Some of the sufficient conditions to get the G-Bi-shadowing action

Mohammed Hussein Obaid Ajam[a,*], Iftichar Mudhar Talb Al-Shara'a[a]

[a]*Department of Mathematics, Collage of Education for Pure Science, University of Babylon, Babylon, Iraq*

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Abstract

The aim of this paper is introduced some examples of a $\mathbb{G}$-bi-shadowing actions on the metric $\mathbb{G}$-space, by study a sufficient conditions of actions to be $\mathbb{G}$-bi-shadowing. We show the $\mathbb{G}$-λ-Contraction actions, $\mathbb{G}$-(λ, L)-Contraction action, and $\mathbb{G}$-Hardy-Rogers contraction action are $\mathbb{G}$-bi-shadowing by proved some theorems.

Keywords: G-space, G-Bi-Shadowing, Sufficient Conditions, G-Contraction.

1. Introduction

The concept of shadowing is of great importance in studying and understanding dynamical systems because it often accounts for the accuracy of a computer simulation of the system being used. Work on it began to be developed by many researchers in recent years as an important link for dynamical systems with stability and chaos. The map with has shadowing property is assumed to have a true orbit fairly close to each pseudo orbit of this map. The researcher who gave the concept of shadowing is [15], see more [13, 1, 2].

Some researchers have evolved the shadowing into the bi-shadowing by assuming that the true orbit be on another maps under specific conditions. The researcher who gave the concept of bi-shadowing is [9]. Later in [3, 4] the researchers studied many relations between the bi-shadowing and other concept.

[5] introduce types of bi-shadowing concept. [6] introduced the concept of $\mathbb{G}$-bi-shadowing on the $\mathbb{G}$-space, and studied the $\mathbb{G}$-chaotic behaviour with $\mathbb{G}$-bi-shadowing.

In this paper we presented the concepts of $\mathbb{G}_{\mathfrak{s}}$-$\lambda$-contraction, $\mathbb{G}_{\mathfrak{s}}$-$(\lambda, L)$-contraction, and $\mathbb{G}_{\mathfrak{s}}$-Hardy-Rogers contraction, then studied the actions satisfies it is $\mathbb{G}$-bi-shadowing.

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*Corresponding author

Email addresses: `mohammed_ajam@science.uoqasim.edu.iq` (Mohammed Hussein Obaid Ajam),
`ifticharalshraa@gmail.com` (Iftichar Mudhar Talb Al-Shara'a)

2.   Preliminaries

Let $\mathbb{G}$ be a group, $\mathbb{X}$ be a Hausdorff topological space and ϕ be a map. Then the triple $(\mathbb{G}, \mathbb{X}, \phi)$ is called topological transformation group.

Definition 2.1. *[12]*
   The map ϕ: $\mathbb{G} \times \mathbb{X} \to \mathbb{X}$ which satisfies:

1. *$\phi(g, .)$ is a homeomorphism of $\mathbb{X}$ for any $g \in \mathbb{G}$,*
2. *$\phi(e, x) = x$ for all $x \in \mathbb{X}$ where e is the identity of the group $\mathbb{G}$,*
3. *$\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$, for all $g_1, g_2 \in \mathbb{G}$, $x \in \mathbb{X}$.*

   *is called an **action** of a group $\mathbb{G}$ on $\mathbb{X}$. And a space $\mathbb{X}$ with an action ϕ of $\mathbb{G}$ is called a $\mathbb{G}$-**space**.*

Definition 2.2. *[12, 10]*
   *The group $\mathbb{G}$ is called **generated by** $\mathbb{S}$ if $\langle \mathbb{S} \rangle = \mathbb{G}$.*

The group $\mathbb{G}$ is called **finitely generated** if the generating set $\mathbb{S}$ is finite.
The generating set $\mathbb{S}$ is called **symmetric** if for any $s \in \mathbb{S}$ then $s^{-1} \in \mathbb{S}$.
In this paper we suppose that $\mathbb{G}$ is a finitely generated group, $\mathbb{X}$ is a metric $\mathbb{G}$-space with metric d, and $\phi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be an action. And we fix finite symmetric generating set s of $\mathbb{G}$.

