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Application of a generalization of Darbo's fixed point theorem via Mizogochi-Takahashi mappings on mixed fractional integral equations involving (k, s)-Riemann-Liouville and Erdélyi-Kober fractional integrals

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Abstract

We have established the solvability of fractional integral equations with both (k, s)-Riemann-Liouville and Erdélyi-Kober fractional integrals using a new generalized version of the Darbo's theorem using Mizogochi-Takahashi mappings and justify the validity of our results with the help of suitable examples.

Keywords: Functional integral equations (FIE); Measure of non-compactness (MNC); Fixed point theorems (FPT).
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1. Introduction

Many authors attempted to extend the well-known Banach contraction principle after its publication. Ameer et al., for example, introduced the concept of generalized multivalued contractions and

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established certain common fixed point results in the class of α_K-complete partial b-metric spaces in [10]. Furthermore, Ameer et al. [11] introduced the concept of Ćirić type rational graphic contraction pair mappings and offered some associated common fixed point results on partial b-metric spaces equipped with a directed graph. They demonstrated several electric circuit equations and fractional differential equations as applications.

Darbo's fixed point theorem is a well-known generalization of the Banach contraction principle. Many researchers believe in this theorem. Matani and Roshan proposed the ideas of multivariate generalized Meir-Keeler condensing operator and multivariate L-function in reference [8], and used the measure of non-compactness to verify several new fixed point theorems. They also used these findings to analyze the solvability of a system of Volterra type functional integral equations in three variables. In addition, Roshan published several expansions of Darbo's fixed point theorem, as well as some conclusions on the existence of coupled fixed points for a specific class of operators in a Banach space (see [9]). He also investigated the existence of a solution for a system of nonlinear functional integral equations as an application.

Functional integral equations play a pivotal role in different fields and many real life problems which can be modelled using integral equations with fractional order in a very effective manner. A fractional derivative is a derivative of any real or complex non-integer order. In recent times, the fixed point theory has applications in different scientific fields. The FIEs have made significant contributions to several real-life problems, e.g., science, engineering, mathematics, and other areas which can be described using all kinds of integral equations of fractional order. Fixed point theorems can be applied in seeking solutions for fractional differentials and integral equations.

In this work, we have established a generalization of Darbo's Fixed point theorem, which is an extension of the work ([3]) and we have applied it to a functional integral equation (FIE) of mixed type.

In this paper, we will be using the following abbreviated forms:
FIE : Fractional integral equation,
MNC: Measure of noncompactness,
NBCC: Nonempty, bounded, closed and convex subset.

In this article, $\mathbf{E}$ is a Banach space with the norm $\| \,.\, \|_{\mathbf{E}}$, $B[\theta, \kappa]$ is a closed ball with center θ and radius κ in $\mathbf{E}$, $\bar{\Lambda}$ is the closure of Λ, ConvΛ is the convex closure of Λ, $\mathfrak{M}_{\mathbf{E}}$ denotes the family of all nonempty and bounded subsets of $\mathbf{E}$ and $\mathfrak{N}_{\mathbf{E}}$ is the family of all relatively compact sets. For more details on fractional calculous and the theory of measure of noncompactness we refer the reader to [12]-[16].

Definition 1.1. *[4] A function $\vartheta : \mathfrak{M}_{\mathbf{E}} \to \mathbb{R}^+ = [0, \infty)$ is called an MNC in $\mathbf{E}$ if:*

(i) $\Lambda \in \mathfrak{M}_{\mathbf{E}}$ *and* $\vartheta(\Lambda) = 0$ *gives* Λ *is precompact.*

(ii) *ker* $\vartheta = \{\Lambda \in \mathfrak{M}_{\mathbf{E}} : \vartheta(\Lambda) = 0\}$ *is nonempty and ker* $\vartheta \subset \mathfrak{N}_{\mathbf{E}}$.

(iii) $\Lambda \subseteq \Lambda_1 \implies \vartheta(\Lambda) \leq \vartheta(\Lambda_1)$.

(iv) $\vartheta\left(\bar{\Lambda}\right) = \vartheta(\Lambda)$.

(v) $\vartheta(Conv\Lambda) = \vartheta(\Lambda)$.

(vi) $\vartheta(\varpi\Lambda + (1-\varpi)\Lambda_1) \leq \varpi\vartheta(\Lambda) + (1-\varpi)\vartheta(\Lambda_1)$ *for all* $\varpi \in [0,1]$.

(vii) *if* $\Lambda_n \in \mathfrak{M}_{\mathbf{E}}$, $\Lambda_n = \bar{\Lambda}_n$, $\Lambda_{n+1} \subset \Lambda_n$ *for all* $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty} \vartheta(\Lambda_n) = 0$, *then* $\Lambda_\infty = \bigcap\limits_{n=1}^{\infty} \Lambda_n \neq \phi$.

The family kerϑ is said to be the *kernel of measure* ϑ. Also, $\Lambda_\infty \in ker\vartheta$ and $\vartheta(\Lambda_\infty) \leq \vartheta(\Lambda_n)$ for any n. So, $\vartheta(\Lambda_\infty) = 0$. This gives $\Lambda_\infty \in$ kerϑ.

