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Caputo definition for finding fractional moments of power law distribution functions

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Abstract

In this paper, we propose a modern technique to derive fractional moment by using Caputo definition of fractional derivative. Such technique represents a modification of fractional moments on a certain type of function which is known by the power law distribution functions. The results have been obtained show that the obtained fractional moments within such functions have closed form.

Keywords: Fractional moment, Caputo definition, Power law distribution functions, Fourier transform.
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1. Introduction

The idea of derivative for integer order may be considered as a special case of fractional order derivative. Fractional derivative is originally starting from some speculations of Leibniz (1695, 1697) and Euler (1730) in 18th century by asking a very smart question, namely, " *What is the meaning of half derivative*" this question led to a fruitful and progressive area of research and applications for wide area of different science, for instance, engineering, physics, social phenomena and many other applications [4].

There are many different definitions for fractional order derivative and integral, the most useful ones which are used in wide range of applications are Riemann-Liouville definition (RL) [4] Caputo derivative [5], Grünwald–Letnikov definition [5] and Manchaud [3]. These definitions shared with some fundamental properties to ensure the generalization of ordinary derivative and integral.

In this work the power law distribution functions are considered and studied in terms of moments. A power law distribution function is a special case of probability distribution function which satisfies

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the following: if α is particular value of specific quantity and if α varies in inverse form as a power of probability of measuring α then α is said to follow the power law, also, drops of stress distributed like the power law behavior, when it is unbalanced, [3].

There are several problems arise when dealing with power law distribution functions, for instance , when studying the moment of distribution of random variable with power law distribution, $F(x) \sim |x|^{-\alpha}, \alpha > 0$ have $F(x)$ is the distribution function, in these cases, moments $E(|x|^{\nu})$ exist if $\nu > \alpha$ and other integer moments of order not less than α is divergent.

In front of this problem, there are many real applications arise in the real life and many fields of science, for instance: the search processes of animals [2], the length distribution of motion of human, plasma devices fluctuation time spending in turbulent flow tracing of groundwater, distribution of cities, earth quacks, forest fires and solar flares [2].

Recently, fractional operators has great attention for applications in science and engineering [3]. Also there are clear attention from researcher forward to the direction of fractional moments of different density functions because of wide applications areas which have been appears in last decades.

Until now, research on FC was limited to the field of mathematics. However, in the last two decades, many applications of FC in different fields of engineering, science, mathematics and economics have been studied. As a result, FC has become an important topic for researchers in various fields. One of these fields is the field of probability and statistics. In the other hand, the power-law distribution functions occur in some abnormal phenomena. For instance, society populations, the distribution of earthquakes, solar flares [6], computer files and the numbers of words in research scientists write [1].

This paper consists of four parts. In the next part, we present several basic concepts that would be useful to our main structures. In part three we have presented our main ideas. In the last part we conclude some results related to our work and refer to some future work.

2. Basic notions and preliminaries

2.1. Definitions

In the following, definitions of the most used forms of fractional derivatives which maintained above, the Caputo definitions, Riemann–Liouville (RL), and the Grünwald–Letnikov (GL), [5] are stated respectively for seek of comparison:

2.1.1. Caputo definition

The Caputo derivative for the function $g(x)$ of order α is given as:

$$D_*^{\alpha} g(t) = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{g^{(m)}(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau \tag{2.1}$$

where $((m-1) < \alpha < m), m$ is integer and α is real number, $\Gamma(.)$ is the well-known gamma function.

2.1.2. Riemann–Liouville

The RL definition for a function $g(x)$ is given as:

$$D_t^{\alpha} g(t) = \frac{1}{\Gamma(m - \alpha)} (\frac{d}{dt}) \int_0^t \frac{g(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau \tag{2.2}$$

where $((m-1) < \alpha < m), m$ is integer and α is real number.

2.1.3. Grünwald–Letnikov

The GL definition for a function $g(x)$ is given as:

$$D_t^{\alpha} g(t) = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)h^{\alpha}} \sum_{k=0}^{\frac{t-\alpha}{h}} (\frac{\Gamma(\alpha+k)}{\Gamma(k+1} g(t-kh) \tag{2.3}$$

where h is the sampling period, $\Gamma(.)$is the well-known gamma function and means the integer part of $\frac{t-\alpha}{h}$, the discrete nature of this definition make it popular to use for fractional derivative and integral

2.1.4. Power Law Distribution Function

The goal functions under consideration in this work are the power law distribution functions which appear in many applications in different areas of science, for instance, the motion of human or animals in searching for their food could be modeled in some power law distribution functions [2] as well as the motion of carried charges in semiconductors [3]. These applications are quite specific for such applications , so the non-central moments may have thereby much specific meaning, which may studied in future work. For a real variable with a continuous power law distribution of the real variable x associated a probability density function $p(x)$ in the interval $(x, x+dx)$ is given by:

$$p(x) = \frac{\alpha-1}{x_0}(\frac{x}{x_0})^{-\alpha}, \alpha > 1 \tag{2.4}$$

where x_0 is the minimum value for x .

2.1.5. Convolution Operation

Convolution property of Foureir transform tells that convolution in some variable corresponds to multiplication in the co-domain. This will be more convenient than directly performing the convolution. Consider two functions $h(x)$ and $g(x)$, and their corresponding Fourier transform $H(f)$ and $G(f)$, the convolution of these functions denoted by : $f = h * g$ is defined as:

$$f(t) = h * g = \int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau \tag{2.5}$$

The convolution theorem stated as follows: The FT of the convolution is the product of the two FT.

2.2. preliminaries

In this section the basic concepts and preliminaries are stated.

