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Approximate additive and quadratic mappings in 2-Banach spaces and related topics

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Abstract

Won–Gil Park [Won–Gil Park, J. Math. Anal. Appl., 376 (1) (2011) 193–202] proved the Hyers–Ulam stability of the Cauchy functional equation, the Jensen functional equation and the quadratic functional equation in 2–Banach spaces. One can easily see that all results of this paper are incorrect. Hence the control functions in all theorems of this paper are not correct. In this paper, we correct these results.

Keywords: Hyers–Ulam Stability, Cauchy Functional Equation, Jensen Functional Equation, Quadratic Functional Equation.
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1. Introduction

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem concerning group homomorphisms was raised by Ulam [2] in 1940 and affirmatively solved by Hyers [3]. The result of Hyers was generalized by Aoki [4] for approximate additive mappings and by Th.M. Rassias [5] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\varepsilon(\| x \|^p + \| y \|^p)$.

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The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to symmetric bi-additive function and is called a *quadratic functional equation* and every solution of the quadratic equation (1.1) is said to be a *quadratic mapping*. Skof [6] proved the Hyers-Ulam stability of the quadratic functional equation (1.1).

The theory of linear 2-normed spaces was first developed by S. Gähler [7] in the mid 1960's, while that of 2-Banach spaces was studied later by S. Gähler [8] and A. White [9]. For more details, the readers refer to the papers [10] - [12].

Definition 1.1. *([7]) Let $\mathcal{X}$ be a real linear space over $\mathbb{R}$ with dim $\mathcal{X} > 1$ and $\|\cdot\ ,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function. Then $(\mathcal{X}, \|\cdot\ ,\cdot\|)$ is called a linear 2-normed space if*

$(2N_1)$ $\|x, y\| > 0$ *and* $\|x, y\| = 0$ *if and only if x and y are linearly dependent;*
$(2N_2)$ $\|x, y\| = \|y, x\|$*;*
$(2N_3)$ $\|\alpha x, y\| = |\alpha|\ \|x, y\|$*;*
$(2N_4)$ $\|x, y+z\| \le \|x, y\| + \|x, z\|$ *for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{R}$.*

If $x \in \mathcal{X}$ and $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$. Moreover, the functions $x \to \|x, y\|$ are continuous functions of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$ [7, 10].

The basic definitions of a 2-Banach space are given in [11] (see also [8, 9]):

(1) A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *convergent sequence* if there is an $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in \mathcal{X}$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if there are $y, z \in \mathcal{X}$ such that y and z are linearly independent, $\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0$ and $\lim_{n,m\to\infty} \|x_n - x_m, z\| = 0$.

(3) A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-*Banach space*.

From now on, let X be a normed linear space and Y a $2-$Banach space. Recently, W. G. Park [1] proved the Hyers–Ulam stability of the Cauchy functional equation, the Jensen functional equation and the quadratic functional equation in 2–Banach spaces. One can easily see that all results of the paper [1] are incorrect. We remained that for $z, y \in Y$, we can not define $\|z\|^r$ and $\|y\|^r$ in relations (2.1), (2.2), (2.3), (2.4), (3.1), (3.2), (3.3), (3.4), (3.5), (3.8), (4.1), (4.2), (4.3) and (4.4) in the paper [1]. So all relations and all results in paper [1] are incorrect. In the present paper, we correct the results of [1].

2. Approximate Additive Mappings

In this section, we investigate the Hyers–Ulam stability of additive mappings from X into Y.

Theorem 2.1. *Let $\theta \in [0, \infty), p, q \in (0, \infty)$ and $p+q < 1$ and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y), z\| \le \theta \|x\|^p \|y\|^q \tag{2.1}$$

for all $x, y \in X$ and all $z \in Y$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), z\| \le \frac{\theta \|x\|^{p+q}}{2 - 2^{p+q}} \tag{2.2}$$

for all $x \in X$ and all $z \in Y$.