Theorem 1.2. *[2, Shauder] Let $\mathbb{E}$ be a Banach space and $\Lambda(\neq \phi) \subseteq \mathbb{E}$ be closed and convex. Then every continuous compact mapping $\Delta : \Lambda \to \Lambda$ has at least one fixed point.*

Theorem 1.3. *[5, Darbo] Let $\mathbb{E}$ be a Banach space and $\Lambda \subseteq \mathbb{E}$ be nonempty, bounded, closed and convex (NBCC). Also, let $\Delta : \Lambda \to \Lambda$ be a continuous mapping. If*

$$\vartheta(\Delta\Pi) \leq \kappa\vartheta(\Pi),\ \Pi \subseteq \Lambda,$$

for a constant $\kappa \in [0, 1)$, then Δ has a fixed point.

With the help of following concepts, we establish our fixed point theorem.

Denote by Ψ the family of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ so that

(1) $\psi(s) = 0 \Leftrightarrow s = 0$.
(2) ψ is nondecreasing and continuous.

Denote by $\mathbf{\Gamma}$ the family of all $\gamma : (0, \infty) \to \mathbb{R}$ so that

(1) γ is continuous and increasing.
(2) for all $\{t_n\} \subseteq (0, \infty)$, $\lim\limits_{n\to\infty} t_n = 1$ iff $\lim\limits_{n\to\infty} \gamma(t_n) = 0$.
(3) for all $\{t_n\} \subseteq (0, \infty)$, $\lim\limits_{n\to\infty} t_n = 0$ iff $\lim\limits_{n\to\infty} \gamma(t_n) = -\infty$.

Note that $\gamma(1) = 0$.

Some examples of elements of $\mathbf{\Gamma}$ are:

(1) $\gamma_1(\varsigma) = \ln(\varsigma)$
(2) $\gamma_2(\varsigma) = -\varsigma^{-\frac{1}{2}} + 1$.

Definition 1.4. *[7] Let $\mathbb{F}$ be the family of all maps $F : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying:*

(1) $\max\{l, m\} \leq F(l, m)$ *for all* $l, m \geq 0$.
(2) F is continuous and nondecreasing.
(3) $F(l_1 + l_2, m_1 + m_2) \leq F(l_1, m_1) + F(l_2, m_2)$.
For example, let $F(\tau, \varsigma) = \tau + \varsigma$, for all $\tau, \varsigma \geq 0$.

Definition 1.5. *The function $\beta : \mathbb{R}_+ \to [0, 1)$ such that* $\limsup_{s\to t^+} \beta(s) < 1$, *for any $t > 0$, is called a Mizogochi-Takahashi mapping. We denote this class by $\mathcal{MT}$.*

Lemma 1.6. *[6] Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by $f(x) = x^\alpha$.*
(1) If $\alpha \geq 0$ and $t_1, t_2 \in I = [a, b]$, where $a, b \geq 0$ and $t_2 > t_1$, then $t_2^\alpha - t_1^\alpha \leq \alpha(t_2 - t_1)$.
(2) If $0 < \alpha < 1$ and $t_1, t_2 \in I$ and $t_2 > t_1$, then $t_2^\alpha - t_1^\alpha \leq (t_2 - t_1)^\alpha$.

2. New Results

In this section, we establish a new fixed point theorem with the help of a new condensing operator which involves Mizogochi-Takahashi mappings. Also, we show that this new fixed point theorem is a generalization of Darbo's fixed point theorem.

Theorem 2.1. *Let $\mathbb{E}$ be a Banach space and let $C \subseteq \mathbb{E}$ be an NBCC. Also, let $T : C \to C$ be a continuous mapping satisfying*

$$\gamma\left[\psi\left\{F\left(\vartheta\left(TD\right),\phi\left(\vartheta\left(TD\right)\right)\right)\right\}\right] \leq \gamma\left[\beta\left[\psi\left\{F\left(\vartheta\left(D\right),\phi\left(\vartheta\left(D\right)\right)\right)\right\}\right]\right] + \gamma\left[\psi\left\{F\left(\vartheta\left(D\right),\phi\left(\vartheta\left(D\right)\right)\right)\right\}\right] \tag{2.1}$$

where $D \subseteq C$, $F \in \mathbb{F}$, $\gamma \in \mathbf{\Gamma}$, $\beta \in \mathcal{MT}$, $\psi \in \Psi$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and ϑ is an arbitrary MNC. Then T has a fixed point in C.

Proof . Define a sequence $(\mathbf{C}_n)$, where $\mathbf{C}_1 = C$ and $\mathbf{C}_{n+1} = \mathrm{Conv}(T\mathbf{C}_n)$ for all $n \geq 1$. Also, $T\mathbf{C}_1 = TC \subseteq C = \mathbf{C}_1$, $\mathbf{C}_2 = \mathrm{Conv}(T\mathbf{C}_1) \subseteq C = \mathbf{C}_1$ and continuing this process, we have $\mathbf{C}_1 \supseteq \mathbf{C}_2 \supseteq \mathbf{C}_3 \supseteq \ldots \supseteq \mathbf{C}_n \supseteq \mathbf{C}_{n+1} \supseteq \ldots$.

If $n_1 \in \mathbb{N}$ satisfying $\vartheta(\mathbf{C}_{n_1}) = 0$, then $\mathbf{C}_{n_1}$ is compact. By Theorem 1.2 it can be observed that T has a fixed point.