Remark 2.3. *For $x \in \mathbb{X}$ and $n \in \mathbb{N}$, we have:*

1. *By Definition 2.1 the image of x by ϕ is $\phi(s, x)$ for $s \in \mathbb{S}$.*
2. *We denote to the inverse image of ϕ by ϕ^{-1}, and the inverse image of x by ϕ is $\phi^{-1}(s, x) = \phi(s^{-1}, x)$, for $s \in \mathbb{S}$.*
3. *The n-iterate of x by ϕ is*

$$\underbrace{\phi(s, \ldots \phi(s, x))}_{n-iterate} = \phi\left(\underbrace{s \ldots s}_{n-items}, x\right) = \phi(ns, x), \quad for\ s \in \mathbb{S},$$

   so we denote to n iterate of ϕ by ϕ^n and $\phi^n(s, .) = \phi(ns, .)$, for $s \in \mathbb{S}$.
4. *The n inverse iterate of x by ϕ is*

$$\underbrace{\phi\left(s^{-1}, \ldots \phi\left(s^{-1}, x\right)\right)}_{n-iterate} = \phi\left(\underbrace{s^{-1} \ldots s^{-1}}_{n-items}, x\right) = \phi\left(ns^{-1}, x\right), \quad for\ s \in s,$$

   so we denote to n inverse iterate of ϕ by ϕ^{-n} and $\phi^{-n}(s, .) = \phi(ns^{-1}, .)$, for $s \in \mathbb{S}$.

Definition 2.4. *[8] For $x \in \mathbb{X}$, the sequence $\mathbb{O}(x) = \{\phi(g, x) \in X \mid g \in \mathbb{G}\}$ which generated by x and a group $\mathbb{G}$ is called $\mathbb{G}$-**orbit of** x for ϕ in $\mathbb{X}$.*

Definition 2.5. *[14] A sequence $\mathbf{x} = \{x_g \in \mathbb{X} \mid g \in \mathbb{G}\}$ is called $\mathbb{G}_s$-**orbit** for ϕ if satisfying*

$$x_{sg} = \phi(s, x_g)\ \ for\ s \in \mathbb{S}\ and\ g \in \mathbb{G}. \tag{2.1}$$

Remark 2.6.

1. *We rewrite a $\mathbb{G}_{\mathbb{s}}$-orbit $\mathbf{x} = \{x_g \in \mathbb{X} \mid g \in \mathbb{G}\}$ in Definition 2.5 as a sequence associated with a subset of integer numbers, then a sequence $\mathbf{x}$ **is became** $\mathbf{x} = \{x_n \in \mathbb{X} \mid n \in \mathbb{I} \subseteq \mathbb{Z}\}$, when the length of an interval $\mathbb{I} \subseteq \mathbb{Z}$ depends on the members of a group $\mathbb{G}$. We can reformulated the Condition (2.1) as follows :*

$$x_{n+1} = \phi(\mathbb{s}, x_n) \ for \ n \in \mathbb{Z}, \ \mathbb{s} \in \mathbb{S}.$$

2. *Note that a sequence $\mathbf{x}$ can be finite or infinite.*

Definition 2.7. *[14] For $\delta > 0$, a sequence $\mathbf{y} = \{y_g \in \mathbb{X} \mid g \in \mathbb{G}\}$ is called $\mathbb{G}_{\mathbb{s}}$-δ-**pseudo orbit** for ϕ if satisfying*

$$\mathrm{d}(y_{\mathbb{s}g}, \phi(\mathbb{s}, y_g)) \leq \delta, \ for \ \mathbb{s} \in \mathbb{S} \ and \ g \in \mathbb{G}. \tag{2.2}$$

Remark 2.8.

1. *As in Remark 2.6, we can reformulated $\mathbf{y} = \{y_n \in \mathbb{X} \mid n \in \mathbb{I} \subseteq \mathbb{Z}\}$ and the condition (2.2) as follows $\mathrm{d}(y_{n+1}, \phi(\mathbb{s}, y_n)) \leq \delta, \ for \ n \in \mathbb{Z}, \ \mathbb{s} \in \mathbb{S}.$*
2. *Note that a sequence $\mathbf{y}$ can be finite or infinite.*
3. *Let $\mathbb{O}(\phi)$, and $\mathbb{O}(\phi, \delta)$ be denote the sets of all (finite or infinite) $\mathbb{G}_{\mathbb{s}}$-orbits for ϕ, and $\mathbb{G}_{\mathbb{s}}$-δ-pseudo orbits.*