Proof . Setting $y = x$ in (2.1), we get $\|f(2x) - 2f(x), z\| \leq \theta\|x\|^{p+q}$ for all $x \in X$ and all $z \in Y$. It follows that

$$\|f(x) - \frac{1}{2}f(2x), z\| \leq \frac{\theta}{2}\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Replacing x by $2^j x$ and dividing 2^j, we obtain

$$\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x), z\| \leq 2^{(p+q-1)(j-1)}\theta\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), z\| \leq \frac{2^{(p+q-1)l} - 2^{(p+q-1)m}}{2 - 2^{p+q}}\theta\|x\|^{p+q} \tag{2.3}$$

for all $x \in X$ and all $z \in Y$. It follows that

$$\lim_{l,m\to\infty} \|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x), z\| = 0$$

for all $x \in X$ and all $z \in Y$. This means that $\{\frac{1}{2^j}f(2^j x)\}$ is a Cauchy sequence in Y. Since Y is a 2−Banach space, the sequence $\{\frac{1}{2^j}f(2^j x)\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by $A(x) := \lim_{j\to\infty} \frac{1}{2^j}f(2^j x)$ for all $x \in X$. It is easy to show that

$$\|A(x + y) - A(x) - A(y), z\| \leq \theta\|x\|^p\|y\|^q \lim_{j\to\infty} 2^{(p+q-1)j} = 0$$

for all $x, y \in X$ and all $z \in Y$. It follows that $A : X \to Y$ is additive. To prove the uniqueness property of A, let $B : X \to Y$ be an additive mapping satisfying (2.2). Then one can show that

$$\|A(x) - B(x), z\| \leq \frac{2^{(p+q-1)j}}{1 - 2^{p+q-1}}\theta\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all $j \in \mathbb{N}$. The right hand side of the last inequality tends to zero as $j \to \infty$. It follows that $A(x) = B(x)$ for all $x \in X$. This completes the proof. □

Theorem 2.2. *Let $\theta \in [0, \infty), p, q \in (0, \infty)$ and $p+q > 1$ and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x + y) - f(x) - f(y), z\| \leq \theta\|x\|^p\|y\|^q \tag{2.4}$$

for all $x, y \in X$ and all $z \in Y$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), z\| \leq \frac{\theta\|x\|^{p+q}}{2^{p+q} - 2}$$

for all $x \in X$ and all $z \in Y$.

Proof . Setting $y = x$ in (2.4), we get $\|f(2x) - 2f(x), z\| \leq \theta\|x\|^{p+q}$ for all $x \in X$ and all $z \in Y$. It follows that

$$\|f(x) - 2f(\frac{x}{2}), z\| \leq \frac{\theta}{2^{p+q}}\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Replacing x by $2^{-j}x$ and multiplying 2^j, we obtain

$$\|2^j f(\frac{x}{2^j}) - 2^{j+1}f(\frac{x}{2^{j+1}}), z\| \leq \frac{2^j\theta}{2^{(p+q)(j+1)}}\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), z\| \leq \frac{1}{2^{p+q-2}}[\frac{1}{2^{(p+q-1)l}} - \frac{1}{2^{(p+q-1)m}}]\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. It follows that

$$\lim_{l,m\to\infty} \|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), z\| = 0$$

for all $x \in X$ and all $z \in Y$. This means that $\{2^j f(\frac{x}{2^j})\}$ is a Cauchy sequence in Y. Since Y is a 2−Banach space, the sequence $\{2^j f(\frac{x}{2^j})\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by $A(x) := \lim_{j\to\infty} 2^j f(\frac{x}{2^j})$ for all $x \in X$. It is easy to show that $A : X \to Y$ is additive. The rest of proof is similar to the proof of Theorem 2.1. □

3. Approximate Jensen Mappings

In this section, we investigate the Hyers–Ulam stability of Jensen mappings from X into Y.

Theorem 3.1. *Let $\theta \in [0, \infty), p, q \in (0, \infty)$ and $p+q < 1$ and let $f : X \to Y$ be a mapping satisfying*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y), z\| \leq \theta\|x\|^p\|y\|^q \tag{3.1}$$

for all $x, y \in X$ and all $z \in Y$. Then there is a unique Jensen mapping $J : X \to Y$ such that

$$\|f(x) - J(x), z\| \leq \frac{1+3^q}{3-3^{p+q}}\theta\|x\|^{p+q} \tag{3.2}$$

for all $x \in X$ and all $z \in Y$.

Proof . Define $g : X \to Y$ by

$$g(x) := f(x) - f(0)$$

for all $x \in X$. Then it is easy to show that $g(0) = 0$ and g satisfies (3.1). That is

$$\|2g(\frac{x+y}{2}) - g(x) - g(y), z\| \leq \theta\|x\|^p\|y\|^q \tag{3.3}$$

for all $x, y \in X$ and all $z \in Y$. Putting $y = -x$ in (3.3) to get

$$\|g(x) + g(-x), z\| \leq \theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. One can show that

$$\|g(x) - \frac{1}{3}g(3x), z\| \leq \frac{1+3^q}{3}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Replacing x by $3^j x$ and dividing 3^j, we obtain