Let $F\left(\vartheta(\mathbf{C}_n), \phi(\vartheta(\mathbf{C}_n))\right) > 0$ for all $n > 0$. By (2.1) we have

$$\begin{aligned}
&\gamma\left[\psi\left\{F\left(\vartheta\left(\mathbf{C}_{n+1}\right),\phi\left(\vartheta\left(\mathbf{C}_{n+1}\right)\right)\right)\right\}\right]\\
&= \gamma\left[\psi\left\{F\left(\vartheta\left(ConvT\mathbf{C}_n\right),\phi\left(\vartheta\left(ConvT\mathbf{C}_n\right)\right)\right)\right\}\right]\\
&= \gamma\left[\psi\left\{F\left(\vartheta\left(T\mathbf{C}_n\right),\phi\left(\vartheta\left(T\mathbf{C}_n\right)\right)\right)\right\}\right]\\
&\leq \gamma\left[\beta\left[\psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\}\right]\right] + \gamma\left[\psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\}\right]\\
&< \gamma\left[\psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\}\right].
\end{aligned}$$

Since γ is increasing, we have,

$$\psi\left\{F\left(\vartheta\left(\mathbf{C}_{n+1}\right),\phi\left(\vartheta\left(\mathbf{C}_{n+1}\right)\right)\right)\right\} < \psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\},$$

i.e., $\{\psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\}\}_{n=1}^{\infty}$ is a positive, decreasing and bounded below sequence of real numbers.

Suppose that

$$\lim_{n\to\infty} \psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\} = r \geq 0.$$

Assume that $r > 0$. As $n \to \infty$, we have

$$\gamma\left(r\right) < \gamma(r)$$

which is a contradiction. So, $\lim_{n\to\infty} \psi\left\{F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right)\right\} = 0$, i.e.,

$$\lim_{n\to\infty} F\left(\vartheta\left(\mathbf{C}_n\right),\phi\left(\vartheta\left(\mathbf{C}_n\right)\right)\right) = 0.$$

Using the property of F we get $\lim_{n\to\infty} \vartheta\left(\mathbf{C}_n\right) = 0 = \lim_{n\to\infty} \phi\left[\vartheta\left(\mathbf{C}_n\right)\right]$.

Since $\mathbf{C}_n \supseteq \mathbf{C}_{n+1}$, by Definition 1.1 we get $\mathbf{C}_\infty = \bigcap_{n=1}^{\infty} \mathbf{C}_n \subseteq C$ is nonempty, closed and convex. Also, $\mathbf{C}_\infty$ is invariant under T. Thus, Theorem 1.2 implies that T has a fixed point in $\mathbf{C}_\infty \subseteq C$. □

Corollary 2.2. *Let $\mathbb{E}$ be a Banach space and let $C \subseteq \mathbb{E}$ be an NBCC. Also, let $T : C \to C$ be a continuous mapping satisfying*

$$\psi\left\{F\left(\vartheta\left(TD\right),\phi\left(\vartheta\left(TD\right)\right)\right)\right\} \leq \beta\left[\psi\left\{F\left(\vartheta\left(D\right),\phi\left(\vartheta\left(D\right)\right)\right)\right\}\right]\psi\left\{F\left(\vartheta\left(D\right),\phi\left(\vartheta\left(D\right)\right)\right)\right\} \tag{2.2}$$

where $D \subseteq C$, $F \in \mathbb{F}$, $\beta \in \mathcal{MT}$, $\psi \in \Psi$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and ϑ is an arbitrary MNC. Then T has a fixed point in C.

Proof . The result follows by taking $\gamma(\varsigma) = \ln \varsigma$ in Theorem 2.1. $\square$

Theorem 2.3. *Let $\mathbb{E}$ be a Banach space and let $C \subseteq \mathbb{E}$ be an NBCC. Also, let $T : C \to C$ be a continuous mapping satisfying*

$$\psi \left\{F\left(\vartheta\left(TD\right), \phi\left(\vartheta\left(TD\right)\right)\right)\right\} \leq k\psi \left\{F\left(\vartheta\left(D\right), \phi\left(\vartheta\left(D\right)\right)\right)\right\} \tag{2.3}$$

where $D \subseteq C$, $F \in \mathbb{F}$, $\psi \in \Psi$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and ϑ is an arbitrary MNC. Then T has a fixed point in C.

Proof . The result follows by taking $\beta(\varpi) = k \in [0, 1)$ in Theorem 2.2. $\square$

Remark 2.4. *Taking $\psi(x) = \frac{x}{2}$, $\phi \equiv 0$ and $F(p, q) = p + q$ in Theorem 2.3, Darbo's Theorem is obtained.*

3. Measure of noncompactness on $C([a, b])$

Let $\mathbf{E} = C(I)$ be the set of real continuous functions on I, where $I = [a, b]$. Then $\mathbf{E}$ is a Banach space with the norm

$$\| \varrho \| = \sup \left\{|\varrho(\varsigma)| : \varsigma \in I\right\}, \ \varrho \in \mathbf{E}.$$

Let $\Lambda(\neq \phi) \subseteq \mathbf{E}$ be bounded. For $\varrho \in \Lambda$ and $\epsilon > 0$, denote by $\omega(\varrho, \epsilon)$ the modulus of the continuity of ϱ, i.e.,

$$\omega(\varrho, \epsilon) = \sup \left\{|\varrho(\varsigma_1) - \varrho(\varsigma_2)| : \varsigma_1, \varsigma_2 \in I, |\varsigma_1 - \varsigma_1| \leq \epsilon\right\}.$$

Further, we define

$$\omega(\Lambda, \epsilon) = \sup \left\{\omega(\varrho, \epsilon) : \varrho \in \Lambda\right\}$$

and

$$\omega_0(\Lambda) = \lim_{\epsilon \to 0} \omega(\Lambda, \epsilon).$$

It is well-known that the function ω_0 is an MNC in $\mathbf{E}$ such that the Hausdorff MNC χ is given by $\chi(\Lambda) = \frac{1}{2}\omega_0(\Lambda)$ (see [4]).