Definition 2.9. *Let $\phi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ and $\psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be an actions. The $\mathbb{G}_{\mathbb{s}}$-**distance** between ϕ and ψ is given by:*

$$\mathrm{d}_0(\phi, \psi) = \sup_{x \in \mathbb{X}} \{\mathrm{d}(\phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\} \ for \ \mathbb{s} \in \mathbb{S}.$$

Definition 2.10. *Let $\mathrm{a}, \mathrm{b} > 0$, the action ϕ is called $\mathbb{G}_{\mathbb{s}}$-**bi-shadowing with** a **and** b **on** $\mathbb{X}$ (for short we denoted by $\mathbb{G}_{\mathbb{s}}$-(a, b)-bi-shadowing) if there exists $0 < \delta \leq \mathrm{b}$ such that for any $\mathbb{G}_{\mathbb{s}}$-δ-pseudo orbit (finite or infinite) $\mathbf{y} = \{y_n \in \mathbb{X} \mid n \in \mathbb{I} \subseteq \mathbb{Z}\} \in \mathbb{O}(\phi, \mathbb{X}, \delta)$ and any action $\psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ satisfying $\mathrm{d}_0(\phi, \psi) \leq -\delta$ then there exists a $\mathbb{G}_{\mathbb{s}}$-orbit $\mathbf{x} = \{x_n \in \mathbb{X} \mid n \in \mathbb{I} \subseteq \mathbb{Z}\} \in \mathbb{O}(\psi, \mathbb{X})$ such that:*

$$\mathrm{d}(x_n, y_n) \leq \mathrm{a}(\delta + \mathrm{d}_0(\phi, \psi)) \leq \mathrm{ab}, \ for \ all \ n \ as \ define \ in \ \mathbf{y}.$$

We will introduced the concepts of $\mathbb{G}_{\mathbb{s}}$-λ-contraction, $\mathbb{G}_{\mathbb{s}}$-(λ, L)-contraction, and $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction.

Definition 2.11. *The action ϕ is called $\mathbb{G}_{\mathbb{s}}$-λ-**contraction** if there exists $0 < \lambda < 1$ such that $\mathrm{d}(\phi(\mathbb{s}, x), \phi(\mathbb{s}, y)) \leq \lambda \, \mathrm{d}(x, y)$ for all $x, y \in \mathbb{X}$, and $\mathbb{s} \in \mathbb{S}$.*

Definition 2.12. *The action ϕ is called $\mathbb{G}_{\mathbb{s}}$-(λ, L)-**contraction**, if there exists constants $0 \leq \lambda < 1$ and $L \geq 0$ such that*

$$\mathrm{d}(\phi(\mathbb{s}, x), \phi(\mathbb{s}, y)) \leq \lambda \mathrm{d}(x, y) + L \, \mathrm{d}(y, \phi(\mathbb{s}, x)), \ for \ all \ x, y \in \mathbb{X} \ and \ \mathbb{s} \in \mathbb{S}.$$

Definition 2.13. *The action ϕ is called $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction action [11] if there exist nonnegative constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ with $0 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1$ such that*

$$\mathrm{d}(\phi(\mathbb{s}, x), \phi(\mathbb{s}, y)) \leq \lambda_1 \mathrm{d}(x, y) + \lambda_2 \mathrm{d}(x, \phi(\mathbb{s}, x)) + \lambda_3 \mathrm{d}(y, \phi(\mathbb{s}, y))$$
$$+ \lambda_4 \mathrm{d}(x, \phi(\mathbb{s}, y)) + \lambda_5 \mathrm{d}(y, \phi(\mathbb{s}, x)),$$

for all $x, y \in \mathbb{X}$, and $\mathbb{s} \in \mathbb{S}$.

Remark 2.14.

1. *A $\mathbb{G}_{\mathbb{s}}$-λ-contraction actions are special case of a $\mathbb{G}_{\mathbb{s}}$-(λ, L)-contraction actions such that every $\mathbb{G}_{\mathbb{s}}$-λ-contraction action is $\mathbb{G}_{\mathbb{s}}$-(λ, L)-**contraction** with $L = 0$.*
2. *A $\mathbb{G}_{\mathbb{s}}$-λ-contraction actions are special case of a $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction actions such that every $\mathbb{G}_{\mathbb{s}}$-λ-contraction action is $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction with $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.*

3.   Main Theorems

In this section, we proved some theorems that showed the actions which $\mathbb{G}_\mathrm{s}$-λ-contraction, $\mathbb{G}_\mathrm{s}$-(λ, L)-contraction, and $\mathbb{G}_\mathrm{s}$-Hardy-Rogers contraction respectively are $\mathbb{G}$-bi-shadowing.