$$\|\frac{1}{3^j}g(3^j x) - \frac{1}{3^{j+1}}g(3^{j+1}x), z\| \leq (1+3^q)3^{(p+q-1)(j-1)}\theta\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\|\frac{1}{3^l}g(3^l x) - \frac{1}{3^m}g(3^m x), z\| \leq \frac{1+3^q}{3-3^{p+q}}[3^{(p+q-1)l} - 3^{(p+q-1)m}]\theta\|x\|^{p+q} \tag{3.4}$$

for all $x \in X$ and all $z \in Y$. It follows that

$$\lim_{l,m\to\infty} \|\frac{1}{3^l}g(3^l x) - \frac{1}{3^m}g(3^m x), z\| = 0$$

for all $x \in X$ and all $z \in Y$. This means that $\{\frac{1}{3^j}g(3^j x)\}$ is a Cauchy sequence in Y. Since Y is a 2−Banach space, the sequence $\{\frac{1}{3^j}g(3^j x)\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by $A(x) := \lim_{j\to\infty} \frac{1}{3^j}g(3^j x) = \lim_{j\to\infty} \frac{1}{3^j}f(3^j x)$ for all $x \in X$. It is easy to see that

$$\|g(x) - A(x), z\| \leq \frac{1+3^q}{3-3^{p+q}}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Define a mapping $J : X \to Y$ by $J(x) := A(x) + f(0)$ for all $x \in X$. Then we have

$$\|f(x) - J(x), z\| = \|g(x) - A(x), z\| \leq \frac{1+3^q}{3-3^{p+q}}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. By using the same technique of proving Theorem 3.1 of [1], one can show that J is a unique Jensen mapping from X into Y such that (3.2). □

Theorem 3.2. *Let $\theta \in [0,\infty), p, q \in (0,\infty)$ and $p + q > 1$ and let $f : X \to Y$ be a mapping such that $f(0) = 0$ and that*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y), z\| \leq \theta\|x\|^p\|y\|^q \tag{3.5}$$

for all $x, y \in X$ and all $z \in Y$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), z\| \leq \frac{1+3^q}{3^{p+q}-3}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$.

Proof . By the same argument as in the proof of Theorem 3.1, we get

$$\|f(x) - 3f(\frac{x}{3}), z\| \leq \frac{1+3^q}{3^{p+q}}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Replacing x by $3^{-j}x$ and multiplying 3^j, we obtain

$$\|3^j f(\frac{x}{3^j}) - 3^{j+1}f(\frac{x}{3^{j+1}}), z\| \leq 1 + 3^q 3^{(p+q)(j+1)}3^j\theta\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}), z\| \leq \frac{1+3^q}{3^{p+q-3}}[\frac{1}{3^{(p+q-1)l}} - \frac{1}{3^{(p+q-1)m}}]\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. It follows that

$$\lim_{l,m\to\infty} \|3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m}), z\| = 0$$

for all $x \in X$ and all $z \in Y$. This means that $\{3^j f(\frac{x}{3^j})\}$ is a Cauchy sequence in Y. Since Y is a 2−Banach space, the sequence $\{3^j f(\frac{x}{3^j})\}$ converges for all $x \in X$. So one can define the mapping $A : X \to Y$ by $A(x) := \lim_{j\to\infty} 3^j f(\frac{x}{3^j})$ for all $x \in X$. It is easy to show that $A : X \to Y$ is additive. The rest of proof is similar to the proof of Theorem 3.1. □

4. Approximate Quadratic Mappings

In 1995, C. Borelli and G.L. Forti [13] obtained the result on the stability theorem for a class of functional equations including the quadratic functional equation. In this section, we investigate the Hyers–Ulam stability of quadratic mappings from normed X into $2-$Banach space Y.

Theorem 4.1. *Let $\theta \in [0,\infty), p, q \in (0,\infty)$ and $p+q<2$ and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y),z\| \le \theta\|x\|^p\|y\|^q \tag{4.1}$$

for all $x, y \in X$ and all $z \in Y$. Then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x),z\| \le \frac{1}{4-2^{p+q}}\theta\|x\|^{p+q}+\frac{1}{3}\|f(0),z\| \tag{4.2}$$

for all $x \in X$ and all $z \in Y$.