4. Existence of solution of an integral equation involving two different fractional integrals

In this part, we shall establish the existence of solution of an integral equation involving both Erdélyi-Kober and (k, s)-Riemann-Liouville fractional equation in $C[1, T]$ with the help of the newly established fixed point theorem.

We find in [6] the definition of the Erdélyi-Kober fractional integral equation of a continuous function f as follows:

$$\mathbf{I}^{\gamma}_{\beta,a} f(\varsigma) = \frac{\beta}{\Gamma(\gamma)} \int_a^{\varsigma} \frac{\xi^{\beta-1} f(\xi)}{(\varsigma^{\beta} - \xi^{\beta})^{1-\gamma}} d\xi, \ \beta > 0, \ 0 < \gamma < 1, \ a > 0.$$

For $a = 1$,

$$\mathbf{I}^{\gamma}_{\beta} f(\varsigma) = \frac{\beta}{\Gamma(\gamma)} \int_1^{\varsigma} \frac{\xi^{\beta-1} f(\xi)}{(\varsigma^{\beta} - \xi^{\beta})^{1-\gamma}} d\xi, \ \beta > 0, \ 0 < \gamma < 1.$$

We find in [1] the definition of the (k, s)-Riemann-Liouville fractional integral equation of a continuous function f as follows:

$$ {}_{k}^{s}\mathbf{J}_{a}^{\alpha}f(\varsigma)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{S}{k}}\Gamma(\frac{\varsigma}{k})}\int_{a}^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}f(\eta)d\eta,\ \varsigma\in[a,b],\ k>0,\ \alpha>0,\ s\in\mathbb{R}\backslash\{1\},\ a>0. $$

For $a=1$,

$$ {}_{k}^{s}\mathbf{J}^{\alpha}f(\varsigma)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{S}{k}}\Gamma(\frac{\varsigma}{k})}\int_{1}^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}f(\eta)d\eta. $$

In this part, we study the fractional integral equation

$$ \Upsilon(\varsigma)=\Pi\left(\varsigma,\mathbf{U}(\varsigma,\Upsilon(\varsigma)),\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma),{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)\right), \tag{4.1} $$

where $0<\gamma,\beta,k<1,\ \alpha>0,\ s\in\mathbb{R}\backslash\{1\}$ and $\varsigma\in I=[1,T]$.

Let

$$ B_{d_0}=\{\Upsilon\in\mathbf{E}:\|\Upsilon\|\leq d_0\}. $$

Assume that

(A) $\Pi:I\times\mathbb{R}^3\to\mathbb{R},\ \mathbf{U}:I\times\mathbb{R}\to\mathbb{R}$ be continuous and there exist constants $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\geq 0$ satisfying

$$ \left|\Pi(\varsigma,\mathbf{U},I_1,I_2)-\Pi(\varsigma,\bar{\mathbf{U}},\bar{I}_1,\bar{I}_2)\right|\leq\alpha_1\left|\mathbf{U}-\bar{\mathbf{U}}\right|+\alpha_2\left|I_1-\bar{I}_1\right|+\alpha_3\left|I_2-\bar{I}_2\right|, $$

for all $\varsigma\in I,\ \mathbf{U},I_1,I_2,\bar{\mathbf{U}},\bar{I}_1,\bar{I}_2\in\mathbb{R}$ and

$$ |\mathbf{U}(\varsigma,J_1)-\mathbf{U}(\varsigma,J_2)|\leq\alpha_4|J_1-J_2|,\ J_1,J_2\in\mathbb{R}, $$

and

$$ \alpha_1\alpha_4<1. $$

(B) There exists $d_0>0$ satisfying

$$ \bar{\Pi}=\sup\left\{|\Upsilon(\varsigma,\mathbf{U},I_1,I_2)|:\varsigma\in I,\mathbf{U}\in[-\hat{\mathbf{U}},\hat{\mathbf{U}}],I_1\in[-\hat{\mathbf{J}},\hat{\mathbf{J}}],I_2\in[-\hat{\mathbf{I}},\hat{\mathbf{I}}]\right\}\leq d_0 $$

where

$$ \hat{\mathbf{U}}=\sup\{|\mathbf{U}|:\varsigma\in I,\Upsilon(\varsigma)\in[-d_0,d_0]\}, $$
$$ \hat{\mathbf{J}}=\sup\left\{\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma)\right|:\varsigma\in I,\Upsilon(\varsigma)\in[-d_0,d_0]\right\} $$

and

$$ \hat{\mathbf{I}}=\sup\{|{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)|:\varsigma\in I,\Upsilon(\varsigma)\in[-d_0,d_0]\}. $$

(C) $|\Pi(\varsigma,0,0,0)|=0$.

(D) there exists a positive solution d_0 of the inequality

$$ \alpha_1\alpha_4 r+\frac{\alpha_2 r}{\Gamma(\gamma+1)}T^{\beta\gamma}+\frac{\alpha_3 r(s+1)^{-\frac{\alpha}{k}}}{\alpha k^{\frac{T}{k}-1}\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\alpha}{k}}\leq r. $$

Theorem 4.1. *If conditions (A)-(D) hold, then 4.1 has a solution in* $\mathbf{E}=C(I)$.