Theorem 3.1. *Let ϕ be a $\mathbb{G}_\mathrm{s}$-λ-contraction action, then ϕ is $\mathbb{G}_\mathrm{s}$-(a,b)-bi-shadowing given by:*

$$\mathrm{a} = \frac{2}{1-\lambda} \ \ and \ \mathrm{b} = (1-\lambda) \tag{3.1}$$

Proof . *Fix $\delta < \frac{(1-\lambda)}{2}$. Let $\{y_g \equiv y_n | g \in \mathbb{G}, n \in \mathbb{Z}\}$ be a given $\mathbb{G}_\mathrm{s}$-δ-pseudo orbit for ϕ, and let $\psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be any action such that $\mathrm{d}_0(\phi, \psi) \le \frac{(1-\lambda)}{2}$.*
It follows that $\delta + \mathrm{d}_0(\phi, \psi) \le \frac{(1-\lambda)}{2} + \frac{(1-\lambda)}{2} = (1-\lambda) = \mathrm{b}$.
Fix $m \in \mathbb{Z}$, consider a $\mathbb{G}_\mathrm{s}$-orbit $\{x_g \equiv x_n | g \in \mathbb{G}, n \in \mathbb{Z}\}$ for ψ and satisfying

$$\mathrm{d}(x_m, y_m) \le \frac{1}{1-\lambda}(\delta + \mathrm{d}_0(\phi, \psi)) \tag{3.2}$$

By induction, for $n = m+1$ then:

$$\begin{aligned}
\mathrm{d}(x_{m+1}, y_{m+1}) &= \mathrm{d}(\psi(\mathrm{s}, x_m), y_{m+1}) \\
&\le \mathrm{d}(\psi(\mathrm{s}, x_m), \phi(\mathrm{s}, x_m)) + \mathrm{d}(\phi(\mathrm{s}, x_m), \phi(\mathrm{s}, y_m)) + \mathrm{d}(\phi(\mathrm{s}, y_m), y_{m+1}) \\
&\le \mathrm{d}_0(\phi, \psi) + \lambda \mathrm{d}(x_m, y_m) + \delta = (\delta + \mathrm{d}_0(\phi, \psi)) + \ \mathrm{d}(x_m, y_m).
\end{aligned}$$

For $n = m+2$ then:

$$\begin{aligned}
\mathrm{d}(x_{m+2}, y_{m+2}) &= \mathrm{d}(\psi(\mathrm{s}, x_{m+1}), y_{m+2}) \\
&\le \mathrm{d}(\psi(\mathrm{s}, x_{m+1}), \phi(\mathrm{s}, x_{m+1})) + \mathrm{d}(\phi(\mathrm{s}, x_{m+1}), \phi(\mathrm{s}, y_{m+1})) + \mathrm{d}(\phi(\mathrm{s}, y_{m+1}), y_{m+2}) \\
&\le \mathrm{d}_0(\phi, \psi) + \lambda \ \mathrm{d}(x_{m+1}, y_{m+1}) + \delta \\
&\le \mathrm{d}_0(\phi, \psi) + \lambda[(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda \mathrm{d}(x_m, y_m)] + \delta \\
&= (1+\lambda)(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^2 \mathrm{d}(x_m, y_m)
\end{aligned}$$