Proof . Putting $y = x$ in (4.1), we get $\|f(2x)-4f(x)+f(0),z\| \le \theta\|x\|^{p+q}$ for all $x \in X$ and all $z \in Y$. So, we get

$$\|f(x)-\frac{1}{4}f(2x)-\frac{1}{4}f(0),z\| \le \frac{\theta}{4}\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Replacing x by $2^j x$ and dividing 4^j, we obtain

$$\|\frac{1}{4^j}f(2^jx)-\frac{1}{4^{j+1}}f(2^{j+1}x),z\| \le 2^{(p+q-2)(j-2)}\theta\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all integers $j \ge 0$. For all integers l, m with $0 \le l < m$, we get

$$\|\frac{1}{4^l}f(2^lx)-\frac{1}{4^m}f(2^mx)-\frac{1}{3}(4^{-l}-4^{-m})f(0),z\| \le [\frac{2^{(p+q-2)l}-2^{(p+q-2)m}}{4-2^{p+q}}]\theta\|x\|^{p+q} \tag{4.3}$$

for all $x \in X$ and all $z \in Y$. It follows that

$$\lim_{l,m\to\infty}\|\frac{1}{4^l}f(2^lx)-\frac{1}{4^m}f(2^mx),z\| = 0$$

for all $x \in X$ and all $z \in Y$. This means that $\{\frac{1}{4^j}f(2^jx)\}$ is a Cauchy sequence in Y. Since Y is a $2-$Banach space, the sequence $\{\frac{1}{4^j}f(2^jx)\}$ converges for all $x \in X$. So one can define the mapping $Q : X \to Y$ by $Q(x) := \lim_{j\to\infty}\frac{1}{4^j}f(2^jx)$ for all $x \in X$. It is easy to see that

$$\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y),z\| \le \theta\|x\|^p\|y\|^q\lim_{j\to\infty}2^{(p+q-2)j} = 0$$

for all $x, y \in X$ and all $z \in Y$. It follows that

$$Q(x+y)+Q(x-y) = 2Q(x)+2Q(y)$$

for all $x, y \in X$. Hence Q is a quadratic mapping. On the other hand, by (4.3), it follows that

$$\|f(x)-Q(x)-\frac{1}{3}f(0),z\| \le \frac{1}{4-2^{p+q}}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. It follows inequality (4.2). Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (4.2). Then we have

$$\|Q(x)-Q'(x),z\| \le \frac{2^{(p+q-2)j+1}}{4-2^{p+q}}\theta\|x\|^{p+q}+\frac{2}{3.4^j}\|f(0),z\|$$

which tends to zero as $j \to \infty$ for all $x \in X$. It follows that $Q = Q'$. □

Theorem 4.2. *Let $\theta \in [0,\infty), p, q \in (0,\infty)$ and $p+q>2$ and let $f : X \to Y$ be a mapping such that $f(0)=0$ and that*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y), z\| \leq \theta\|x\|^p\|y\|^q \tag{4.4}$$

for all $x, y \in X$ and all $z \in Y$. Then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x), z\| \leq \frac{1}{2^{p+q}-4}\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$.

Proof . Putting $y=x$ in (4.4), we get $\|f(2x)-4f(x), z\| \leq \theta\|x\|^{p+q}$ for all $x \in X$ and all $z \in Y$. So we get

$$\|f(x)-4f(\frac{x}{2}), z\| \leq \frac{\theta}{2^{p+q}}\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. Replacing x by $2^{-j}x$ and multiplying 4^j, we obtain

$$\|4^j f(\frac{x}{2^j})-4^{j+1}f(\frac{x}{2^{j+1}}), z\| \leq \frac{4^j}{2^{(p+q)(j+1)}}\theta\|x\|^{p+q}$$

for all $x \in X$, all $z \in Y$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\|4^l f(\frac{x}{2^l})-4^m f(\frac{x}{2^m}), z\| \leq \frac{1}{2^{p+q-4}}[\frac{1}{2^{(p+q-1)l}}-\frac{1}{2^{(p+q-1)m}}]\theta\|x\|^{p+q}$$

for all $x \in X$ and all $z \in Y$. It follows that

$$\lim_{l,m\to\infty}\|4^l f(\frac{x}{2^l})-4^m f(\frac{x}{2^m}), z\| = 0$$

for all $x \in X$ and all $z \in Y$. This means that $\{4^j f(\frac{x}{2^j})\}$ is a Cauchy sequence in Y. Since Y is a $2-$Banach space, the sequence $\{4^j f(\frac{x}{2^j})\}$ converges for all $x \in X$. So one can define the mapping $Q : X \to Y$ by $Q(x) := \lim_{j\to\infty} 4^j f(\frac{x}{2^j})$ for all $x \in X$. It is easy to show that $Q : X \to Y$ is a quadratic mapping. The rest of proof is similar to the proof of Theorem 4.1. □

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