Proof . Define the operator $\mathbf{Q}:\mathbf{E}\to\mathbf{E}$ as follows:

$$(\mathbf{Q}\Upsilon)(\varsigma)=\Pi\left(\varsigma,\mathbf{U}(\varsigma,\Upsilon(\varsigma)),\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma),{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)\right).$$

Step 1: We prove that the function $\mathbf{Q}$ maps B_{d_0} into B_{d_0}. Let $\Upsilon\in B_{d_0}$. We have

$$\begin{aligned}
&\left|(\mathbf{Q}\Upsilon)(\varsigma)\right|\\
&\le\left|\Pi\left(\varsigma,\mathbf{U}(\varsigma,\Upsilon(\varsigma)),\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma),{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)\right)-\Pi(\varsigma,0,0,0)\right|+\left|\Pi(\varsigma,0,0,0)\right|\\
&\le\alpha_1\left|\mathbf{U}(\varsigma,\Upsilon(\varsigma))-0\right|+\alpha_2\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma)-0\right|+\alpha_3\left|{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)-0\right|+\left|\Pi(\varsigma,0,0,0)\right|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma\right|\\
&=\left|\frac{\beta}{\Gamma(\gamma)}\int_1^{\varsigma}\frac{\xi^{\beta-1}\Upsilon(\xi)}{(\varsigma^{\beta}-\xi^{\beta})^{1-\gamma}}d\xi\right|\\
&\le\frac{\beta}{\Gamma(\gamma)}\int_1^{\varsigma}\frac{\xi^{\beta-1}\left|\Upsilon(\xi)\right|}{(\varsigma^{\beta}-\xi^{\beta})^{1-\gamma}}d\xi\\
&<\frac{\beta d_0}{\Gamma(\gamma)}\int_1^{\varsigma}\frac{\xi^{\beta-1}}{(\varsigma^{\beta}-\xi^{\beta})^{1-\gamma}}d\xi\\
&<\frac{d_0}{\Gamma(\gamma+1)}T^{\beta\gamma},
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)\right|\\
&=\left|\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma}{k}}\Gamma(\frac{\varsigma}{k})}\int_1^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}\Upsilon(\eta)d\eta\right|\\
&\le\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma}{k}}\Gamma(\frac{\varsigma}{k})}\int_1^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}\left|\Upsilon(\eta)\right|d\eta\\
&\le\frac{d_0(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma}{k}}\Gamma(\frac{\varsigma}{k})}\int_1^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}d\eta\\
&\le\frac{d_0(s+1)^{-\frac{\alpha}{k}}}{\alpha k^{\frac{T}{k}-1}\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\alpha}{k}}.
\end{aligned}$$

Hence, $\parallel\Upsilon\parallel<d_0$ gives

$$\parallel\mathbf{Q}\Upsilon\parallel<\alpha_1\alpha_4 d_0+\frac{\alpha_2 d_0}{\Gamma(\gamma+1)}T^{\beta\gamma}+\frac{\alpha_3 d_0(s+1)^{-\frac{\alpha}{k}}}{\alpha k^{\frac{T}{k}-1}\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\alpha}{k}}\le d_0.$$

Due to the assumption (D) $\mathbf{Q}$ maps B_{d_0} into B_{d_0}.

Step 2: We prove that $\mathbf{Q}$ is continuous on B_{d_0}. Let $\epsilon>0$ and $\Upsilon,\bar{\Upsilon}\in B_{r_0}$ such that $\parallel\Upsilon-\bar{\Upsilon}\parallel<\epsilon$. We have

$$\begin{aligned}
&\left|(\mathbf{Q}\Upsilon)(\varsigma)-(\mathbf{Q}\bar{\Upsilon})(\varsigma)\right|\\
&\le\left|\Pi\left(\varsigma,\mathbf{U}(\varsigma,\Upsilon(\varsigma)),\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma),{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)\right)-\Pi\left(\varsigma,\mathbf{U}(\varsigma,\bar{\Upsilon}(\varsigma)),\mathbf{I}_{\beta}^{\gamma}\bar{\Upsilon}(\varsigma),{}_{k}^{s}\mathbf{J}^{\alpha}\bar{\Upsilon}(\varsigma)\right)\right|\\
&\le\alpha_1\left|\mathbf{U}(\varsigma,\Upsilon(\varsigma))-\mathbf{U}(\varsigma,\bar{\Upsilon}(\varsigma))\right|+\alpha_2\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma)-\mathbf{I}_{\beta}^{\gamma}\bar{\Upsilon}(\varsigma)\right|+\alpha_3\left|{}_{k}^{s}\mathbf{J}^{\alpha}\Upsilon(\varsigma)-{}_{k}^{s}\mathbf{J}^{\alpha}\bar{\Upsilon}(\varsigma)\right|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma)-\mathbf{I}_{\beta}^{\gamma}\bar{\varUpsilon}(\varsigma)\right|\\
&=\left|\frac{\beta}{\Gamma(\gamma)}\int_{1}^{\varsigma}\frac{\xi^{\beta-1}(\varUpsilon(\xi)-\bar{\varUpsilon}(\xi))}{(\varsigma^{\beta}-\xi^{\beta})^{1-\gamma}}d\xi\right|\\
&\leq\frac{\beta}{\Gamma(\gamma)}\int_{1}^{\varsigma}\frac{\xi^{\beta-1}\left|\varUpsilon(\xi)-\bar{\varUpsilon}(\xi)\right|}{(\varsigma^{\beta}-\xi^{\beta})^{1-\gamma}}d\xi\\
&<\frac{\beta\epsilon}{\Gamma(\gamma)}\int_{1}^{\varsigma}\frac{\xi^{\beta-1}}{(\varsigma^{\beta}-\xi^{\beta})^{1-\gamma}}d\xi\\
&<\frac{\epsilon}{\Gamma(\gamma+1)}T^{\beta\gamma},
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma)-{}_{k}^{s}\mathbf{J}^{\alpha}\bar{\varUpsilon}(\varsigma)\right|\\
&=\left|\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma}{k}}\Gamma(\frac{\varsigma}{k})}\int_{1}^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}\left(\varUpsilon(\eta)-\bar{\varUpsilon}(\eta)\right)d\eta\right|\\
&\leq\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma}{k}}\Gamma(\frac{\varsigma}{k})}\int_{1}^{\varsigma}\left(\varsigma^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^{s}\left|\varUpsilon(\eta)-\bar{\varUpsilon}(\eta)\right|d\eta\\
&<\frac{\epsilon(s+1)^{-\frac{\alpha}{k}}}{\alpha k^{\frac{T}{k}-1}\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\alpha}{k}}.
\end{aligned}$$