For $n = m+3$ then:

$$\begin{aligned}
\mathrm{d}(x_{m+3}, y_{m+3}) &= \mathrm{d}(\psi(\mathrm{s}, x_{m+2}), y_{m+3}) \\
&\le \mathrm{d}(\psi(\mathrm{s}, x_{m+2}), \phi(\mathrm{s}, x_{m+2})) + \mathrm{d}(\phi(\mathrm{s}, x_{m+2}), \phi(\mathrm{s}, y_{m+2})) + \mathrm{d}(\phi(\mathrm{s}, y_{m+2}), y_{m+3}) \\
&\le \mathrm{d}_0(\phi, \psi) + \lambda \ \mathrm{d}(x_{m+2}, y_{m+2}) + \delta \\
&\le \mathrm{d}_0(\phi, \psi) + \lambda[(1+\lambda)(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^2 \mathrm{d}(x_m, y_m)] + \delta \\
&\le (\delta + \mathrm{d}_0(\phi, \psi)) + \lambda[(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^2 \mathrm{d}(x_m, y_m)] \\
&\le (\delta + \mathrm{d}_0(\phi, \psi)) + \lambda(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^2(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^3 \mathrm{d}(x_m, y_m) \\
&\le (1 + \lambda + \lambda^2)(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^3 \mathrm{d}(x_m, y_m).
\end{aligned}$$

In general, we obtain

$$\begin{aligned}
\mathrm{d}(x_{m+n}, y_{m+n}) &\le (1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1})(\delta + \mathrm{d}_0(\phi, \psi)) + \lambda^n \mathrm{d}(x_m, y_m) \\
&\le \frac{1}{1-\lambda}(\delta + \mathrm{d}_0(\phi, \psi)) + \mathrm{d}(x_m, y_m).
\end{aligned}$$

Since $\lambda < 1$ and using the conditions (3.1) and (3.2) we have

$$\mathrm{d}\left(x_{m+n}, y_{m+n}\right) \le \frac{2}{1-\lambda}\left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) = \mathrm{a}\left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right).$$

Since $m \in \mathbb{Z}$ is arbitrary, then $\mathrm{d}\left(x_{m+n}, y_{m+n}\right) \le \mathrm{a}\left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right)$ for $n \in \mathbb{Z}$. $\square$

Now, we generalize the result in Theorem 3.1 to $\mathbb{G}$-(λ, L)-contraction actions.

Theorem 3.2. *Let ϕ be a $\mathbb{G}_\mathrm{s}$-(λ, L)-contraction action for $0 < \lambda < 1$, $L \ge 0$, and $\lambda + L < 1$. And let an action ϕ satisfy the conditions below:*

i) *For any $\mathbb{G}_\mathrm{s}$-δ-pseudo orbit $\{y_g \equiv y_n | g \in \mathbb{G},\ n \in \mathbb{Z}\}$ of ϕ with $\delta < \frac{(1-\lambda-L)}{2}$, then the series $S = \sum_{n=0}^{\infty} \mathrm{d}\left(\phi\left(\mathrm{s}, y_n\right), y_n\right)$ is convergent.*

ii) *For every action $\psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ satisfying $\mathrm{d}_0\left(\phi, \psi\right) \le \frac{(1-\lambda-L)}{2}$, the following inequality is satisfied:*

$$L\ S < \delta + \mathrm{d}_0\left(\phi, \psi\right). \tag{3.3}$$

Then the action ϕ is $\mathbb{G}_\mathrm{s}$-(a,b)-bi-shadowing given by

$$\mathrm{a} = \frac{2}{1-\lambda-L} \quad and \quad \mathrm{b} = \left(1-\lambda-L\right). \tag{3.4}$$

Proof . *Fix $\delta < \frac{(1-\lambda-L)}{2}$. Let $\{y_g \equiv y_n | g \in \mathbb{G},\ n \in \mathbb{Z}\}$ be a given $\mathbb{G}_\mathrm{s}$-δ-pseudo orbit for ϕ, and let $\psi : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ be any action such that $\mathrm{d}_0\left(\phi, \psi\right) \le \frac{(1-\lambda-L)}{2}$. It follows that*

$$\delta + \mathrm{d}_0\left(\phi, \psi\right) \le \frac{(1-\lambda-L)}{2} + \frac{(1-\lambda-L)}{2} = \left(1-\lambda-L\right) = \mathrm{b}.$$

Consider a $\mathbb{G}_\mathrm{s}$-orbit $\{x_g \equiv x_n | g \in \mathbb{G},\ n \in \mathbb{Z}\}$ for ψ satisfying Definition 2.10, and we use (3.3) to choose $m \in \mathbb{Z}$ such that x_m with the following property:

$$\mathrm{d}\left(x_m, y_m\right) + \frac{L\ S}{1-\lambda-L} \le \frac{\delta + \mathrm{d}_0\left(\phi, \psi\right)}{1-\lambda-L}. \tag{3.5}$$

