonacci Noor iteration scheme*, J. Interdiscip. Math., 24 (2021) 775–779.

[14] S.H. Malih, *Random fixed point under contractive condition of integral type*, J. Phys. Conf. Ser. 1897 (2021) 12035.

[15] S.H. Malih, *Random Fixed Point on Ishikawa Random Iteration Under Fibonacci Sequence*, J. Phys. Conf. Ser. 1879 (2021) 32044.

[16] S.H. Malih and S.S. Abed, *Convergence and stability of some random iterative schemes*, J. Phys.: Conf. Ser. 1963 (2021) 12112.

[17] S.H. Malih and S.S. Abed, *Convergence of random iterative scheme to a common random fixed points*, J. Phys. Conf. Ser. 1963 (2021) 12113.

[18] B. Nanjaras and B. Panyanak, *Demiclosed principle for asymptotically nonexpansive mappings in CAT (0) spaces*, Fixed Point Theory Appl. 2010 (2010) 268780.

[19] Y. Niwongsa and B. Panyanak, *Noor iterations for asymptotically nonexpansive mappings in CAT (0) spaces*, Int. J. Math. Anal. 4 (2010) 645–656.

[20] H. Zhou, R.P. Agarwal, Y.J. Cho and Y.S. Kim, *Nonexpansive mappings and iterative methods in uniformly convex Banach spaces*, Georg. Math. J. 9 (2002) 591–600.