Hence, $\parallel \varUpsilon-\bar{\varUpsilon}\parallel<\epsilon$ gives

$$\left|\left(\mathbf{Q}\varUpsilon\right)(\varsigma)-\left(\mathbf{Q}\bar{\varUpsilon}\right)(\varsigma)\right|<\alpha_1\alpha_4\epsilon+\frac{\alpha_2\epsilon}{\Gamma(\gamma+1)}T^{\beta\gamma}+\frac{\alpha_3\epsilon(s+1)^{-\frac{\alpha}{k}}}{\alpha k^{\frac{T}{k}-1}\Gamma(\frac{1}{k})}\left(T^{s+1}-1\right)^{\frac{\alpha}{k}}.$$

As $\epsilon\to 0$ we get $\left|\left(\mathbf{Q}\varUpsilon\right)(\varsigma)-\left(\mathbf{Q}\bar{\varUpsilon}\right)(\varsigma)\right|\to 0$. This shows that $\mathbf{Q}$ is continuous on B_{d_0}.

Step 3: An estimate of $\mathbf{Q}$ with respect to ω_0: Assume that $\varOmega(\neq\phi)\subseteq B_{d_0}$. Let $\epsilon>0$ be arbitrary and choose $\varUpsilon\in\varOmega$ and $\varsigma_1,\varsigma_2\in I$ such that $|\varsigma_2-\varsigma_1|\leq\epsilon$ and $\varsigma_2\geq\varsigma_1$.

Now,

$$\begin{aligned}
&\left|\left(\mathbf{Q}\varUpsilon\right)(\varsigma_2)-\left(\mathbf{Q}\varUpsilon\right)(\varsigma_1)\right|\\
&=\left|\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_2),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_2)\right)-\varPi(\varsigma_1,\mathbf{U}\left(\varsigma_1,\varUpsilon(\varsigma_1)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)\right|\\
&\leq\left|\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_2),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_2)\right)-\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_2),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)\right|\\
&+\left|\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_2),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)-\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)\right|\\
&+\left|\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)-\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_1,\varUpsilon(\varsigma_1)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)\right|\\
&+\left|\varPi(\varsigma_2,\mathbf{U}\left(\varsigma_1,\varUpsilon(\varsigma_1)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)-\varPi(\varsigma_1,\mathbf{U}\left(\varsigma_1,\varUpsilon(\varsigma_1)\right),\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1),{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right)\right|\\
&\leq\alpha_3\left|{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_2)-{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right|+\alpha_2\left|\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_2)-\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1)\right|\\
&+\alpha_1\left|\mathbf{U}\left(\varsigma_2,\varUpsilon(\varsigma_2)\right)-\mathbf{U}\left(\varsigma_1,\varUpsilon(\varsigma_1)\right)\right|+\omega_{\varPi}(I,\epsilon)\\
&\leq\alpha_3\left|{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_2)-{}_{k}^{s}\mathbf{J}^{\alpha}\varUpsilon(\varsigma_1)\right|+\alpha_2\left|\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_2)-\mathbf{I}_{\beta}^{\gamma}\varUpsilon(\varsigma_1)\right|\\
&+\alpha_1\alpha_4\left|\varUpsilon(\varsigma_2)-\varUpsilon(\varsigma_1)\right|+\omega_{\varPi}(I,\epsilon)
\end{aligned}$$

where

$$\omega_{\Pi}(I,\epsilon)=\sup\left\{\begin{array}{c}|\Pi(\varsigma_2,\mathbf{U},\mathbf{I}_1,\mathbf{I}_2)-\Pi(\varsigma_1,\mathbf{U},\mathbf{I}_1,\mathbf{I}_2)|:|\varsigma_2-\varsigma_1|\leq\epsilon;\varsigma_1,\varsigma_2\in I;\\ \mathbf{U}\in[-\hat{\mathbf{U}},\hat{\mathbf{U}}];\mathbf{I}_1\in[-\hat{\mathbf{J}},\hat{\mathbf{J}}];\mathbf{I}_2\in[-\hat{\mathbf{I}},\hat{\mathbf{I}}]\end{array}\right\}.$$