By induction, for $n = m+1$:

$$\begin{aligned}
\mathrm{d}\left(x_{m+1}, y_{m+1}\right) &= \mathrm{d}\left(\psi\left(\mathrm{s}, x_m\right), y_{m+1}\right) \\
&\le \mathrm{d}\left(\psi\left(\mathrm{s}, x_m\right), \phi\left(\mathrm{s}, x_m\right)\right) + \mathrm{d}\left(\phi\left(\mathrm{s}, x_m\right), \phi\left(\mathrm{s}, y_m\right)\right) + \mathrm{d}\left(\phi\left(\mathrm{s}, y_m\right), y_{m+1}\right) \\
&\le \mathrm{d}_0\left(\phi, \psi\right) + \lambda \mathrm{d}\left(x_m, y_m\right) + L\ \mathrm{d}\left(x_m, \phi\left(\mathrm{s}, y_m\right)\right) + \delta \\
&\le \left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \lambda \mathrm{d}\left(x_m, y_m\right) + L\ \left[\mathrm{d}\left(x_m, y_m\right) + \left(y_m, \phi\left(\mathrm{s}, y_m\right)\right)\right] \\
&\le \left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \left(\lambda + L\right) \mathrm{d}\left(x_m, y_m\right) + L\ \mathrm{d}\left(y_m, \phi\left(\mathrm{s}, y_m\right)\right).
\end{aligned}$$

For $n = m+2$:

$$\begin{aligned}
\mathrm{d}\left(x_{m+2}, y_{m+2}\right) &= \mathrm{d}\left(\psi\left(\mathrm{s}, x_{m+1}\right), y_{m+2}\right) \\
&\le \mathrm{d}\left(\psi\left(\mathrm{s}, x_{m+1}\right), \phi\left(\mathrm{s}, x_{m+1}\right)\right) + \mathrm{d}\left(\phi\left(\mathrm{s}, x_{m+1}\right), \phi\left(\mathrm{s}, y_{m+1}\right)\right) + \mathrm{d}\left(\phi\left(\mathrm{s}, y_{m+1}\right), y_{m+2}\right) \\
&\le \mathrm{d}_0\left(\phi, \psi\right) + \lambda \mathrm{d}\left(x_{m+1}, y_{m+1}\right) + L \mathrm{d}\left(x_{m+1}, \phi\left(\mathrm{s}, y_{m+1}\right)\right) + \delta \\
&\le \left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \lambda \mathrm{d}\left(x_{m+1}, y_{m+1}\right) + L\ \left[\mathrm{d}\left(x_{m+1}, y_{m+1}\right) + \left(y_{m+1}, \phi\left(\mathrm{s}, y_{m+1}\right)\right)\right] \\
&\le \left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \left(\lambda + L\right) \mathrm{d}\left(x_{m+1}, y_{m+1}\right) + L\ \mathrm{d}\left(y_{m+1}, \phi\left(\mathrm{s}, y_{m+1}\right)\right). \\
&\le \left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \left(\lambda + L\right)\ \left[\left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \left(\lambda + L\right) \mathrm{d}\left(x_m, y_m\right) + L\ \mathrm{d}\left(y_m, \phi\left(\mathrm{s}, y_m\right)\right)\right] \\
&\quad + L\ \mathrm{d}\left(y_{m+1}, \phi\left(\mathrm{s}, y_{m+1}\right)\right). \\
&\le \left(1 + \left(\lambda + L\right)\right)\left(\delta + \mathrm{d}_0\left(\phi, \psi\right)\right) + \left(\lambda + L\right)^2 \mathrm{d}\left(x_m, y_m\right) + L\ \left(\lambda + L\right) \mathrm{d}\left(y_m, \phi\left(\mathrm{s}, y_m\right)\right) \\
&\quad + L\ \mathrm{d}\left(y_{m+1}, \phi\left(\mathrm{s}, y_{m+1}\right)\right).
\end{aligned}$$