Theorem 2.1. *[1] "Let X be a non-negative r.v. with f as the corresponding c.f. The moment m_k exists if and only if the k^{th} derivative of f exists at the point $t = 0$".*

Generally, the k^{th} moment is defined for integer $k > 0$ as following:

$$\begin{aligned}
E(x^k) &= \int_{-\infty}^{\infty} x^k p(x)dx \\
&= \int_{x_0}^{\infty} x^k p(x)dx \\
&= \int_{x_0}^{\infty} x^k \frac{\alpha-1}{x_0}(\frac{x}{x_0})^{-\alpha}dx \\
&= \frac{\alpha-1}{x_0^{\alpha-1}}\int_{x_0}^{\infty} x^{k-\alpha}dx \\
&= x_0^k(\frac{\alpha-1}{\alpha-1-k}) \text{ for } k<\alpha-1
\end{aligned}$$

If $k = 1$, the first moment is defined while k^{th} moment of order $k > 1$ is infinite, therefore it is convenient to be use Caputo derivative in finding the fractional moments for these cases. The lack of moments leads to limitation of using such probability density functions, namely, the CF or logarithmic functions using moments.[2]

3. Methodology

3.1. Derivation of fractional moments

In order to find moments of random variable it is usually using the higher order derivative of characteristic functions which defined as following: If $F(x)$ represents the probability of the random variable X for all values less than x is the CDF and $p(x)$ is the PDF. The Fourier transform of $p(x)$, denoted as $Fp = \phi(u)$ is the first characteristic functions of first kind represented as:

$$\phi(\nu) = E(e^{i\nu t}) = \int_{-\infty}^{\infty} e^{i\nu t}p(x)dx \tag{3.1}$$

where $\nu \in \mathbb{R}$ and the integer moments of $X$is the expectation of the function $g(X) = X^k$ with $k = 1, 2, 3, \cdots$ These integer moments denoted by $E(X^k)$. To simplify the work with the fractional order and generalization to the complex order, the Taylor expansion is used in association with this property:

$$E(iX)^k = \frac{d^k\phi(\nu)}{d\nu^k}\,|_{\nu=0} \tag{3.2}$$

and the FT of the function $p(x)$ is given by:

$$\phi(\nu) = \sum_{k=0}^{\infty}(E(iX)^k)\frac{\nu^k}{k!} \tag{3.3}$$

Singh et. al. in [6] propose a closed form to evaluate the fractional derivative of Fourier transform of some real valued function based on the Caputo derivative. In this paper, the fractional moments of order $\alpha > 0$ is handle by using the closed form evaluated in [6].

By using properties of Fourier transform, it have been shown in [2] that moment M_ν of order ν is derived from the CDF by the derivative as following:

$$M_\nu = \frac{d^\nu\phi(t)}{du^\nu}\,|_{u=0} \tag{3.4}$$

3.2. Results and Discussion

In this part, we use the closed form derived above to the power law distribution function. If we look at the definition of Caputo in 2.1 as a convolution operation of two functions 2.5 as:

$$D_*^\alpha g(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} * \frac{dx(t)}{dt} \tag{3.5}$$

and taking the Fourier transform of 3.5 we get:

$$\mathcal{F}[D_*^\alpha g(t)] = \mathcal{F}[t^{-\alpha}/(\Gamma(1-\alpha)) * (dx(t))/dt]$$

$$\mathcal{F}[D_*^\alpha g(t)] = F[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}]F[(dx(t))/dt] \tag{3.6}$$

Then the Fourier transform of fractional derivative 3.5 is given by:

$$F[D_*^\alpha g(t)](\omega)] = (i\omega)^\alpha F[\frac{dx(t)}{dt}] \tag{3.7}$$

Therefore 3.7 represents a closed form for Fourier transform for the Caputo derivative of order α where $0 < \alpha < 1$. Apply 3.7 for the power law distribution defined in 2.4 we need to find the derivative as following:

$$\frac{dp(x)}{dx} = -(\frac{x}{x_0})^{1-\alpha}, \quad \alpha > 1. \tag{3.8}$$

The Fourier transform for 3.8 with utility of properties of Fourier transform gives:

$$F[-(\frac{x}{x_0})^{1-\alpha}](k) = \frac{-1}{(x_0)^{1-\alpha}} F((x_0)^{1-\alpha}) = \int_0^\infty (x)^{1-\alpha} e^{-2\pi ikx} dk = -\frac{(ix^{-\alpha})}{2\pi} \tag{3.9}$$

According to 3.4, the fractional moment of order ν is the fractional derivative of the Fourier transform given in 3.9 above, which is the desired result.

4. Conclusion and future works

The power series expansion which used to represent the power law distribution function may be useful in the direction of finding the Fourier transform of another types of functions. The above result can be used to find the fractional moment for the general form of the power law distribution functions. For future work, one may consider the fractional moments for order of complex number as well.

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References

[1] G. Laue, *Remarks on the Relation between Fractional Moments and Fractional Derivatives of Characteristic Functions*, J. Appl. Probab. 17(2) (1980) 456–466.

[2] G. Cottone and M. Di Paola, *On the use of fractional calculus for the probabilistic characterization of random variables*, Probab. Egin. Mech. Phys. A 389 (2010) 909—920.

[3] I.B. Bapna and N. Mathur, *Application of Fractional Calculus in Statistics*, Int. J. Contemp. Math. Sci. 7(18) (2012) 849–856.

[4] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, New York: Academic, 1974.

[5] M. Ganji, N. Eghbali and F. Gharari, *Some Fractional Special Functions and Fractional Moments*, Gen. Math. Notes 18(2) (2013) 120–127.

[6] K. Singh, R. Saxena and S. Kumar *Caputo-base fractional derivative in fractional Fourier transform domain*, IEEE J. Emerg. Select Topics in Circ. Syst. 3(3) (2013) 23650151.