Also,

$$\begin{aligned}
&|{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_2)-{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_1)|\\
&=\left|\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_2}{k}}\Gamma(\frac{\varsigma_2}{k})}\int_1^{\varsigma_2}\left(\varsigma_2^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\Upsilon(\eta)d\eta-\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_1}{k}}\Gamma(\frac{\varsigma_1}{k})}\int_1^{\varsigma_1}\left(\varsigma_1^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\Upsilon(\eta)d\eta\right|\\
&\leq\left|\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_2}{k}}\Gamma(\frac{\varsigma_2}{k})}\int_1^{\varsigma_2}\left(\varsigma_2^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\Upsilon(\eta)d\eta-\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_2}{k}}\Gamma(\frac{\varsigma_2}{k})}\int_1^{\varsigma_1}\left(\varsigma_1^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\Upsilon(\eta)d\eta\right|\\
&+\left|\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_2}{k}}\Gamma(\frac{\varsigma_2}{k})}-\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_1}{k}}\Gamma(\frac{\varsigma_1}{k})}\right|\int_1^{\varsigma_1}\left(\varsigma_1^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\left|\Upsilon(\eta)\right|d\eta\\
&\leq\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_2}{k}}\Gamma(\frac{\varsigma_2}{k})}\left|\int_1^{\varsigma_2}\left(\varsigma_2^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\Upsilon(\eta)d\eta-\int_1^{\varsigma_1}\left(\varsigma_1^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\Upsilon(\eta)d\eta\right|\\
&+\left|\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_2}{k}}\Gamma(\frac{\varsigma_2}{k})}-\frac{(s+1)^{1-\frac{\alpha}{k}}}{k^{\frac{\varsigma_1}{k}}\Gamma(\frac{\varsigma_1}{k})}\right|\parallel\Upsilon\parallel\int_1^{\varsigma_1}\left(\varsigma_1^{s+1}-\eta^{s+1}\right)^{\frac{\alpha}{k}-1}\eta^s\left|\Upsilon(\eta)\right|d\eta\\
&\leq\frac{(s+1)^{-\frac{\alpha}{k}}\parallel\Upsilon\parallel}{(s+2)k^{\frac{\varsigma_2}{k}}\gamma\left(\frac{\varsigma_2}{k}\right)}\left[2(\varsigma_2^{s+1}-\varsigma_1^{s+1})^{\frac{\alpha}{k}}+(\varsigma_2^{s+1}-1)^{\frac{\alpha}{k}}-(\varsigma_1^{s+1}-1)^{\frac{\alpha}{k}}\right]\\
&+(s+1)^{-\frac{\alpha}{k}}\left|\frac{1}{k^{\frac{\varsigma_2}{k}}\gamma\left(\frac{\varsigma_2}{k}\right)}-\frac{1}{k^{\frac{\varsigma_1}{k}}\gamma\left(\frac{\varsigma_1}{k}\right)}\right|\parallel\Upsilon\parallel\frac{k}{\alpha}\left(T^{s+1}-1\right)^{\frac{\alpha}{k}}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_2)-\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_1)\right|\\
&=\left|\frac{\beta}{\Gamma(\varpi)}\int_1^{\varsigma_2}\frac{\xi^{\beta-1}\Upsilon(\xi)}{\left(\varsigma_2^{\beta}-\xi^{\beta}\right)^{1-\gamma}}d\xi-\frac{\beta}{\Gamma(\gamma)}\int_1^{\varsigma_1}\frac{\xi^{\beta-1}\Upsilon(\xi)}{\left(\varsigma_1^{\beta}-\xi^{\beta}\right)^{1-\gamma}}d\xi\right|\\
&\leq\frac{\parallel\Upsilon\parallel}{\Gamma(\gamma+1)}\left[2\left(s_2^{\beta}-s_1^{\beta}\right)^{\gamma}+\left(s_2^{\beta}-1\right)^{\gamma}-\left(s_1^{\beta}-1\right)^{\gamma}\right].
\end{aligned}$$

As $\epsilon\to 0$, then $\varsigma_2\to\varsigma_1$ and so, $|{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_2)-{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_1)|\to 0$ and $\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_2)-\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_1)\right|\to 0$. Hence,

$$\begin{aligned}
&|(\mathbf{Q}\Upsilon)(\varsigma_2)-(\mathbf{Q}\Upsilon)(\varsigma_1)|\\
&\leq\alpha_3\,|{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_2)-{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_1)|+\alpha_2\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_2)-\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_1)\right|\\
&+\alpha_1\alpha_4\omega(\Upsilon,\epsilon)+\omega_{\Pi}(I,\epsilon),
\end{aligned}$$

i.e.

$$\omega(\mathbf{Q}\Upsilon,\epsilon)\leq\alpha_3\,|{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_2)-{}_k^s\mathbf{J}^{\alpha}\Upsilon(\varsigma_1)|+\alpha_2\left|\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_2)-\mathbf{I}_{\beta}^{\gamma}\Upsilon(\varsigma_1)\right|+\alpha_1\alpha_4\omega(\Upsilon,\epsilon)+\omega_{\Pi}(I,\epsilon).$$

By the uniform continuity of Π on $I\times[-\hat{\mathbf{U}},\hat{\mathbf{U}}]\times[-\hat{\mathbf{J}},\hat{\mathbf{J}}]\times[-\hat{\mathbf{I}},\hat{\mathbf{I}}]$ we have $\omega_{\Pi}(I,\epsilon)\to 0$, as $\epsilon\to 0$.