For $n = m + 3$:

$$d(x_{m+3}, y_{m+3}) = d(\psi(s, x_{m+2}), y_{m+3})$$
$$\leq d(\psi(s, x_{m+2}), \phi(s, x_{m+2})) + d(\phi(s, x_{m+2}), \phi(s, y_{m+2})) + d(\phi(s, y_{m+2}), y_{m+3})$$
$$\leq (\delta + d_0(\phi, \psi)) + (\lambda + L) d(x_{m+2}, y_{m+2}) + L d(y_{m+2}, \phi(s, y_{m+2})).$$
$$\leq (\delta + d_0(\phi, \psi)) + (\lambda + L) [(1 + (\lambda + L))(\delta + d_0(\phi, \psi)) + (\lambda + L)^2 d(x_m, y_m)$$
$$+ L (\lambda + L) d(y_m, \phi(s, y_m)) + L d(y_{m+1}, \phi(s, y_{m+1}))] + L d(y_{m+2}, \phi(s, y_{m+2}))$$
$$\leq (1 + (\lambda + L) + (\lambda + L)^2)(\delta + d_0(\phi, \psi)) + (\lambda + L)^3 d(x_m, y_m) + L (\lambda + L)^2 d(y_m, \phi(s, y_m))$$
$$+ L (\lambda + L) d(y_{m+1}, \phi(s, y_{m+1})) + L d(y_{m+2}, \phi(s, y_{m+2})).$$

In general, we obtain

$$d(x_{m+n}, y_{m+n}) \leq (\delta + d_0(\phi, \psi)) \sum_{k=0}^{n-1} (\lambda + L)^k + (\lambda + L)^n d(x_m, y_m)$$
$$+ \sum_{k=0}^{n-1} L (\lambda + L)^{n-k-1} d(y_{m+k}, \phi(s, x_{m+k}))$$

Note, if we write

$$\sum_{k=0}^{n-1} L (\lambda + L)^{n-k-1} d(y_{m+k}, \phi(s, x_{m+k})) = \sum_{k=0}^{n-1} a_k b_{n-k-1}$$

$a_n = d(y_{m+k}, \phi(s, x_{m+k}))$, *and* $b_n = L (\lambda + L)^n$ *for* $n \geq 0$, *and if*

$$c_n = \sum_{k=0}^{n-1} a_k b_{n-k-1}$$

then Theorem 8.46 in [7] implies that the series $\sum_{k=0}^{\infty} c_n$ *is convergent and*

$$\sum_{k=0}^{\infty} c_n = \frac{L S}{1 - \lambda - L}.$$

Therefore, by the conditions (3.4) and (3.5) and since $\lambda + L < 1$ *we have*

$$d(x_{m+n}, y_{m+n}) \leq (\delta + d_0(\phi, \psi)) \sum_{n=0}^{\infty} (\lambda + L)^n + (\lambda + L)^n d(x_m, y_m) + \sum_{n=0}^{\infty} c_n$$
$$\leq \frac{\delta + d_0(\phi, \psi)}{1 - \lambda - L} + (x_m, y_m) + \frac{L S}{1 - \lambda - L} \leq \frac{\delta + d_0(\phi, \psi)}{1 - \lambda - L} + \frac{\delta + d_0(\phi, \psi)}{1 - \lambda - L}$$
$$= \frac{2 (\delta + d_0(\phi, \psi))}{1 - \lambda - L} = a (\delta + d_0(\phi, \psi)).$$

Since $m \in \mathbb{Z}$ *is arbitrary, then* $d(x_{m+n}, y_{m+n}) \leq a(\delta + d_0(\phi, \psi))$ *for* $n \in \mathbb{Z}$. $\square$

Theorem 3.3. *Let ϕ be a $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction action as in Definition 2.13, then it is $\mathbb{G}_{\mathbb{s}}$-(λ, L)-contraction such that*

$$\lambda = \frac{\lambda_1 + \lambda_2 + \lambda_4}{1 - \lambda_3 - \lambda_4} \ and \ L = \frac{\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} \tag{3.6}$$

Proof .