Taking $\sup_{\Upsilon \in \Omega}$ and $\epsilon \to 0$ we get,

$$\omega_0(\mathbf{Q}\Omega) \leq \alpha_1 \alpha_4 \omega_0(\Omega),$$

Thus, by Remark 2.4, $\mathbf{Q}$ has a fixed point in $\Omega \subseteq B_{d_0}$, i.e., equation (4.1) has a solution in $\mathbf{E}$. □

Now, we shall consider an example of integral equations involving both (k,s)-Riemann-Liouville and Erdélyi-Kober fractional integrals and study the existence of solution of it on $C[1,2]$.

Example 4.2. Consider the following equation

$$\Upsilon(\varsigma) = \frac{\Upsilon(\varsigma)}{7+\varsigma^2} + \frac{\mathbf{I}^{\frac{1}{3}}_{\frac{1}{3}} \Upsilon(\varsigma)}{6} + \frac{{}^{\frac{1}{3}}_{\frac{1}{3}}\mathbf{J}^{\frac{2}{3}} \Upsilon(\varsigma)}{400} \tag{4.2}$$

for $\varsigma \in [1,2] = I$.

Here,

$$\mathbf{I}^{\frac{1}{3}}_{\frac{1}{3}} \Upsilon(\varsigma) = \frac{1}{3\Gamma(\frac{1}{3})} \int_1^{\varsigma} \xi^{-\frac{2}{3}} \left(\varsigma^{\frac{1}{3}} - \xi^{\frac{1}{3}}\right)^{-\frac{2}{3}} \Upsilon(\xi) d\xi$$

and

$${}^{\frac{1}{3}}_{\frac{1}{3}}\mathbf{J}^{\frac{2}{3}} \Upsilon(\varsigma) = \frac{3^{3\varsigma+1}}{4\Gamma(3\varsigma)} \int_1^{\varsigma} \xi^{\frac{1}{3}} \left(\varsigma^{\frac{4}{3}} - \xi^{\frac{4}{3}}\right) \Upsilon(\xi) d\xi.$$

Also, $\Pi(\varsigma, \mathbf{U}, \mathbf{I}_1, \mathbf{I}_2) = \mathbf{U} + \frac{\mathbf{I}_1}{6} + \frac{\mathbf{I}_2}{400}$ and $\mathbf{U}(\varsigma, \Upsilon) = \frac{\Upsilon}{7+\varsigma^2}$. It is trivial that both Π and $\mathbf{U}$ are continuous and

$$|\mathbf{U}(\varsigma, J_1) - \mathbf{U}(\varsigma, J_2)| \leq \frac{|J_1 - J_2|}{8}$$

and

$$\left|\Pi(\varsigma, \mathbf{U}, \mathbf{I}_1, \mathbf{I}_2) - \Pi(\varsigma, \bar{\mathbf{U}}, \bar{\mathbf{I}}_1, \bar{\mathbf{I}}_2)\right| \leq \left|\mathbf{U} - \bar{\mathbf{U}}\right| + \frac{1}{6}\left|\mathbf{I}_1 - \bar{\mathbf{I}}_1\right| + \frac{1}{400}\left|\mathbf{I}_2 - \bar{\mathbf{I}}_2\right|.$$

Therefore, $\alpha_1 = 1,\ \alpha_2 = \frac{1}{6},\ \alpha_3 = \frac{1}{400},\ \alpha_4 = \frac{1}{8}$ and $\alpha_1 \alpha_4 = \frac{1}{8} < 1$.

If $\| \Upsilon \| \leq d_0$, then

$$\hat{\mathbf{U}} = \frac{d_0}{8},$$

$$\hat{\mathbf{J}} = \frac{3\left(2^{\frac{1}{3}} - 1\right)^{\frac{1}{3}} d_0}{\Gamma(\frac{1}{3})}$$

and

$$\hat{\mathbf{I}} = \frac{3^8 \left(2^{\frac{4}{3}} - 1\right)^2 d_0}{32\Gamma(3)}.$$

Further,

$$|\Pi(\varsigma, \mathbf{U}, \mathbf{I}_1, \mathbf{I}_2)| \leq \frac{d_0}{8} + \frac{1}{6}\left\{\frac{3\left(2^{\frac{1}{3}} - 1\right)^{\frac{1}{3}} d_0}{\Gamma(\frac{1}{3})}\right\} + \frac{1}{400}.\frac{3^8 \left(2^{\frac{4}{3}} - 1\right)^2 d_0}{32\Gamma(3)} \leq d_0.$$

If we choose $d_0 = 2$, then

$$\hat{\mathbf{U}} = \frac{1}{4},\ \hat{\mathbf{J}} = \frac{\left(2^{\frac{1}{3}} - 1\right)^{\frac{1}{3}}}{\Gamma(\frac{1}{3})},\ \hat{\mathbf{I}} = \frac{3^8 \left(2^{\frac{4}{3}} - 1\right)^2}{16\Gamma(3)}$$

which gives

$$\bar{\Pi} \leq 2.$$

On the other hand, assumption (D) is also satisfied for $d_0 = 2$.

We observe that all the assumptions from $(A) - (D)$ of Theorem 4.1 are satisfied. By Theorem 4.1 it can be said that equation (4.2) has a solution in $\mathbf{E} = C(I)$.

5. Conclusion

Here, the solvability of fractional integral equations with both (k, s)-Riemann-Liouville and Erdélyi-Kober fractional integrals using a new generalized version of the Darbo's theorem using Mizogochi-Takahashi mappings has been studied. Also, justify the validity of our results with the help of suitable examples. This method can be applied to different types of integral equations involving different fractional integrals.

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