$$\mathrm{d}\left(\phi\left(\mathbb{s},x\right),\phi\left(\mathbb{s},y\right)\right)$$
$$\leq \lambda_1 \mathrm{d}\left(x, y\right) + \lambda_2 \mathrm{d}\left(x, \phi\left(\mathbb{s},x\right)\right) + \lambda_3 \mathrm{d}\left(y, \phi\left(\mathbb{s},y\right)\right) + \lambda_4 \mathrm{d}\left(x, \phi\left(\mathbb{s},y\right)\right) + \lambda_5 \mathrm{d}\left(y, \phi\left(\mathbb{s},x\right)\right)$$
$$\leq \lambda_1 \mathrm{d}\left(x, y\right) + \lambda_2 \ \left[\mathrm{d}\left(x, y\right) + \left(y, \phi\left(\mathbb{s},x\right)\right)\right] + \lambda_3 \ \left[\mathrm{d}\left(y, \phi\left(\mathbb{s},x\right)\right) + \left(\phi\left(\mathbb{s},x\right), \phi\left(\mathbb{s},y\right)\right)\right]$$
$$+ \lambda_4 \ \left[\mathrm{d}\left(x, y\right) + \mathrm{d}\left(y, \phi\left(\mathbb{s},x\right)\right) + \mathrm{d}\left(\phi\left(\mathbb{s},x\right), \phi\left(\mathbb{s},y\right)\right)\right] + \lambda_5 \mathrm{d}\left(y, \phi\left(\mathbb{s},x\right)\right)$$
$$\leq \left(\lambda_1 + \lambda_2 + \lambda_4\right) \mathrm{d}\left(x, y\right) + \left(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5\right) \mathrm{d}\left(y, \phi\left(\mathbb{s},x\right)\right) + \left(\lambda_3 + \lambda_4\right) \mathrm{d}\left(\phi\left(\mathbb{s},x\right), \phi\left(\mathbb{s},y\right)\right),$$

It follows that

$$\mathrm{d}\left(\phi\left(\mathbb{s},x\right), \phi\left(\mathbb{s},y\right)\right) \leq \frac{\lambda_1 + \lambda_2 + \lambda_4}{1 - \lambda_3 - \lambda_4} \ \mathrm{d}\left(x, y\right) + \frac{\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} \mathrm{d}\left(y, \phi\left(\mathbb{s},x\right)\right)$$

Take

$$\lambda = \frac{\lambda_1 + \lambda_2 + \lambda_4}{1 - \lambda_3 - \lambda_4} \ and \ L = \frac{\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4},$$

Then by assumption $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1$ we have $L \geq 0$ and $\lambda < 1$.
This shows that ϕ is $\mathbb{G}_{\mathbb{s}}$-(λ, L)-contraction. $\square$

Theorem 3.4. *Let ϕ be $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction action satisfying (i) and (ii) of Theorem 3.2. Then the action ϕ is $\mathbb{G}_{\mathbb{s}}$-$(\mathbb{a}, \mathbb{b})$-bi-shadowing provided that $\lambda_1 + 2 \ \lambda_2 + 2 \ \lambda_3 + 3 \ \lambda_4 + \lambda_5 < 1$ such that*

$$\mathbb{a} = 2\frac{1 - \lambda_3 - \lambda_4}{1 - \lambda_1 - 2\lambda_2 - 2\lambda_3 - 3\lambda_4 - \lambda_5} \ and \ \mathbb{b} = \frac{1 - \lambda_1 - 2\lambda_2 - 2\lambda_3 - 3\lambda_4 - \lambda_5}{1 - \lambda_3 - \lambda_4} \tag{3.7}$$

Proof . *Let ϕ be a $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction action and satisfying (i) and (ii) of Theorem 3.2. By Theorem 3.3 a $\mathbb{G}_{\mathbb{s}}$-Hardy-Rogers contraction action is $\mathbb{G}_{\mathbb{s}}$-(λ, L)-contraction then ϕ is $\mathbb{G}_{\mathbb{s}}$-bi-shadowing provided that*

$$\lambda + L = \frac{\lambda_1 + 2 \ \lambda_2 + \lambda_3 + 2 \ \lambda_4 + \lambda_5}{1 - \lambda_3 - \lambda_4} < 1,$$

that is $\lambda_1 + 2 \ \lambda_2 + 2 \ \lambda_3 + 3 \ \lambda_4 + \lambda_5 < 1$.
Moreover, we can find the values of $\mathbb{a}$ and $\mathbb{b}$ in (3.7) substituting λ and L which given by (3.6) in (3.4). $\square$

4. conclusion

The $\mathbb{G} - \lambda$ -Contraction actions, $\mathbb{G} - (\lambda, L)$- Contraction action with some conditions, and $\mathbb{G}-$Hardy-Rogers contraction action with some conditions are $\mathbb{G}$-bi-shadowing

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